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ITERATIVE OPTIMAL PLASTIC DESIGN OF STEEL FRAMES*

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1. INTRODUCTION

It is well known that the three basic conditions to be satisfied by any plastic design of a steel structure are [1] equilibrium, [2] mechanism and [3] yield. A number of optimal plastic design methods have been developed that take these three conditions into account in a variety of ways. For example, some methods[2),16)~19)] based on the static theorem simultaneously satisfy the equilibrium and yield conditions in the design process, with a subsequent check of the mechanism condition. Conversely, other methods[3),8),9),12),14)] based on the kinematic theorem simultaneously satisfy the equilibrium and mechanism conditions, with a subsequent check of the yield condition. There are still other proposed methods[5),6)] where, in a highly efficient technique, all three conditions are simultaneously considered. While all of these methods have no intrinsic difficulties, they become increasing difficult to apply as the complexity of the structure increases[10)].

This study presents a kinematic approach to the optimal plastic design of framed structures using linear programming. Here, the design process involves the minimization of an objective criterion (i.e., total steel weight) while satisfying the equilibrium and mechanism conditions i.e., the limit equilibrium condition that the structure may not fail in any of the all possible collapse modes prior to the specified design ultimate load level.

Theoretically, the constraints to the design should include the complete set of limit equilibrium equations pertaining to all possible collapse modes for the structure. While the design of simple structures poses no problem in this regard, there are considerable difficulties with the condition of limit equilibrium in the case of complex structures having a large number of possible collapse modes. Not only is the determination of all such modes a tedious task, but the total inclusion of the corresponding limit equilibrium equations as constraints to the design often makes the problem size prohibitive. It is said[12)] that the number of all possible collapse modes for a five-story, five-bay frame is about 70,000. In attempting to overcome these problems, it has been recognized[12),14)] that it is not likely that all collapse modes will be critical for any one design and, therefore, that only a limited number of equilibrium constraints need be considered. Even then, aside from the difficulties in finding these critical modes, the question still remains as to how many and which modes to take as constraints to the design.

The present paper develops an iterative optimal plastic design technique[13)] that overcomes the above noted difficulties with the limit equilibrium condition. Essentially, the procedure involves the performance of a series of minimum weight design where, at any one stage, the limit equilibrium equation pertaining to the critical collapse mode for the previous design is added to the constraint set for the next design. The initial constraints to the first optimal design pertain to the elementary mechanisms excluding joint mechanisms. Upon succeeding iterations, the final design is achieved when a subsequent strength analysis finds the collapse load for the design to be greater than or equal to the specified design load level. Both the plastic design and strength analysis processes are formulated as linear programming (LP) problems. The optimal plastic designs for three example frames are presented. Results are achieved using a specially devised computer program (Fig. 1) that determines the optimal design and its active equilib-

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rium constraints, and, in addition, also identifies the sequence in which the various active collapse modes become critical to the design. In every case, convergence to the final design is very rapid and the number of equilibrium constraints considered by the design is considerably less than the total number of possible collapse modes for the structure. Larger and more complex examples that all possible collapse modes can not be determined by a hand work are readily possible, but it is believed that those presented are sufficient to illustrate the simplicity and efficiency of the approach[10].

2. OPTIMAL PLASTIC DESIGN

Based on the kinematic approach, the optimal plastic design of a steel frame with flexural members may be stated as the following LP problem[5),6)]:

minimize:

$$Z=\sum_{h=1}^{n} M_{ph}L_h \quad \cdots\cdots(1a)$$

subject to:

$$\sum_{h=1}^{n} C_{rh}M_{ph} \geq \lambda_0 e_r \qquad (r=1, 2, \cdots, p) \quad \cdots(1b)$$

and:

$$M_{ph} \geq 0 \quad \cdots\cdots(1c)$$

where Eq. (1a) represents the selected objective criterion of minimum total weight of the frame, Eqs. (1b) define the limit equilibrium conditions pertaining to all possible collapse modes. h=index referring to the number of design variables in a structure; n=number of design variables, M_{ph}; C_{rh}=coefficients indicating the contribution of M_{ph} in the r-th equilibrium condition; M_{ph}= the unknown design plastic moment; L_h=the member length over which M_{ph} is constant; Z= the total weight of the frame at some scale; p= total number of possible collapse mechanisms; e_r=measure of external work by service loads in a collapse mechanism r; r=index referring to possible collapse mechanisms; λ_0=specified design load factor on the service loads (i.e., proportional loading assumed).

It should be noted, however, that Eqs. (1b) are classified by limit equilibrium equations pertaining to elementary mechanisms excluding joint mechanisms and combined mechanisms, respectively. That is,

$$\sum_{h=1}^{n} a_{ih}M_{ph} \geq \lambda_0 e_i \qquad (i=1, 2, \cdots, q) \quad \cdots\cdots(1d)$$

$$\sum_{h=1}^{n} b_{kh}M_{ph} \geq \lambda_0 e_k \qquad (k=1, 2, \cdots, p-q) \quad \cdots\cdots(1e)$$

where i, k=indexes referring to elementary mechanisms and combined mechanisms, respectively; a_{ih}, b_{kh}=coefficients indicating the contribution of M_{ph} in the i-th elementary mechanism and in the k-th combined mechanism, respectively; e_i, e_k=measures of external work by service loads in a elementary mechanism i and a combined mechanism k; q=total number of elementary mechanisms excluding joint mechanisms, i.e., beam and panel mechanisms.

The problem posed by Eqs. (1a), (1c), (1d) and (1e) involves the determination of the optimal set of design plastic moments M_{ph} $(h=1, 2, \cdots, n)$. Here, all possible collapse modes are considered and, as such, the need to check the yield condition is eliminated.

3. COLLAPSE LOAD ANALYSIS

Based on the kinematic theorem[15], the collapse load analysis of a flexural frame may be stated as the following LP problem[10]:

minimize:

$$\lambda_k=\sum_{j=1}^{s} (M_{pj}^{+}\theta_{kj}^{+}+M_{pj}^{-}\theta_{kj}^{-}) \quad \cdots\cdots(2a)$$

subject to:

$$\theta_{kj}^{+}-\theta_{kj}^{-}-\sum_{i=1}^{m} t_{ki}\theta_{ij}=0 \qquad (j=1, 2, \cdots, s) \quad \cdots\cdots(2b)$$

$$\sum_{i=1}^{m} t_{ki}e_i=1 \quad \cdots\cdots(2c)$$

and:

$$\theta_{kj}^{+}, \theta_{kj}^{-} \geq 0 \quad \cdots\cdots(2d)$$

where Eq. (2a) defines the internal energy dissipated in forming the critical collapse mode for a unit value of external work, Eqs. (2b) express the condition that the relative rotations of the plastic-hinge sections in the critical collapse mode are each a linear combination of the relative rotations associated with the elementary mechanisms for the frame, and Eq. (2c) defines the condition that the external work is equal to unity. The quantities in Eqs. (2) are defined as follows: j=index referring to critical sections; m=number of elementary mechanisms; s=number of critical sections; λ_k=collapse load factor corresponding to collapse of the frame in critical collapse mode k; θ_{ij}=relative rotation of section j in elementary mechanism i; t_{ki}=factor defining the way and extent to which elementary mechanism i enters the mechanism combination forming the critical collapse mode k. Because a standard LP constraint is that the variables be non-negative, t_{ki} is transposed as $t_{ki}=t'_{ki}-t$. t'_{ki}= value of t_{ki} in positive solution space for transposed LP problem; t=positive constant used to transpose LP problem into positive solution space. M_{pj}^{+}, M_{pj}^{-}=the plastic moment capacities of the positive and negative direction at the section j, respectively; $\theta_{kj}^{+}, \theta_{kj}^{-}$=absolute values of relative rotation of the positive and negative direction at plastic-hinge section j in collapse mode k, respectively. All other quantities are as defined for Eqs. (1).

Here, the set of plastic moment capacities $M_{pj}^{\pm}$ ($j=1, 2, \cdots, s$) are known and the analysis involves the determination of the collapse load factor, λ_k, the set of relative rotations defining the critical collapse mode k, $\theta_{kj}^{\pm}$ ($j=1, 2, \cdots, s$), and the set of factors defining the combination of elementary mechanisms forming the critical collapse mode k, t_{ki} ($i=1, 2, \cdots, m$).

4. ITERATIVE OPTIMAL PLASTIC DESIGN

The iterative optimal plastic design is performed as follows (see Fig. 1):

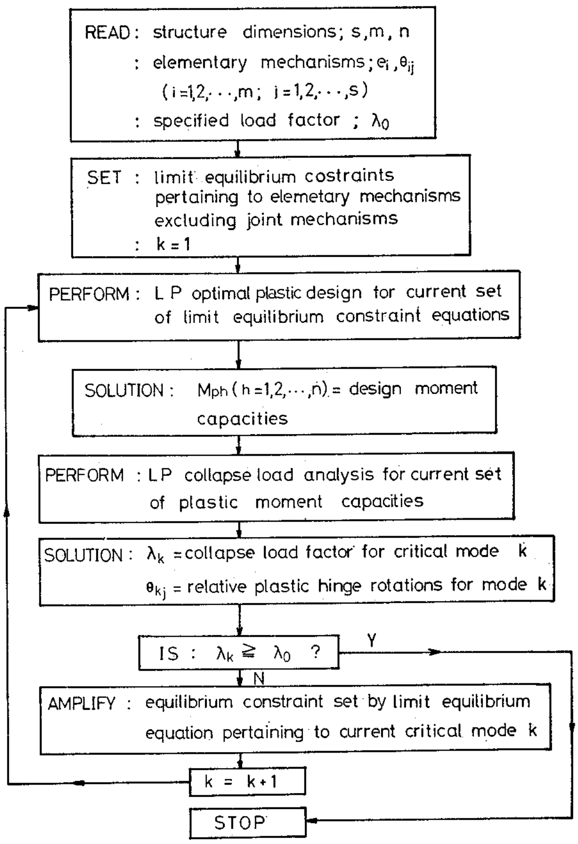

Fig. 1 Flow Chart: Iterative Optimal Plastic Design.

(1) Construct the elementary mechanisms with θ_{ij} and e_i ($i=1, 2, \cdots, m$; $j=1, 2, \cdots, s$) as the input data.

(2) Initially take the limit equilibrium equations pertaining to the elementary mechanisms excluding joint mechanisms ($i=1, 2, \cdots, q$) as the equilibrium constraints and perform an optimal plastic design to determine a set of M_{ph} ($h=1, 2, \cdots, n$).

That is, the first optimal design may be expressed by the following non-dimensionalized matrix form.

minimize:

$$z=\boldsymbol{L}\boldsymbol{x} \quad \cdots\cdots(3a)$$

subject to:

$$\left.\begin{array}{l} \boldsymbol{a}_1\boldsymbol{x} \geqslant \lambda_0 \\ \boldsymbol{a}_2\boldsymbol{x} \geqslant \lambda_0 \\ \cdots\cdots \\ \boldsymbol{a}_q\boldsymbol{x} \geqslant \lambda_0 \end{array}\right\} \quad \cdots\cdots(3b)$$

$$\boldsymbol{x} \geqslant 0 \quad \cdots\cdots(3c)$$

where Eqs. (3b) correspond to Eqs. (1d), respectively. $\boldsymbol{L}=[L_1 L_2 \cdots L_n]/L$; $z=Z/WL^2$; $\boldsymbol{a}_i=[a_{i1}a_{i2} \cdots a_{in}]WL/e_i$; $a_{ih}=\sum_{j=s_{h-1}+1}^{s_h}(\theta_{ij}^{+}+\theta_{ij}^{-})$ ($h=1, 2, \cdots, n$; $i=1, 2, \cdots, q$); s_h=the maximum number of critical sections which are controlled by the design variable M_{ph}, for instance, $s_1=4$, $s_2=7$, $s_3=11$, $s_4=14$ in the frame as shown in Fig. 6. Therefore, $s_0=0$ and $s_n=s$; W, L =a standard service load and a standard member length;

$$\boldsymbol{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=\frac{1}{WL}\begin{pmatrix} M_{p1} \\ M_{p2} \\ \vdots \\ M_{pn} \end{pmatrix}$$

Solving Eq. (3), the set of initial solutions $\boldsymbol{x}_0$ is found to be the set of the lowest values among the design moments because of adopting independent elementary mechanisms as the equilibrium constraints.

(3) Perform a collapse load analysis to determine the collapse load factor and the critical collapse mode corresponding to the set of plastic moments found in the previous design.

That is, the collapse load factor λ_k and the critical collapse mode $\boldsymbol{b}_k$ corresponding to the set of plastic moment $\boldsymbol{x}_{k-1}$ found in the k-th optimal design are determined by the k-th collapse load analysis as follows:

$$\lambda_k=\boldsymbol{b}_k\boldsymbol{x}_{k-1} \quad \cdots\cdots(4a)$$

$$\boldsymbol{b}_k=[b_{k1}b_{k2}\cdots b_{kn}]WL/e_k \quad \cdots\cdots(4b)$$

where $b_{kh}=\sum_{j=s_{h-1}+1}^{s_h}(\theta_{kj}^{+}+\theta_{kj}^{-})$ $(h=1, 2, \cdots, n)$;

$$\boldsymbol{x}_{k-1}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{k-1}=\frac{1}{WL}\begin{pmatrix} M_{p1} \\ M_{p2} \\ \vdots \\ M_{pn} \end{pmatrix}_{k-1},$$

$$M_{p1}=M_{pj}^{+}=M_{pj}^{-} \quad (j=1, 2, \cdots, s_1),$$
$$M_{p2}=M_{pj}^{+}=M_{pj}^{-} \quad (j=s_1+1, \cdots, s_2),$$
$$\vdots$$
$$M_{pn}=M_{pj}^{+}=M_{pj}^{-} \quad (j=s_{n-1}+1, \cdots, s),$$

It should be noted that Eq. (4a) expresses a hyperplane with the slope $\boldsymbol{b}_k$ passing through the point $\boldsymbol{x}_{k-1}$ in the $x_1, x_2, \cdots, x_n$-space, geometrically and that Eq. (4b) defines a collapse in the weakest part of the structure.

(4) Depending on the values of λ_k found in step 3, proceed in one of the two following ways:

(a) if $\lambda_k<\lambda_0$, the structure may fail in the collapse mode $\boldsymbol{b}_k$ prior to the specified ultimate load level, as the hyperplane expressed by Eq. (4a) is located in the outside of the k-th feasible design region even if it passes through the extreme point $\boldsymbol{x}_{k-1}$ on the boundary lines of region. Therefore, in order that the structure may not fail in the collapse mode $\boldsymbol{b}_k$, add to the previous set of equilibrium constraints a new limit equilibrium equation pertaining to collapse mode found in step 3 (i.e., $\boldsymbol{b}_k\boldsymbol{x}\geqslant\lambda_0$) and perform another optimal design to determine a new set of design moments $\boldsymbol{x}_k$. Return to step 3.

That is, the $(k+1)$-th optimal design may be formulated as the following LP problem.

minimize:

$$z=\boldsymbol{L}\boldsymbol{x} \quad \cdots\cdots(5a)$$

subject to:

$$\left.\begin{array}{l} \boldsymbol{a}_1\boldsymbol{x}\geq\lambda_0 \\ \boldsymbol{a}_2\boldsymbol{x}\geqslant\lambda_0 \\ \cdots\cdots \\ \boldsymbol{a}_q\boldsymbol{x}\geqslant\lambda_0 \\ \boldsymbol{b}_1\boldsymbol{x}\geqslant\lambda_0 \\ \boldsymbol{b}_2\boldsymbol{x}\geqslant\lambda_0 \\ \cdots\cdots \\ \boldsymbol{b}_{k-1}\boldsymbol{x}\geqslant\lambda_0 \end{array}\right\} \quad \cdots\cdots(5b)$$

$$\boldsymbol{b}_k\boldsymbol{x}\geqslant\lambda_0 \quad \cdots\cdots(5c)$$

$$\boldsymbol{x}\geqslant 0 \quad \cdots\cdots(5d)$$

where Eqs. (5b) express the previous set of equilibrium constraints and Eq. (5c) defines the new limit equilibrium condition.

Here, it should be noted that Eq. (5c) represents an active constraint for the $(k+1)$-th optimal design.

As it is obvious that the values $\boldsymbol{x}_k$ found in Eqs. (5) satisfy the new limit equilibrium condition Eq. (5c),

$$\boldsymbol{b}_k\boldsymbol{x}_k\geqslant\lambda_0 \quad \cdots\cdots(6a)$$

On the other hand, from Eq. (4a) and $\lambda_k<\lambda_0$,

$$\boldsymbol{b}_k\boldsymbol{x}_{k-1}<\lambda_0 \quad \cdots\cdots(6b)$$

Therefore,

$$\boldsymbol{b}_k\boldsymbol{x}_k>\boldsymbol{b}_k\boldsymbol{x}_{k-1} \quad \cdots\cdots(6c)$$

Eq. (6c) means that the previous hyperplane with the slope $\boldsymbol{b}_k$ passed through the point $\boldsymbol{x}_{k-1}$ is lifted up in parallel to the new hyperplane passed through the point $\boldsymbol{x}_k$ in the feasible design region. Because it is found that $x_1, x_2, \cdots, x_n$-intercepts of the new hyperplane defined by the left-hand side of Eq. (6c) are greater than those of the hyperplane expressed by the right-hand side of Eq. (6c), respectively.

Repeating step 3 and 4 (i.e., $k=1, 2, \cdots$), the required feasible design region excluding inactive constraints is finally formed by adding to the previous constraints Eqs. (5b) the new active constraint Eq. (5c).

(b) if $\lambda_k\geq\lambda_0$, the structure may not fail in any of the all possible collapse modes prior to the specified ultimate load level. Geometrically, the hyperplane Eq. (4a) determined by the k-th collapse load analysis is located in the inside or on the one of boundary lines forming the feasible design region defined by Eqs. (5b). Actually, $\lambda_k=\lambda_0$, i.e., the collapse mode $\boldsymbol{b}_k$ coincides with the one of the previous collapse modes $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_q, \boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_{k-1}$ and the set of final optimal design moments $\boldsymbol{x}_k$ has been found to be the set of correct solutions.

To prove the iterative approach mentioned above geometrically, a simple example as shown in Fig. 2 that has been investigated elsewhere[15] is examined in detail.

It is required to find the design moments x_1 $(=M_{p1}/WL)$ and x_2 $(=M_{p2}/WL)$ which minimize the objective function $z=3x_1+2x_2$ while satisfying the limit equilibrium conditions correspond-

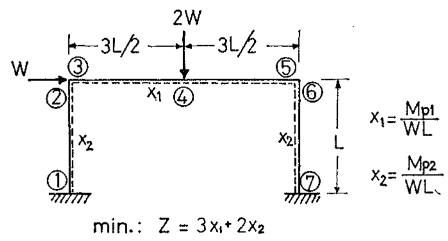

Fig. 2 Frame Geometry and Service Loading.

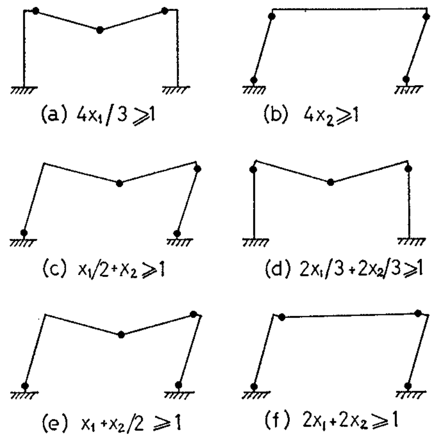

Fig. 3 Possible Collapse Modes

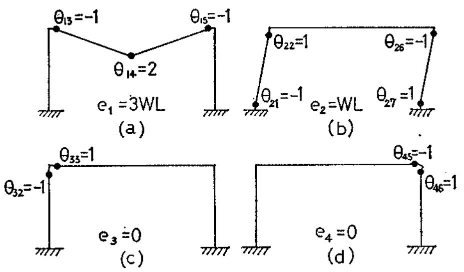

Fig. 4 Elementary Mechanisms

ing to all possible collapse modes shown in Fig. 3 ($\lambda_0=1$ is assumed). Although the solution to this problem may be easily found by only one LP application as shown in Fig. 5 (c), the proposed method is firstly started with selecting the elementary mechanisms as shown in Fig. 4 with their θ_{ij} and e_i values (it is assumed that positive moments and plastic rotations are associated with tension occurring on the side of the frame closest to the dotted line shown in Fig. 2).

Performing the first optimal design by adopting two elementary mechanisms in Fig. 4 (a), (b) (or Fig. 3 (a), (b)) to be the initial constraints, i.e., the lines ⓐ, ⓑ in Fig. 5 (a), the solution is found to be the extreme point A ($x_1=0.75$, $x_2=0.25$) as shown in Fig. 5 (a). The first collapse load analysis is then performed with θ_{ij} and e_i values in Fig. 4 and the straight line ⓒ′ that passes through the point A is determined as follows:

$$\lambda_1=\frac{1}{2}x_1+x_2=0.625 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)$$

It is noted, however, that the straignt line ⓒ′ in Fig. 5 (a) is located in the outside of the

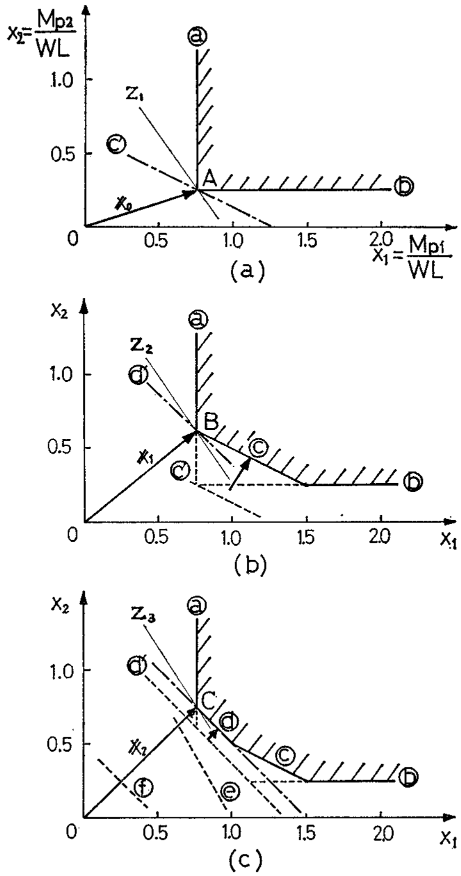

Fig. 5 Geometrical Interpretation

feasible design region, i.e., $\lambda_1<\lambda_0$ and therefore, the frame will fail in the collapse mode shown in Fig. 3 (c) prior to the specified load level.

Adding to the previous constraints, i.e., lines ⓐ, ⓑ in Fig. 5 (a) a new constraint $x_1/2+x_2\geq 1$ to strengthen the weakness of the frame in the second optimal design, it is found that the dotted line ⓒ′ in Fig. 5 (b) has been lifted up in parallel to the line ⓒ which represents an active constraint. The second collapse load analysis is then performed and the line ⓓ′ in Fig. 5 (b) that passes through the point B ($x_1=0.75$, $x_2=0.625$) is found to be:

$$\lambda_2=\frac{2}{3}x_1+\frac{2}{3}x_2=0.917 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (8)$$

Eq. (8) indicates, however, that the line ⓓ′ in Fig. 5 (b) is still placed in the outside of the feasible design region, i.e., $\lambda_2<\lambda_0$. The dotted line ⓓ′ in Fig. 5 (c) is again lifted up to the line ⓓ to perform the third optimal design. It is confirmed that a new straight line $\lambda_3=2x_1/3+$

$2x_2/3=1$ determined by the third collapse load analysis coincides with the line ⓓ in Fig. 5 (c), i.e., $\lambda_3=\lambda_0$ and that the extreme point C ($x_1=0.75$, $x_2=0.75$) is the final optimal solution.

As evidenced by the geometrical interpretation as shown in Fig. 5, the set of final results of this approach is necessarily found converged to the set of correct solutions. Because the approach represents a design by finding and strengthening the weakest part of the structure, i.e., the critical collapse mode.

5. EXAMPLES

(1) Example 1

In order to illustrate a fairy complex frame, the two-story, one-bay plane frame[6] made of perfectly plastic prismatic bars as shown in Fig. 6 is to be designed to resist a given set of ultimate loads (here, a specified design load factor $\lambda_0=1$ is assumed). In this example, the constraints to the design should include theoretically sixty limit equilibrium equations corresponding to all possible collapse mechanisms as identified in the reference[6].

The frame has $s=14$ critical sections, $N=6$ statical indeterminacies and, therefore, $m=s-N=8$ elementary mechanisms that are taken to be as shown in Fig. 7.

Initially taking four elementary mechanisms

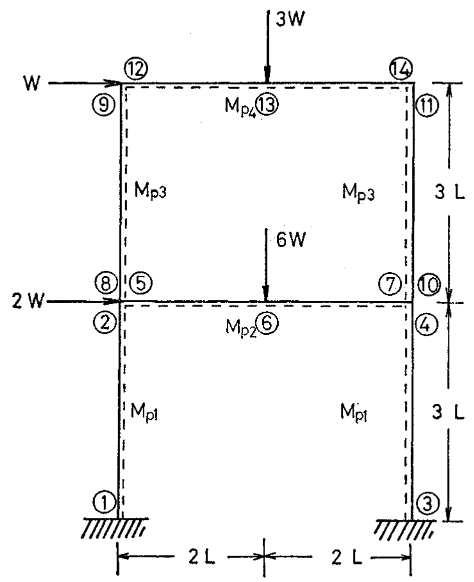

Fig. 6 Example 1: Frame Geometry and Service Loading.

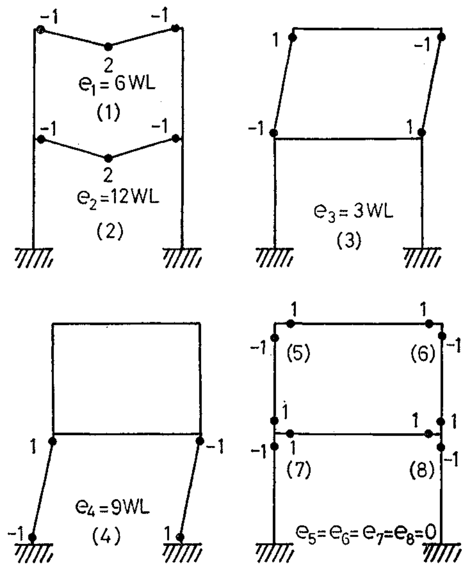

Fig. 7 Example 1: Elementary Mechanisms

($i=1, 2, 3, 4$) excluding joint mechanisms shown in Fig. 7 to be the limit equilibrium constraints, the first optimal plastic design problem can be formulated using Eqs. (1a), (1c) and (1d) as follows:

minimize:

$$Z=(3+3)M_{p1}L+4M_{p2}L+(3+3)M_{p3}L+4M_{p4}L \quad \cdots\cdots(9a)$$

subject to:

$$\left.\begin{array}{ll} 4M_{p4}\geq 6WL & (i=1) \\ 4M_{p2}\geq 12WL & (i=2) \\ 4M_{p3}\geq 3WL & (i=3) \\ 4M_{p1}\geq 9WL & (i=4) \end{array}\right\} \quad \cdots\cdots(9b)$$

$$M_{p1},\ M_{p2},\ M_{p3},\ M_{p4}\geq 0 \quad \cdots\cdots(9c)$$

By using the simplex algorithm, the solution to the above problem is found to be: $M_{p1}=2.25\,WL$, $M_{p2}=3.0\,WL$, $M_{p3}=0.75\,WL$, $M_{p1}=1.5\,WL$ and $Z=36.0\,WL^2$. Therefore, taking the M_{ph} ($h=1, 2, 3, 4$) found above to be the plastic moment capacities of the critical sections, $M^{\pm}_{pj}$ ($j=1, 2, \cdots, 14$), and adopting the elementary mechanisms given in Fig. 7 with θ_{ij} and e_i ($i=1, 2, \cdots, 8$; $j=1, 2, \cdots, 14$) values, a collapse load analysis is performed to determine a new limit equilibrium constraint for the second optimal design. The critical collapse mode so found is $k=1$ in Fig. 8. It is noted that from Fig. 9 for $k=1$ that the first design violated the condition of limit equilibrium since $\lambda_k<\lambda_0$ for elementary collapse mechanisms excluding joint mechanisms (i.e., $\lambda_1=$

$0.667\lambda_0$). From Fig. 8 for $k=1$, the corresponding limit equilibrium equation is:

$$4M_{p3}+2M_{p4}\geq 9\,WL \quad\cdots\cdots\cdots\cdots(10)$$

Therefore, a second optimal design is then per-

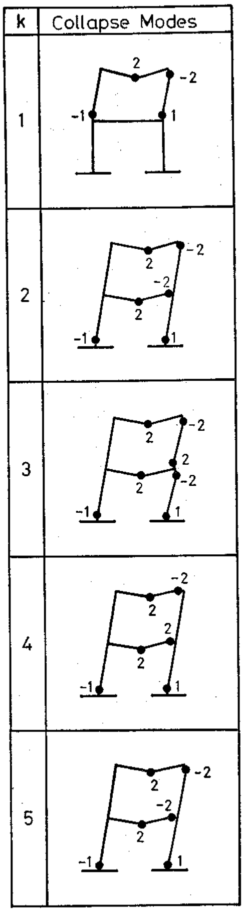

Fig. 8 Example 1: Critical Collapse Modes (k)

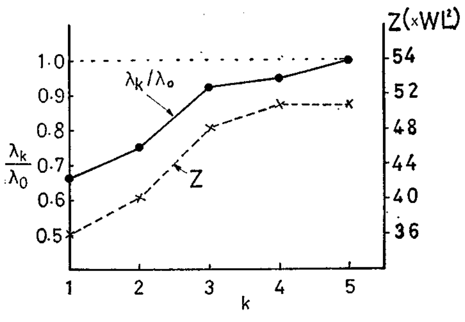

Fig. 9 Example 1: Relative Safety (λ_k/λ_0) and Total Weight (Z) versus Collapse Modes (k)

formed with Eq. (10) added to the initial equilibrium constraint set expressed by Eqs. (9b). Subsequent collapse load analysis of the resulting design to determine the current critical collapse mode $k=2$ indicates, however, that the limit equilibrium condition is still violated, as shown in Fig. 9 for $k=2$ (i.e., $\lambda_2=0.750\lambda_0$).

The iterative procedure is then continued and the limit equilibrium condition is eventually found satisfied for the fifth optimal plastic design, as evidenced in Fig. 9 for $k=5$. It is noted that $\lambda_k=\lambda_0$ for the collapse mode $k=5$ found to be critical for the final design, and from Fig. 8, that this mode corresponds to the limit equilibrium constraint $k=2$ found in the design process. In fact, performance of an alternate collapse load analysis[10] for the final design determines that $\lambda_k=\lambda_0$ for all modes shown in Fig. 8, thereby indicating a design with a great deal of plastic adaptability and, hence, optimality. The set of final design moments and the total weight of the frame are found to be: $M_{p1}=3\,WL$, $M_{p2}=4.5\,WL$, $M_{p3}=M_{p4}=1.5\,WL$ and $Z=51\,WL^2$. These solutions agree with the values found in the reference[6] and the procedure considers only eight collapse modes among sixty possible collapse modes.

(2) Example 2

An optimal plastic design is required for the one-bay, four-story frame as shown in Fig. 10 for

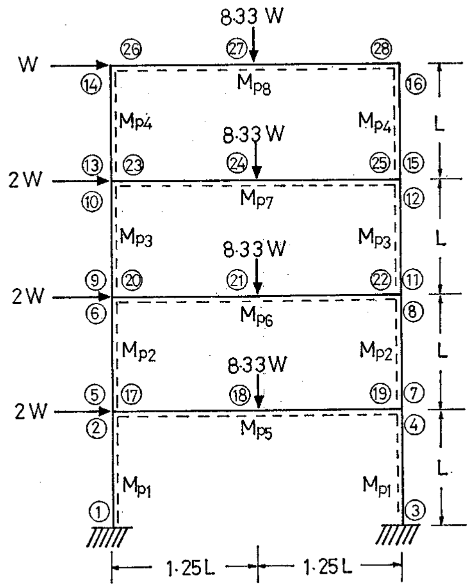

Fig. 10 Example 2: Frame Geometry and Service Loading.

which $W=1.6$ ton and $L=4$ m. A design load factor $\lambda_0=1.8$ is specified for adequate ultimate safety. The $m=s-N=28-12=16$ elementary mechanisms selected for the structure are shown in Fig. 11 with their θ_{ij} and e_i values.

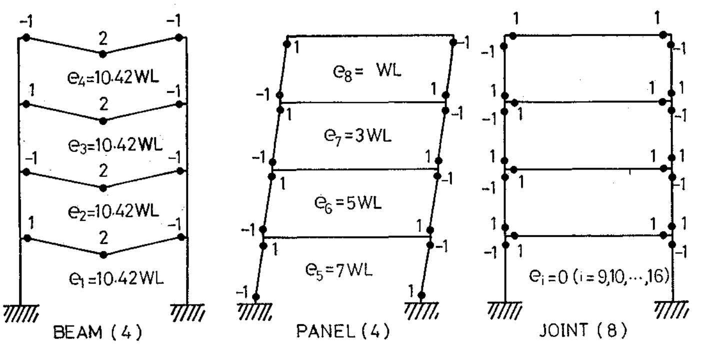

Fig. 11 Example 2: Elementary Mechanisms

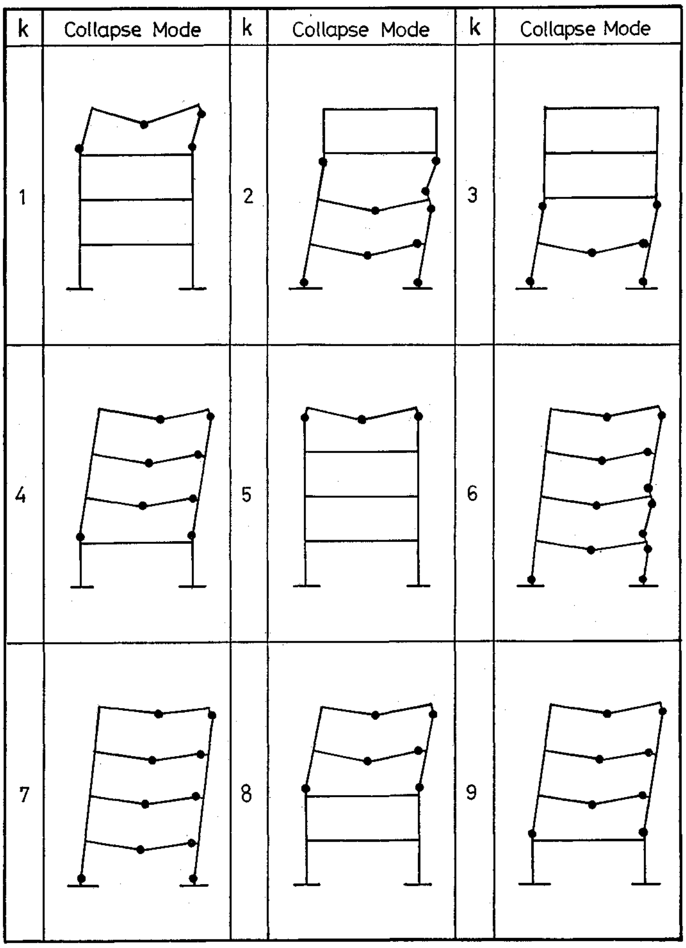

Fig. 12 Example 2: Critical Collapse Modes (k)

The first optimal design is performed with eight limit equilibrium equations corresponding to beam and panel mechanisms in Fig. 11. The condition of limit equilibrium is then checked for the set of design moments resulting for the first optimal design by performing collapse load analysis. The corresponding critical collapse mode is $k=1$ in Fig. 12. From Fig. 13 for $k=1$, it is noted that the design violates the condition of limit equilibrium since $\lambda_k<\lambda_0$ for its critical collapse mode (i.e., $\lambda_1=0.543\lambda_0$). The limit equilibrium equation pertaining to mode $k=1$ in Fig. 12 is then added to the constraint set, and the design procedure is continued until the limit equilibrium condition is found satisfied for the ninth optimal design, as evidenced by the fact that $\lambda_k=\lambda_0$ for the collapse mode $k=9$ found to be critical for this design as shown in Fig. 13.

It is noted that Fig. 13 shows a slight decrease in the collapse load factor λ_7 and no increase in the total weight Z for $k=7$, although the seventh optimal design has been performed with the new constraint corresponding to mechanism $k=6$ in Fig. 12 added to the previous constraints. This phenomenon is due to the degeneracy[4] in the LP problem, which is defined by forming the weight compatible mechanism[15] without contribution of mechanism $k=6$ in Fig. 12. Because it is confirmed that the weight function $Z=2(M_{p1}+M_{p2}+M_{p3}+M_{p4})L+2.5(M_{p5}+M_{p6}+M_{p7}+M_{p8})L$ agrees with the composed mechanism of mechanisms $i=3, 4, 5$ in Fig. 11 and $k=2, 3, 4, 5, 6$ in Fig. 12 multiplied by the coefficients 1/4, 1/8, 3/16 and 1/2, 1/8, 3/8, 5/8, 0, respectively and that those mechanisms are satisfied as equations of limit equilibrium conditions in the seventh optimal design. However, although the degeneracy has been arised in the seventh optimal design, the set of seventh optimal solutions is correct[4] and therefore, the set of

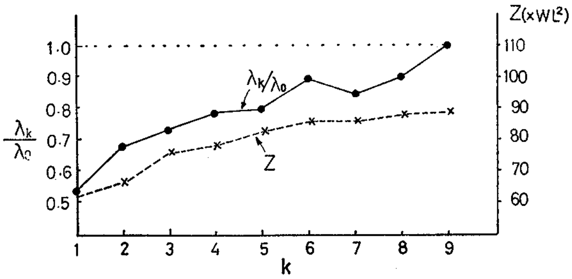

Fig. 13 Example 2: Relative Safety (λ_k/λ_0) and Total Weight (Z) versus Collapse Modes (k).

final optimal solutions is independent of the degeneracy in LP problem.

The set of final solutions is found to be: $M_{p1}=3.835\,WL=24.54$ t-m, $M_{p2}=4.176\,WL=26.73$ t-m, $M_{p3}=1.350\,WL=8.64$ t-m, $M_{p4}=4.689\,WL=30.01$ t-m, $M_{p5}=8.010\,WL=51.26$ t-m, $M_{p6}=5.526\,WL=35.37$ t-m, $M_{p7}=5.814\,WL=37.21$ t-m, $M_{p8}=4.689\cdot WL=30.01$ t-m and $Z=88.197\,WL^2=2\,257.84$ t-m^2. Equilibrium constraints corresponding to mechanisms $i=4, 7$ in Fig. 11 and $k=2, 4, 5, 6, 7, 8, 9$ in Fig. 12 were satisfied as equations in the final design.

(3) Example 3

The four-bay, two-story frame shown in Fig. 14 is to be designed to resist applied load for the two following cases ($\lambda_0=1$ is assumed).

(a) Case I: The relative design moment capacities for the frame have been selected such that the resulting optimal design will represent a strong-column weak-beam design, i.e., $M_{p1}=1.5\cdot M_p$, $M_{p2}=2.5M_p$, $M_{p3}=2M_p$, $M_{p4}=M_p$, $M_{p5}=1.5M_p$, $M_{p6}=M_p$ (M_p: a standard design moment capacity).

The frame has 20 elementary mechanisms as shown in Fig. 15. Therefore, the first optimal design is performed with 10 limit equilibrium conditions corresponding to beam and panel mechanisms ($i=1, 2, \cdots, 10$) in Fig. 15. The collapse mode becoming critical for the resulting first optimal design is found to be the beam mechanism denoted by solid lines in Fig. 16 with $\lambda_1=\lambda_0$. Performance of an alternate collapse load analysis of this design determines that $\lambda_1=\lambda_0$ for the three beam mechanisms denoted by broken lines in Fig. 16 and, therefore, it is very close to being a truly optimum strong-column weak-beam design (i.e., all four beam mechanisms having the possibility of forming). The set of final design moments and the total weight are found to be: $M_{p1}=M_{p5}=1.250\,WL$, $M_{p2}=2.083\,WL$, $M_{p3}=1.667\,WL$, $M_{p4}=M_{p6}=0.833\,WL$ and $Z=30.938\,WL^2$.

(b) Case II: The frame shown in Fig. 14 is to be designed with the added requirement that the beam and column sizes prevail con-

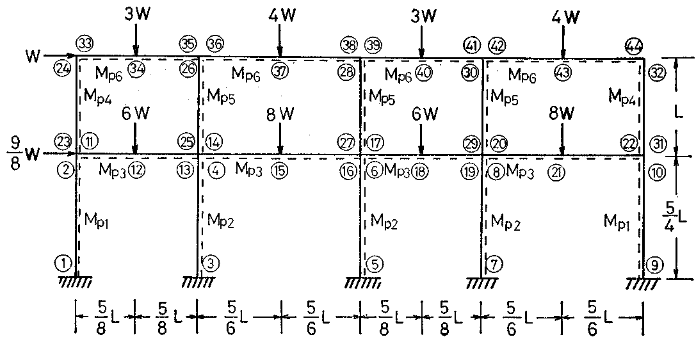

Fig. 14 Example 3: Frame Geometry and Service Loading.

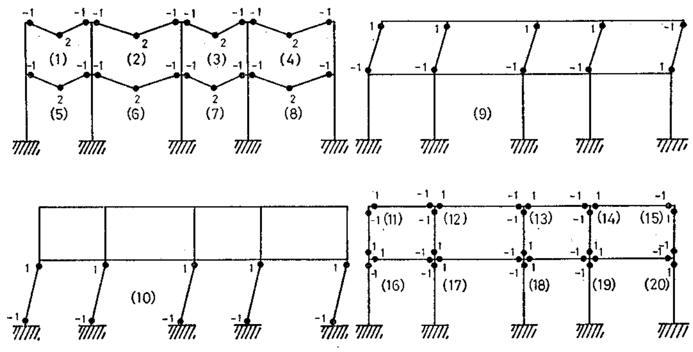

Fig. 15 Example 3: Elementary Mechanisms

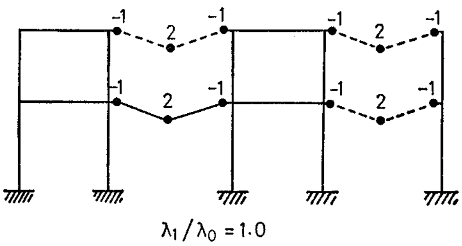

Fig. 16 Example 3: Critical Collapse Modes (k) (Case I)

stant at each story, i.e., $M_{p1}=M_{p2}=\alpha_1$, $M_{p3}=\alpha_2$, $M_{p4}=M_{p5}=\alpha_3$, $M_{p6}=\alpha_4$ (α_1, α_2, α_3, α_4: the design moment capacities as independent variables).

The iterative design procedure is conducted, and the final design satisfying all limit equilibrium conditions is found to correspond to the sixth optimal design. The collapse modes becoming critical at the various stages of the design are shown in Fig. 17, and the corresponding convergence to the required limit equilibrium conditions that $\lambda_k \geq \lambda_0$ for all possible modes is indicated in Fig. 18. The final design is such that $\lambda_k=\lambda_0$ for five of six modes (i.e., $k=1, 2, 4, 5, 6$) shown in Fig. 17. The set of final solutions are given as follows: $M_{p1}=M_{p2}=0.279\,WL$, $M_{p3}=2.082\,WL$, $M_{p4}=M_{p5}=0.143\,WL$, $M_{p6}=1.063\,WL$ and $Z=20.804\,WL^2$.

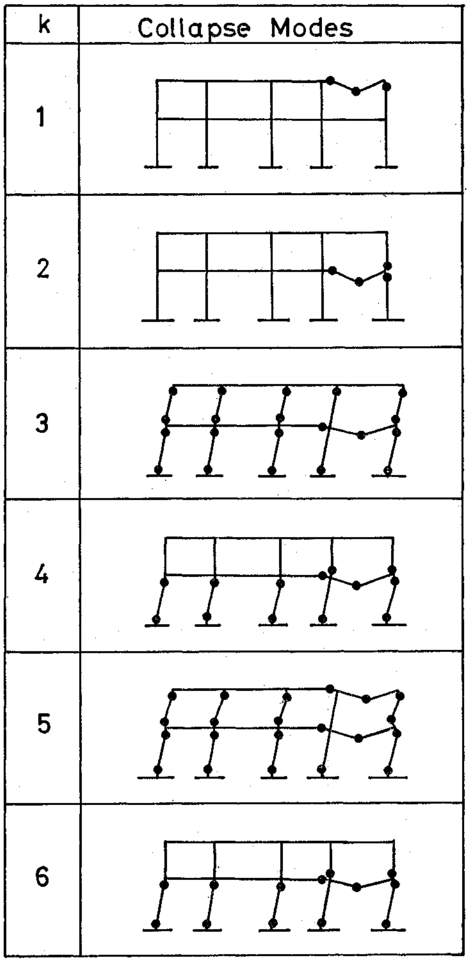

Fig. 17 Example 3: Critical Collapse Modes (k) (Case II)

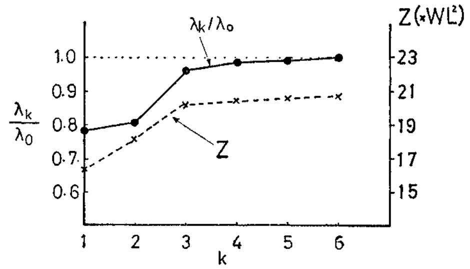

Fig. 18 Example 3: Relative Safety (λ_k/λ_0) and Total Weight (Z) versus Collapse Modes (k) (Case II)

6. CONCLUSIONS

The paper has presented an iterative approach to the optimal plastic design of steel frames wherein the limit equilibrium constraint set is progressively enlarged by means of a series of analysis and design iterations until the condition of limit equilibrium is found satisfied for all possible collapse modes. The procedure identifies and considers only those collapse modes that are critical to the design, and, therefore, circumvents the need to consider a great many of the total number of possible collapse modes. In fact, the maximum number of collapse modes that need ever be considered is equal to the number of independent elementary mechanisms for the frame.

The proposed method represents a design by tracing and reinforcing the weakness of the structure. The analysis and design phases are both formulated as linear programming problem and, as such, the procedure readily lends itself to efficient computerization and may be of great advantage to the larger and more complex frame in which all possible collapse modes can not be easily idendified by a hand task. It is of interest to note that at most $(n+1)$ or $(n+2)$ computing cycles have been required for all examples worked to date. Using an IBM 360/195 computer of Japan IBM Company, the solutions for the three examples presented were achieved in 11 sec, 18 sec, 12 sec (Case I) and 21 sec (Case II), respectively.

With but minor revision, the method has direct application to the optimal plastic design of structures subjected to variable repeated loading[5),6),11)] and may be extended to the designs allowing for the effect of axial forces[1),7)].

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