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ON A NEW EDDY MODEL IN TURBULENT SHEAR FLOW

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1. INTRODUCTION

Turbulent flows may be classified into two categories by the fact that the flow would be directly influenced by solid boundaries or not: the wall turbulent flow and the free turbulent flow. Because of complicated boundary conditions the research on wall turbulence has been more backward as compared with that on free turbulence.

Air-tunnel experiments conducted by NACA group are one of the representative studies on wall turbulence in early the 1950's. In this group Schubauer, Klebanoff et al[1] worked at the boundary layer flow and Laufer[2] dealt with the pipe flows, both obtaining noteworthy results. As peculiar characteristics of turbulent shear flow different from those of free turbulence had been recognized by experiments, a few kinds of an eddy model were proposed by some researchers to explain the behaviours of wall turbulence.

In 1956 Townsend[3] presented so called 'attached eddy model' by applying his theory of large eddy motion derived from the experimental results on grid or wake turbulence to the wall turbulent flow. He tried to explain the experimental facts obtained by Laufer by making use of this model of which a rotating axis oriented to longitudinal direction. Following Townsend's concept, Grant[4] devised a similar eddy model in 1958. These models are constructed so as to satisfy the measured data of spatial correlations of turbulence, but even in qualitative aspects these include something unreasonable. Early in the 1960's Willmarth et al[5], Corcos[6] and the others distinguished more clearly the structures of wall turbulence by the analysis of time-space correlations of the fluctuating pressure in a boundary layer flow. Besides a rapid progress in measurement of turbulence in open channel flow has been made in the last ten years by adopting a hot-film anemometer and a hydrogen bubble tracer.

There were several attempts to describe mechanism of turbulence production by making use of so called 'hairpin eddy model'[7] which has been considered for transition from laminar flow to turbulent flow. In 1966 Willmarth et al[8] proposed 'average model of vortex line' in order to describe structures of wall turbulence qualitatively. The vortex line in this model was assumed to be on a flat plane which inclined at an angle θ to the wall as shown in Fig. 1. The model was verified by detailed measurements of time-space correlation due to Favre[9] and Sternberg[10], and the angle of inclination of vortex line θ was fairly evaluated.

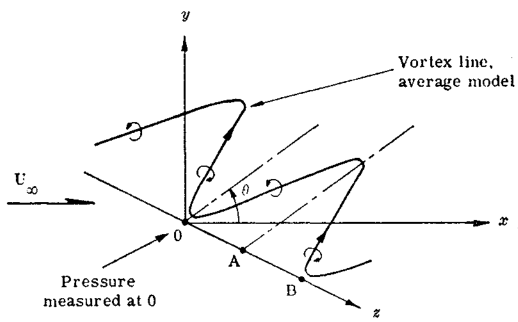

Fig. 1 Average model vortex line (after Willmarth et al[8])

In 1967 a research group of Stanford University represented by Kline et al[11] offered a noteworthy eddy model so that a mechanism of wall turbulence production could be reasonably explained. This model was based on detailed measurement of bursting processes obtained by improved a hydrogen bubble technique which was thought to be the most effective method of measurements of wall turbulence. In the model low-speed streaks with vorticities in the direction of z-axis are lifted up and then stretched with travelling as to have an inclination toward the wall. The vortex line is eventually broken up and more chaotic motions of the vortex appear to produce turbu-

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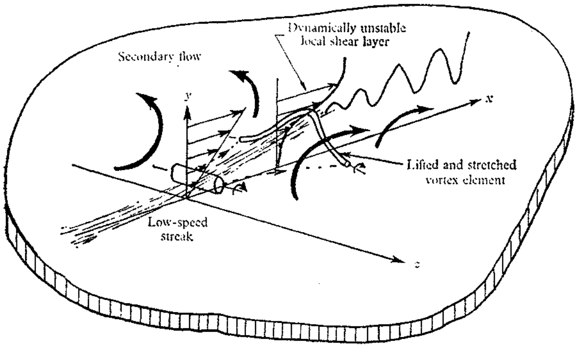

Fig. 2 Mechanics of streak breakup (after Kline et al[11])

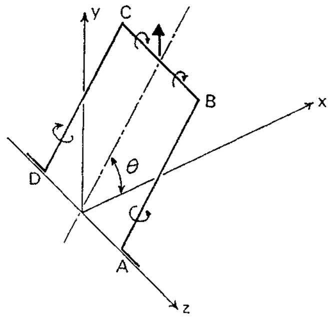

Fig. 3 A π-eddy model

lence as shown in Fig. 2.

Such an eddy model which is called a horseshoe vortex model has been applied to the description of turbulence characteristics. For instance, Yokoshi et al[12] tried to explain the structure of turbulence in actual river by using the eddy model and concluded that a boiling phenomenon in the river would be caused by this eddy.

Early in the 1970's the efforts to elucidate the structures of wall turbulence have been made by the combined means of point measurement with hot-film anemometers and of flow visualization with hydrogen bubble tracers. The interesting researches have been done by Corino et al[13], Kim et al[14], Grass[15], Clark[16], Wallace et al[17] and Willmarth et al[18]. Despite of recent results that production mechanism of bursting has been indicated clearly by analysis of experimental data and proposal of reasonable eddy models, none of them looks to be refined to explain the behaviours of wall turbulence quantitatively yet.

In consideration of the situation mentioned above, a simple eddy model to describe quantitatively the characteristics of wall turbulence in open channel flow is proposed and it is verified by analytical investigations of experimental results obtained by Laufer and the authors.

2. π-EDDY MODEL

(1) Constitution of an Eddy Model and its Formulation

Due to the fact that a horseshoe vortex model proposed already by Willmarth or other reseachers is fairly reasonable in qualitative aspects, as mentioned above, this horseshoe vortex model is accepted as an original eddy model in the following discussions. But, since it is fairly difficult to obtain exact expression of the horseshoe vortex model, in order to discuss quantitatively its behaviours, a simplified eddy model which has the angular vortex lines as described in Fig. 3 may be considered here. As the vortex line of this simplified eddy model has a Π-shape, this model may be called 'a π-eddy model'.

As shown in Fig. 3 a square vortex line which has an angle of inclination θ toward the x-axis is assumed for two-dimensional turbulent shear flow. Its legs $\overline{AB}$ and $\overline{CD}$ are in a plane parallel to the $x-y$ plane, and its top $\overline{BC}$ is parallel to the z-axis. The condition $\theta=0$ represents an incipient stage of a π-eddy which coincides with a hairpin eddy and an attached eddy. However, due to insufficiency of the observed data in the wall layer ($y^+\equiv yU^*/\nu=5\sim10$, for the thickness of viscous sublayer), an equilibrium condition under which the π-eddy is lifted up from the bottom and fully developes is considered here.

Now, a particular eddy is observed in relation to the coordinate travelling with convective velocity of the eddy U_c. Because the distance $\overline{BC}$ is negligible compared with $\overline{AB}$ as indicated by Kline et al[11], contribution of the vortices along the line $\overline{BC}$ to turbulence production can be ignored except that the line $\overline{BC}$ will suffer lift force. On considering the mean eddy scales, a vortex tube $\overline{AB}$ may be assumed to have an elliptical cross section with a long radius a in the x-direction and a short radius b in the z-direction. This assumption has been also done for an attached eddy model proposed by Townsend[3], and verified by the measurements of spatial correlations due to Laufer[23] and other experimenters. Consequently it is concluded that the π-eddy shows an elliptic motion with angular velocity ω. (Fig. 4)

Relationships between the coordinates ($x(t)$,

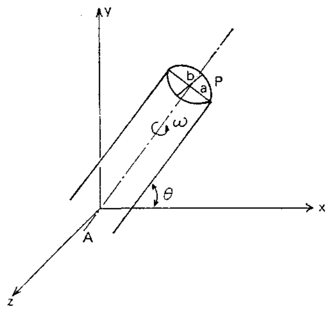

Fig. 4 Vortex line of a π-eddy.

$y(t)$, $z(t)$) of an arbitrary point on the surface of a vortex tube and (x_0, y_0, z_0) of a point P on the rotating axis in the circuit will be given by

$$\left.\begin{aligned} x(t)&=x_0(t)-a\cos\omega t\sin\theta \\ y(t)&=y_0(t)+a\cos\omega t\cos\theta \\ z(t)&=z_0(t)+b\sin\omega t \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(1)$$

It is expected that the point $P(x_0, y_0, z_0)$ will be lifted up along the vortex line due to acceleration of the vortex stretching carried by lift force for $\overline{BC}$ and by main shear flow, and that it will be descended due to depression of the stretching, besides the length scale of the eddy in the z-direction may be assumed to be invariable because the vortex stretching in the x-direction by the main flow $U(y)$ is dominant.

When a head of equilibrium π-eddy is infinitesimally displaced by disturbance from B to B' as indicated in Fig. 5, the vortex line $\overline{AB}$ shifts to $\overline{AB'}$ by stretching. By the fact that the circulation in any circuit moving with the fluid is invariant, it can be concluded that the angular velocity of the vortex ω changes into $(\omega+\Delta\omega)$. So long as a π-eddy under the equilibrium condition does not instantaneously disappear by stretching, some apparent resistances must be thought to work upon the eddy, resulted from complex interactions between the mean flow and turbulence. Analytical investigation of bursting and sweeping phenomena which characterize the

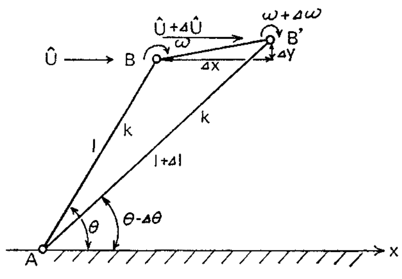

Fig. 5 Vortex stretching of a π-eddy.

mechanism of wall turbulence production will surely contribute to evaluate such a resisting force of the eddy model.

But, as the first step for the present, this apparent resisting force may be simply assumed to be proportional to a displacement length of the vortex line.

The acting forces are given by

$$\left.\begin{aligned} &\text{Lift force : } L=\rho\hat{U}R^2\omega l_1 \\ &\text{Drag force: } D=C_d\hat{U}^2Rl_1 \\ &\text{Resistance : } F=kl' \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(2)$$

where $\hat{U}$: relative velocity $(U-U_c)$, R: a radius of the vortex on $\overline{BC}$, l_1: the half length of $\overline{BC}$, C_d: a drag coefficient, k: a proportional constant and l': stretched length of $\overline{AB}$.

Considering balance of forces in each direction at point B, the following relationships will be given as explained by Fig. 6,

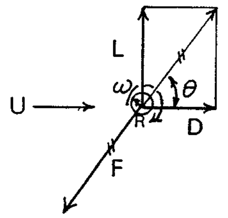

Fig. 6 Force balance of a π-eddy.

$$\left.\begin{aligned} kl'\cos\theta&=C_d\hat{U}^2Rl_1 \\ kl'\sin\theta&=\rho\hat{U}R^2\omega l_1 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

Taking account of the perturbation, the force balance at point B' is obtained in the same manner as

$$\left.\begin{aligned} L&=\rho\left(\hat{U}+\frac{d\hat{U}}{dy}\Delta y\right)(R+\Delta R)^2(\omega+\Delta\omega)l_1 \\ D&=C_d\left(\hat{U}+\frac{d\hat{U}}{dy}\Delta y\right)^2(R+\Delta R)l_1 \\ F&=k(l'+\Delta l) \end{aligned}\right\} \quad (4)$$

Inertia force$=-\rho\pi R^2 l_1((\Delta\ddot{x}),\ (\Delta\ddot{y}))$

where

$$\left.\begin{aligned} \Delta x&=\Delta l\cos\theta-l\sin\theta\Delta\theta \\ \Delta y&=\Delta l\sin\theta+l\cos\theta\Delta\theta \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots(5)$$

$$\Delta\ddot{x}\equiv d^2(\Delta x)/dt^2,\ \Delta\ddot{y}\equiv d^2(\Delta y)/dt^2$$

And the equations of vorticity and mass conservation are given by the followings, respectively,

$$S\omega\sim R^2\omega=\text{const.} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(6)$$

$$\rho Sl=\text{const.} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

where S denotes a cross sectional area of the vortex and l is the length of $\overline{AB}$.

Using Eqs. (3) to (7), the following equation is obtained.

$$\begin{pmatrix}(\Delta\ddot{l})\\(\Delta\ddot{\theta})\end{pmatrix}=-\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\Delta l\\ \Delta\theta\end{pmatrix} \quad\cdots\cdots(8)$$

where

$$A=\frac{k}{\rho\pi R^2 l_1}+\frac{C_d\hat{U}^2\cos\theta}{2\rho\pi Rl}-\left(\frac{2C_d\hat{U}\cos\theta}{\rho\pi R}+\frac{\omega}{\pi}\sin\theta\right)\frac{d\hat{U}}{dy}\sin\theta\quad\cdots\cdots(9)$$

$$B=-\left(\frac{2C_d\hat{U}\cos\theta}{\rho\pi R}+\frac{\omega}{\pi}\sin\theta\right)\frac{d\hat{U}}{dy}l\cos\theta\quad\cdots\cdots(10)$$

$$C=-\frac{C_d\hat{U}^2\sin\theta}{2\rho\pi Rl}+\left(\frac{2C_d\hat{U}\sin\theta}{\rho\pi R}-\frac{\omega}{\pi}\cos\theta\right)\frac{d\hat{U}}{dy}\sin\theta\quad\cdots\cdots(11)$$

$$D=\left(\frac{2C_d\hat{U}\sin\theta}{\rho\pi R}-\frac{\omega}{\pi}\cos\theta\right)\frac{d\hat{U}}{dy}l\cos\theta \quad (12)$$

Upon assuming that the convective velocity of a π-eddy, U_c, is nearly equal to local mean flow velocity, U, according to the experimental results obtained by Favre et al[9] or Sternberg[10], the relative velocity $\hat{U}$ becomes zero and hence $d\hat{U}/dy=dU/dy$. Putting these relations to use, Eqs. (9) to (12) can be simplified as

$$A=\frac{k}{\rho\pi R^2 l_1}-\frac{\omega}{\pi}\sin^2\theta\frac{dU}{dy}\quad\cdots\cdots(13)$$

$$B=-\frac{\omega}{\pi}l\sin\theta\cos\theta\frac{dU}{dy}\quad\cdots\cdots(14)$$

$$C=-\frac{\omega}{\pi}\sin\theta\cos\theta\frac{dU}{dy}\quad\cdots\cdots(15)$$

$$D=-\frac{\omega}{\pi}l\cos^2\theta\frac{dU}{dy}\quad\cdots\cdots(16)$$

$\dot{x}_0$ and $\dot{y}_0$ are represented by

$$\left.\begin{aligned}\dot{x}_0&=(\Delta\dot{l})\cos\theta-l\sin\theta(\Delta\dot{\theta})\\ \dot{y}_0&=(\Delta\dot{l})\sin\theta+l\cos\theta(\Delta\dot{\theta})\end{aligned}\right\}\quad\cdots\cdots(17)$$

Eq. (8) being a linear equation with a symmetric matrix, it can be easily solved, and hence $\dot{x}_0$ and $\dot{y}_0$ are obtained from Eq. (17). Well, considering that the apparent resisting force is included only in term A, it may be concluded that a perturbation along a vortex line is represented by $(\Delta\dot{l})$. Thus $(\Delta\dot{\theta})$ can be regarded as negligible compared with $(\Delta\dot{l})$ in Eq. (17), and θ is assumed to be independent of time t for simplifying the analysis.

Differentiating Eq. (1) by t and making use of Eqs. (8) and (17) in consideration of the above mentioned, the formulation of a π-eddy model will be obtained as

$$u(t)=a\omega\sin\omega t\sin\theta+A_0\omega_0\cos(\omega_0 t+\delta)\cos\theta\quad\cdots\cdots(18)$$

$$v(t)=-a\omega\sin\omega t\cos\theta+A_0\omega_0\cos(\omega_0 t+\delta)\sin\theta\quad\cdots\cdots(19)$$

$$w(t)=b\omega\cos\omega t\quad\cdots\cdots(20)$$

where,

$$\omega_0=\sqrt{\frac{k}{\rho\pi R^2 l_1}-\frac{\omega}{\pi}\sin^2\theta\frac{dU}{dy}}\quad\cdots\cdots(21)$$

and, A_0 and δ are constants. As turbulence is composed of a mixture of eddies with various size, the right hand sides of Eqs. (18) to (20) are expanded into Fourier series of ω in which coefficients a^2, b^2 and A_0^2 indicate contribution of the power.

It is noticed from Eq. (21) that for larger velocity gradient dU/dy or angular velocity ω a resisting force against the vortex stretching apparently becomes more feeble so that the vortex line may be more easily raised with a longer period of the perturbation and in an extreme case a π-eddy may disappear by quick stretching without any vibration. However, it is difficult to evaluate the actual value of ω_0 at present.

(2) Turbulence Intensities and Reynolds Shear Stresses

By making use of Eqs. (18) to (20) and orthogonality of the trigonometrical functions due to $\omega_0\neq\omega$, turbulence intensities and Reynolds shear stresses can be obtained as follows:

$$\bar{u}^2=(a\omega)^2\frac{\sin^2\theta}{2}+(A_0\omega_0)^2\frac{\cos^2\theta}{2}\quad\cdots\cdots(22)$$

$$\bar{v}^2=(a\omega)^2\frac{\cos^2\theta}{2}+(A_0\omega_0)^2\frac{\sin^2\theta}{2}\quad\cdots\cdots(23)$$

$$\bar{w}^2=(b\omega)^2/2\quad\cdots\cdots(24)$$

$$\overline{uv}=-\frac{1}{2}\{(a\omega)^2-(A_0\omega_0)^2\}\sin\theta\cos\theta\quad\cdots(25)$$

$$\overline{uw}=0\quad\cdots\cdots(26)$$

$$\overline{vw}=0\quad\cdots\cdots(27)$$

Eqs. (26) and (27) obviously indicate that there is no correlation between u and w and between v and w, that is, the flow under consideration can be treated as two-dimensional turbulent shear flow. Especially, the relationship $\overline{uw}=0$ has been confirmed by authors' experiment for open channel flow.

Taking notice that $-\overline{uv}=\frac{1}{2}(a^2\omega^2-A_0^2\omega_0^2)\sin\theta\cos\theta\geq 0$, the following unequality can be obtained,

$$\frac{a}{A_0}\geq\frac{\omega_0}{\omega}\quad\cdots\cdots(28)$$

It is suggested from this functional relationship that the perturbation effect of vortex-stretching is secondary in comparison with the effect of main rotating motion. The correlation coefficient between u and v is given by,

$$k=\frac{-\overline{uv}}{\sqrt{\overline{u}^2}\sqrt{\overline{v}^2}}=\frac{(a^2\omega^2-A_0^2\omega_0^2)\sin\theta\cos\theta}{\sqrt{\{(a\omega)^2\sin^2\theta+(A_0\omega_0)^2\cos^2\theta\}\{(a\omega)^2\cos^2\theta+(A_0\omega_0)^2\sin^2\theta\}}}<1 \quad\cdots\cdots(29)$$

and the above unequality indicates one of the actual characteristics of turbulence.

Well, from Eqs. (22) and (23) the difference between $\bar{u}^2$ and $\bar{v}^2$ is given by

$$\bar{u}^2-\bar{v}^2=\frac{1}{2}\{(a\omega)^2-(A_0\omega_0)^2\}\{\sin^2\theta-\cos^2\theta\} \quad\cdots\cdots(30)$$

As it is obviously recognized that $\bar{u}^2$ is larger than $\bar{v}^2$ on the average for turbulent shear flow, almost all of π-eddies have to be applicable in the following range of the inclination angle:

$$\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2} \quad\cdots\cdots(31)$$

(3) Spectrum Density Functions of Energy

The above investigation has been limited only to a specified eddy element, but the characteristics of turbulence as a whole contributed by all of the eddies should be made clear. In order to attain this purpose, the conception of energy spectrum must be introduced, together with adoption of space wave number in Eulerian expression instead of the above Lagrangian form.

Now it is assumed that there exists so called turbulence cascade process, which a large scale eddy produced in the mean flow by Reynolds stress successively transports its turbulence energy into a small scale eddy[19]. In other words, the turbulence similarity (Reynolds similarity) is assumed to be realized for any eddy element. When direct-viscous dissipation during the cascade process can be neglected, the transport of turbulence energy for each eddy should be equal to its final dissipation rate ε.

A rate of work done by mean effective viscosity of eddy $(a^2+b^2)\omega/2$ against eddy motion can be approximately given by application of Oseen's law for a circular cylinder[20];

$$W\sim 4\pi(a^2+b^2)\omega/2(\bar{u}^2+\bar{v}^2+\bar{w}^2)/3 \quad\cdots\cdots(32)$$

When the second term in right hand side of Eq. (22) or (23) will be ignored due to $a\omega\gg A_0\omega_0$, the dissipation rate ε per unit mass and unit time can be written by

$$\varepsilon=\frac{K(1+e^2)^2}{3e}a^2\omega^2 \quad\cdots\cdots(33)$$

where $e\equiv b/a$, and K is a constant.

Let a vortex line inclining by θ degree to the x-axis as shown in Fig. 7 represent by one-dimensional wave number k in the x-direction. Since the wave length λ in the x-direction is equal to $2a/\sin\theta$, k becomes

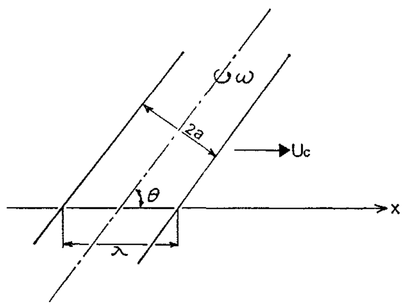

Fig. 7 Vortex element of a π-eddy.

$$k=\frac{2\pi}{\lambda}=\frac{\pi}{a}\sin\theta \quad\cdots\cdots(34)$$

Substituting Eq. (34) into Eq. (33), then the angular velocity will be given by

$$\omega^2=\frac{(3e/K)^{2/3}}{(1+e^2)^{4/3}}(\pi\sin\theta)^{-4/3}\varepsilon^{2/3}k^{4/3} \quad\cdots\cdots(35)$$

Therefore, a spectrum density function of ω^2 will be obtained for $k_0<k<k_\infty$.

$$G(k)=\frac{d\omega^2}{dk}=\frac{4}{3}\frac{(3e/K)^{2/3}}{(1+e^2)^{4/3}}(\pi\sin\theta)^{-4/3}\varepsilon^{2/3}k^{1/3} \quad\cdots\cdots(36)$$

k_0 and k_∞ are the lower and upper limits of wave number within a range where the turbulent cascade process can be realized, respectively, and the wave numbers beyond these limits hardly contribute to this process. It has been also verified by Inoue's investigation[21] that $G(k)$ is proportional to $\varepsilon^{2/3}k^{1/3}$ as shown in Eq. (36).

Now, on the assumption that ω_0 is so inappreciable compared with ω in the cascade process of turbulent energy, turbulence intensities can be written in terms of the wave number.

$$\bar{u}^2(k)\simeq\frac{\sin^2\theta}{2}\left(\frac{\pi}{k}\sin\theta\right)^2G(k)\equiv E_u(k) \quad\cdots\cdots(37)$$

$$\bar{v}^2(k)\simeq\frac{\cos^2\theta}{2}\left(\frac{\pi}{k}\sin\theta\right)^2G(k)\equiv E_v(k) \quad\cdots\cdots(38)$$

$$\bar{w}^2(k)\simeq\frac{e^2}{2}\left(\frac{\pi}{k}\sin\theta\right)^2G(k)\equiv E_w(k) \quad\cdots\cdots(39)$$

where $E_u(k)$, $E_v(k)$ and $E_w(k)$ are spectrum density functions of turbulent energy expressed by wave number for u', v' and w', respectively, and given as

$$E_u(k)=\frac{2}{3}\frac{(3e/K)^{2/3}}{(1+e^2)^{4/3}}(\pi)^{2/3}(\sin\theta)^{8/3}\varepsilon^{2/3}k^{-5/3} \quad \cdots\cdots(40)$$

$$E_v(k)=\frac{2}{3}\frac{(3e/K)^{2/3}}{(1+e^2)^{4/3}}(\pi)^{2/3}(\sin\theta)^{2/3} \times(\cos\theta)^2\varepsilon^{2/3}k^{-5/3} \quad \cdots\cdots(41)$$

$$E_w(k)=\frac{2}{3}\frac{e^2(3e/K)^{2/3}}{(1+e^2)^{4/3}}(\pi)^{2/3}(\sin\theta)^{2/3}\varepsilon^{2/3}k^{-5/3} \quad \cdots\cdots(42)$$

As shown by Eqs. (40) to (42), the local isotropic theory proposed by Kolmogoroff that each spectrum density function is in proportion to $-5/3$ power of the wave number k in the cascade process ($k_0<k<k_\infty$) may be realized.

(4) Intensity of Turbulent Energy

Here, the inclination angle of a vortex line for each eddy element θ will be evaluated. Differing with the wave numbers, θ can be calculated from the time-space correlations in the x- and y-directions of filtered fluctuation components. The observed values of θ for a large eddy have been given by wind-tunnel experiments conducted by Willmarth et al[8], Favre et al[9] and Sternberg[10]. Referring to Fig. 8 given by Sternberg, θ for a large eddy in a turbulent boundary layer flow takes a value larger than 20 degrees at the outer edge of a viscous sublayer, and rapidly increasing with y, it may be around 90 degrees in main flow region. From this fact θ for a representative eddy may be given as a function of y/h where

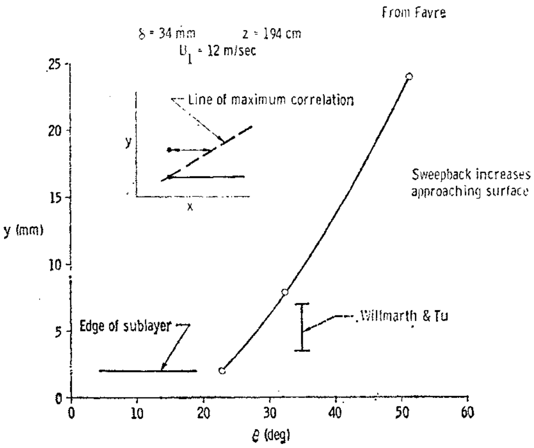

Fig. 8 Eddy inclination to the wall (after Sternberg[10])

h is the flow depth in open channel. Moreover, dependence of θ upon wave number k makes it more difficult to evaluate θ precisely. Thus, for simplifying, θ is assumed to have a uniform distribution among the eddies so as to contribute to the turbulence intensity on an average. On this assumption, spectra of turbulent energy can be numerically calculated by averaging Eqs. (40) to (42) within $\pi/4<\theta<\pi/2$. In those equations e is fairly assumed to be constant because similarity of the turbulent structures is discerned in the cascade region $k_0<k<k_\infty$ and hardly affected by vortex stretching.

Well, Eq. (6) should be valid in stretching vortex, then resulting in

$$a^2e\omega=\text{const.} \quad \cdots\cdots(43)$$

Taking account that the vortex stretching would bring about extreme distortion of the cross section of a vortex line, that is $e^2\ll 1$, the following will be deduced from Eq. (43);

$$\varepsilon\sim\frac{\left(1+e^2+\dfrac{A_0^2}{a^2}\right)^2}{e}a^2\omega^3\sim e^{-1}a^2\omega^3$$

Then, the spectrum density functions of turbulent energy will be rewritten by

$$E_u(k)\sim k^{-1},\ E_v(k)\sim k^{-1} \quad \cdots\cdots(44)$$

$$E_w(k)\sim k \quad \cdots\cdots(45)$$

By applying a transfer function of Heisenberg's form, Tchen[22] also obtained the same expression as Eq. (44) for a production region of turbulent energy. By making use of a π-eddy model proposed here, Eq. (45) different from Eq. (44) has been obtained on the basis of an assumption that vortex stretching does not affect the spanwise components as indicated by Eq. (24).

Now, let a constant e for the cascade region relate to Taylor's integral scales (mean eddy scales) designated as L_x in the x-direction and L_z in the z-direction. Since L_z/L_x is given as 0.5 for isotropic turbulence in which $e=1$, a value of e for any integral scale will be estimated as

$$e\simeq 2L_z/L_x \quad \cdots\cdots(46)$$

According to Laufer's experiment ($R_e=3.08\times10^4$) for a two-dimensional channel flow[23], it was shown that the value of L_z/L_x became about 0.33 independently of y/h. Substituting this value into Eq. (46), e to be adopted here becomes 0.66.

Using this constant value of e, a numerical analysis of Eqs. (40) to (42) averaged for $\pi/4<\theta<\pi/2$ yields the followings about relative turbulence intensities:

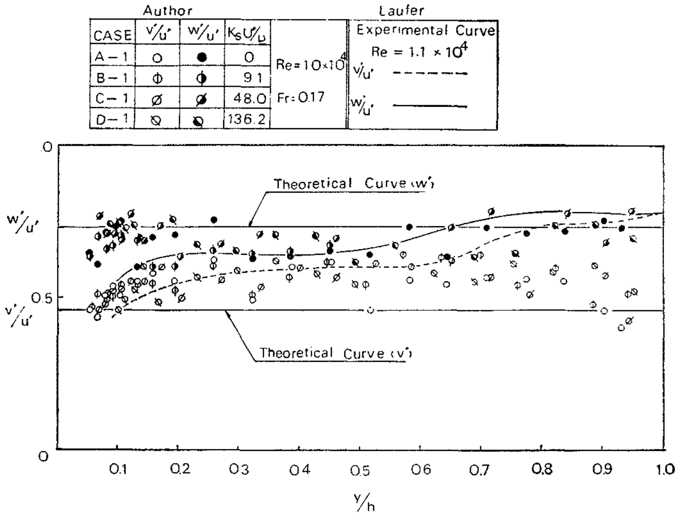

Fig. 9 Relative turbulence intensities.

$$v'/u'=0.455 \text{ and } w'/u'=0.726 \quad\cdots\cdots(47)$$

These theoretical values are compared with author's experimental results for open channel flows* and with Laufer's data for closed channel in Fig. 9. In this figure A-1, B, C-1 and D-1 denote the runs for hydraulically smooth, incompletely rough and completely rough bed, respectively. A good agreement between theoretical turbulence intensities and observed ones due to the authors is recognized, especially near the wall boundary and free surface. From the fact that the experimental values of v'/u' or w'/u' obtained by the authors show a convex or concave variation with y/h independently of the wall conditions while the theoretical values are invariant with the flow depth, the existing conception that the structures of turbulence approach to be isotropic with increase of distance from the wall cannot be applied to open channel shear flow.

On the other hand, the turbulent flow in a closed channel indicates a characteristic of isotropic turbulence around the pipe axis as shown by Laufer's results, and therefore it may be suggested that a free surface has a peculiar effect upon the structure of turbulence.

(5) Turbulence Intensity

If the distribution of the rms velocity fluctuations in the x-direction u' is determined, v' and w' can be directly calculated from Eq. (47). Within the range of the inclination angle given by Eq. (31), Eq. (40) can be written by using $e=0.66$ as

$$E_u(k)=2.295\left[\frac{e}{K(1+e^2)^2}\right]^{2/3}\times\varepsilon^{2/3}k^{-5/3}$$
$$=1.074(\varepsilon/K)^{2/3}k^{-5/3} \quad (48)$$
$$\equiv C\varepsilon^{2/3}k^{-5/3} \quad\cdots\cdots(48)'$$

According to Kolmogoroff[24] C is assumed to be a universal constant to be determined by experiment. Grant et al[25] made a turbulence measurement of a tidal channel flow with very high Reynolds number ($R_e=3\times10^8$) in which the cascade process was perfectly achieved, and he obtained $C=0.47\pm0.02$. Lawn[26] measured a rate of turbulent energy dissipation in a pipe flow and obtained $C=0.53$ for turbulent shear flow.

For the turbulent shear flow under consideration, $C=0.53$ may be used. Thus, integrating of Eq. (48)′ yields

$$u'^2\simeq C\varepsilon^{2/3}\int_{k_0}^{k_\infty}k^{-5/3}dk=\frac{3}{2}C\varepsilon^{2/3}(k_0^{-2/3}-k_\infty^{-2/3}) \quad\cdots\cdots(49)$$

when $k_0\ll k_\infty$, Eq. (49) reduces to

$$\frac{u'}{U_*}=\sqrt{\frac{3C}{2}}\frac{1}{U_*}\left(\frac{\varepsilon}{k_0}\right)^{1/3} \quad\cdots\cdots(50)$$

where U_* is friction velocity.

Denoting P as turbulent energy production and T as diffusion, the turbulent energy equation can be written by neglecting higher order terms:

$$P=T+\varepsilon \quad\cdots\cdots(51)$$

As indicated by Townsend[3], Laufer[28] and the authors[27], T in Eq. (51) is so small compared with the other terms in a wall region. Then, the approximation that $\varepsilon\simeq P$ yields

$$\varepsilon\simeq P=-\overline{uv}\frac{\partial U}{\partial y} \quad\cdots\cdots(52)$$

Assuming the mean flow with logarithmic velocity distribution and substituting Eq. (52) into Eq. (50), it results in

$$\frac{u'}{U_*}=\sqrt{\frac{3C}{2}}\frac{C_1}{\kappa^{1/3}}\left(\frac{1}{k_0y}-\frac{1}{k_0h}\right)^{1/3} \quad\cdots\cdots(53)$$

where κ is Karman's constant ($=0.4$) and C_1 is a constant nearly equal to 1.0 ($\varepsilon\equiv C_1P$). The lower limit of wave number k_0 (for a large eddy) where the turbulent cascade process begins to realize is concerned with a geometric scale of turbulent shear flow, as pointed out by some researchers.

* A report on author's experimental investigation for open channel turbulent flow is going to be published.

For two-dimensional shear flow in open channel under consideration the flow depth h is taken as a geometric scale. Then,

$$k_0^{-1} \sim h \text{ or } hk_0 \equiv C_2 \quad \cdots\cdots(54)$$

Therefore, Eq. (53) will be reduced to

$$\frac{u'}{U_*} = 1.662\sqrt{C} \cdot C_3\left(\frac{1}{y/h} - 1\right)^{1/3} \quad \cdots\cdots(55)$$

where $C_3 \equiv C_1 \cdot C_2^{-1/3}$

Since Eq. (53) is valid only for $y \ll h$, Eq. (55) becomes by using $C=0.53$,

$$\frac{u'}{U_*} = 1.2C_3\left(\frac{y}{h}\right)^{-1/3} \quad \cdots\cdots(56)$$

For $y/h \simeq 1$ it can be concluded that turbulent energy dissipation becomes the same order as its diffusion and so Eq. (56) is no longer valid. From Eq. (56) it is noticed that turbulence intensity is proportional to $-1/3$ power of y/h and that $u'/U_* = O(1)$, where O is order notation.

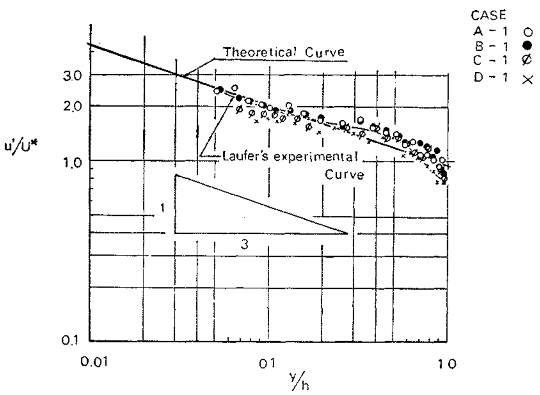

Fig. 10 Turbulence intensity of u

The observed turbulence intensities obtained by the author's experiment are plotted in Fig. 10 against y/h, together with an experimental curve given by Laufer. A theoretical curve shown in the figure was obtained by determining C_3 in Eq. (56) so as to satisfy the observed values of u'/U_* in the wall region. In this manner u'/U_* can be expressed as

$$\frac{u'}{U_*} = 0.95\left(\frac{y}{h}\right)^{-1/3} \quad \cdots\cdots(57)$$

Therefore, $C_3 = C_1 \cdot C_2^{-1/3} = 0.79$. Being C_1 nearly equal to 1.0, C_2 becomes equal to about 2.0. It is obvious from Fig. 10 that the theoretical curve slightly deviates from the experimental values with increase of y/h, and that a π-eddy model represents the structures of wall turbulence with a fair accuracy.

3. FURTHER DISCUSSIONS

(1) Application of a π-Eddy Model to a Larger Eddy Scale

The formulation of turbulence production mentioned above has been proved to be valid only in the turbulent cascade region ($k_0 < k < k_\infty$). A region where $k > k_\infty$ is so called a viscous region to which Heisenberg's theory may be applied. However, this region with high wave number hardly contributes to turbulent energy, only playing an important role on the micro-turbulence such as energy dissipation.

Hence, consider here a larger scale of eddy ($k < k_0$). For k smaller than k_0 a vortex stretching has a great influence on production of turbulent energy and a spectrum functions may be so complicated. One of them has been given by Eqs. (44) and (45). In a region where $0 < k < k_\infty$ it seems most appropriate to use an interpolated formula due to Karman[28] as an energy spectrum function:

$$E_u(k) \sim (k^2 + k_0^2)^{-5/6} \quad \cdots\cdots(58)$$

One of the authors[29] has found that Eq. (58) has a fairly good agreement with the observed results for shear flow in closed conduit. Doing the same operation as applied to Eq. (48), $E_u(k)$ in this case can be obtained as

$$E_u(k) = C\varepsilon^{2/3}(k^2 + k_0^2)^{-5/6} \text{ for } 0 < k < k_\infty \cdots(59)$$

$E_v(k)$ and $E_w(k)$ are also given by the same representation as Eq. (59).

Now, considering the correlation function and the spectra, the following two equations can be obtained.

$$E_u(0) = \frac{2}{\pi} L_x u'^2 = C\varepsilon^{2/3} k_0^{-5/3} \quad \cdots\cdots(60)$$

$$u'^2 = \int_0^\infty E_u(k)dk \simeq \frac{C}{2} B\left(\frac{1}{2}, \frac{1}{3}\right)\varepsilon^{2/3} k_0^{-2/3} \quad (61)$$

where $B(x, y)$ is Beta function. From Eqs. (60) and (61) $L_x k_0$ can be easily given by

$$L_x k_0 = \pi / B(1/2, 1/3) = 0.746 \quad \cdots\cdots(62)$$

From Eq. (54) L_x/h becomes

$$L_x/h = 0.37 \quad \cdots\cdots(63)$$

(2) Applicable Range of a π-Eddy Model

Finally, applicability of a π-eddy model will be discussed. From the above description it is concluded that the π-eddy model can fairly explains the structures of wall turbulence by averaging within the range of inclination angle given by Eq. (31). Therefore, this model is not applicable to a region with smaller inclination angle of eddies (that is $0 \leqq \theta < \pi/4$). According to the experiments made by Favre et al[9] this turbulent field is located very close to the wall and the flow in this field shows constant shear stress. Monin et al[19] found that the turbulence intensity in this region could be represented as a universal function of y^+, and obtained the following results by analyzing a lot of existing data:

For $y^+ \to \infty$, $u'/U_* \to 2.3$, $v'/U_* \to 0.9$ and $w'/U_* \to 1.7$. Hence, $v'/u' \to 0.39$ and $w'/u' \to 0.74$ ......(64)

These values shows a good agreement with the theoretical ones given by Eq. (47), and so, as far as y^+ is large enough, the π-eddy model can be applied even to the region with constant shear stress. However, for smaller values of y^+ ($y^+ \leqq 30$), it was suggested that $v'/u' \simeq 0.25$ and $w'/u' \simeq 0.54$. Therefore, it is reasonable to apply an eddy model with $\theta = 0$ such as an attached eddy model or a hair-pin eddy to this region.

4. CONCLUSIONS

In this paper some characteristics of wall turbulence has been analyzed quantitatively by application of a horseshoe vortex model which has been confirmed to exist in the qualitative aspect.

Since a π-eddy model proposed here has been so simplified, it cannot predict perfectly the quantitative properties of wall turbulence, but it succeeds to describe the macro-turbulence structures to an extent. The results obtained here are summarized as follows;

1) An important characteristic of a turbulent shear flow that $v' < w' < u'$ can be deduced from the π-eddy model.

2) It is proved that turbulence intensity based on this model satisfies the $-1/3$ power law against y/h and coincides with experimental results.

3) The flow depth h, the mean eddy scale L_x and a reciprocal of the lowest wave number in the turbulent cascade region k_0^{-1} are of the same order of magnitude. However, this model leaves something to be improved because it is inapplicable to a region close to the wall and cannot discern the effects of the wall boundaries.

The bursting and sweeping phenomena which play an important part in production of turbulence for $y^+ < 70$ cannot be explained by π-eddy model. So further attempts to reform an eddy model should be made for describing clearly these actual phenomena.

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