PROC. OF JSCE,
NO. 229, SEPT. 1974

ANALYTICAL SOLUTION OF HYDRODYNAMIC PRESSURE WITH REFLECTIVE CONDITION AT RESERVOIR BOTTOM DURING EARTHQUAKES

By Tomoyasu NAKAGAWA* *and Tadashi* HATANO**

1. INTRODUCTION

For the study of fundamental characteristics of hydrodynamic pressure due to earthquakes, one of the authors has once presented a solution of hydrodynamic pressure generated by harmonic motion of an upright rigid wall at one end of reservoir[1]. The governing equation employed there was the two-dimensional wave equation (1.1) for the velocity potential f subject to boundary condition (1.2).

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{W_0}{gK}\frac{\partial^2 f}{\partial t^2} \quad \cdots\cdots(1.1)$$

$$\left.\begin{aligned} &\text{(i)} \quad -\left.\frac{\partial f}{\partial y}\right|_{y=-h}=0 \\ &\text{(ii)} \quad -\left.\frac{\partial f}{\partial x}\right|_{x=0}=\frac{\alpha g}{\omega}\cos\omega t \\ &\text{(iii)} \quad \frac{W_0}{g}\frac{\partial^2 f}{\partial t^2}+W_0\left.\frac{\partial f}{\partial y}\right|_{y=0}=0 \end{aligned}\right\} \quad \cdots\cdots(1.2)$$

The hydrodynamic pressure $\sigma=\frac{W_0}{g}\frac{\partial f}{\partial t}$ was represented as

$$\begin{aligned}\sigma=&\frac{4\alpha W_0}{j_0}\frac{\sinh k_0 h}{\sinh 2k_0h+2k_0h} \\ &\times\cosh k_0(y+h)\cos(\omega t-j_0x) \\ &+\sum_{m=1}^{r}\frac{4\alpha W_0}{j_m}\frac{\sin k_m'h}{\sin 2k_m'h+2k_m'h} \\ &\times\cos k_m'(y+h)\cos(\omega t-j_mx) \\ &-\sum_{m=r+1}^{\infty}\frac{4\alpha W_0}{j_m'}\frac{\sin k_m'h}{\sin 2k_m'h+2k_m'h} \\ &\times\cos k_m'(y+h)e^{-j_m'x}\sin\omega t \quad \cdots\cdots(1.3)\end{aligned}$$

Here, x denotes the horizontal direction upstream along the free surface of reservoir at rest, and y the vertical direction upward along the wall at rest with the depth h of water. The wall is assumed to be rigid and in harmonic motion of $\frac{\alpha g}{\omega^2}\sin\omega t$. The constants k_0, k_m', j_0, j_m, and j_m' are determined as follows: k_0 is the unique real number satisfying $k_0\tanh k_0h=\omega^2/g$, $k_0>0$; k_m' ($m=1, 2, \cdots$) are those real numbers satisfying $k_m'\tan k_m'h=-\omega^2/g$, $0<k_1'<k_2'<k_3'<\cdots$; r is the largest integer of indices m such that $c^2>k_m'^2$; $j_0^2=c^2+k_0^2$; $j_m^2=c^2-k_m'^2$; and $j_m'^2=k_m'^2-c^2$, where $c^2=W_0\omega^2/gK$ in which W_0 is the unit weight of water and K the bulk modulus of water.

The solution was claimed to be an improvement of the result by Westergaard[2] in some respects, who first pointed out the existence of hydrodynamic pressure during earthquakes. However, it seemed to be not enough satisfactory because of a strong resonance inherent to it caused by the elasticity of water. Later, one of the authors observed that in vibration experiments of real dams and in laboratory experiments the pressure at resonance frequencies did not rise significantly[3]. Furthermore, these experiments even revealed no resonance at all in those cases when the reservoir bottom was covered by pressure-absorbing materials such as fine sand. These results suggested that the boundary condition (i) of (1.2) was not adequate since it caused a complete reflection of hydrodynamic pressure at the reservoir bottom.

In order to overcome this difficulty, the following condition taken from the theory of acoustics was proposed to replace (i) of (1.2):

$$\frac{W_0}{g}\left.\frac{\partial f}{\partial t}\right|_{y=-h}=\frac{W_0}{g}c_0\beta\left.\frac{\partial f}{\partial y}\right|_{y=-h} \quad \cdots\cdots(1.4)$$

in which the parameter $\beta=(W_1c_1/g)/(W_0c_0/g)$ is the ratio of acoustic impedances of the bottom material to water where W_1 and c_1 are the unit

* Central Research Institute of Electric Power Industry
** Dr. of Eng., Central Research Institute of Electric Power Industry

weight and the sound velocity of the bottom material, respectively, and c_0 the sound velocity of water.[3] From the point of view of (1.4), the condition (i) of (1.2) is a special case of it in that β is set to infinity. The parameter β assumes around 5.0 if the bottom material is rock, or effectively 1.0 if the bottom is covered by sedimental materials such as sand or silt; in the latter case the sound pressure passes through the sedimental layer and no reflection occurs*.

By the above investigation, we concluded that the hydrodynamic pressure would be best approximated by the wave equation subject to the following boundary conditions:

i) Degree of reflection at the reservoir bottom must be determined in terms of acoustic impedances.
ii) At the contact of water and wall surface, the normal component of velocity of water particle to the wall surface is set equal to the velocity of wall in the same direction.
iii) Water surface is free.

We have already incorporated the above boundary conditions in a proposed numerical method for the coupled vibration of arch dam with reservoir water[4]. The numerical result has shown that these conditions were reasonable. However, the analytic solution of hydrodynamic pressure in the case of incomplete reflection would be still necessary for the purpose of theoretical study of its phenomenon. In the following, we shall present the solution, and give discussion on it.

2. ANALYTICAL SOLUTION OF HYDRODYNAMIC PRESSURE

Suppose that

$$f(x,\ y,\ t)=e^{i\omega t}X(\xi)\,Y(\eta)\Big|_{\substack{\xi=x/h\\ \eta=(y+h)/h}} \quad \cdots\cdots\cdots(2.1)$$

satisfies (1.1). By the separation of variables with the separation parameter λ^2,

$$\left.\begin{array}{ll} X''(\xi)+(\lambda^2+\Omega^2)X(\xi)=0\,, & \xi>0 \\ \text{where } \Omega^2=W_0\omega^2h^2/gK & \end{array}\right\} \quad \cdots\cdots(2.2)$$

and

$$Y''(\eta)-\lambda^2Y(\eta)=0\,,\quad 0<\eta<1 \cdots\cdots\cdots\cdots\cdots(2.3)$$

are obtained. The parameter λ^2 is required to satisfy

$$\left.\begin{array}{l} (\lambda+iq)(\lambda-s)e^{2\lambda}-(\lambda-iq)(\lambda+s)=0 \\ \text{with } q=\dfrac{\omega h}{\beta}\sqrt{\dfrac{W_0}{gK}} \text{ and } s=\dfrac{\omega^2h}{g} \end{array}\right\} \quad (2.4)$$

due to the fact that (2.1) with $Y(\eta)=C_1e^{\lambda\eta}+C_2e^{-\lambda\eta}$, the general solution to (2.3), is subject to (iii) of (1.2), and (1.4). It will be shown in Section 3 that (2.4) has a root $\lambda=0$ and infinite pairs of complex roots $\pm\lambda$ in the second and the fourth quadrants of the complex plane. Let λ_0, λ_1, λ_2, $\cdots$ denote those roots which lie in the second quadrant in ascending order of their imaginary parts. These are the set of eigenvalues for our problem.

The eigenfunctions belonging to λ_m are

$$Y_m(\eta)=\cosh\lambda_m\eta+\frac{iq}{\lambda_m}\sinh\lambda_m\eta \quad \cdots\cdots\cdots(2.5)$$

and $X(\xi)$ corresponding to $Y_m(\eta)$ are given as

$$X_m(\xi)=\exp(-\mu_m\xi)\,,\quad \mu_m=\sqrt{-\Omega^2-\lambda_m^2}$$
$$(\mathrm{Re}\ \mu_m>0)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2.6)$$

(The term $\exp(+\mu_m\xi)$ must be discarded from $X_m(\xi)$ since otherwise $X_m(\xi)$ is not bounded as $\xi\to\infty$.)

Since $f_m=\exp(i\omega t)X_m(\xi)Y_m(\eta)$ with $\xi=x/h$ and $\eta=(y+h)/h$ is a solution to (1.1), so is the linear combination of $\{f_m\}$ with arbitrary coefficients $\{a_m\}$ such that

$$f(x,\ y,\ t)=e^{i\omega t}\sum_{m=0}^{\infty}a_me^{-\mu_m\xi}\left(\cosh\lambda_m\eta+\frac{iq}{\lambda_m}\sinh\lambda_m\eta\right)\Bigg|_{\substack{\xi=x/h\\ \eta=(y+h)/h}} \cdots(2.7)$$

Suppose that (2.7) further satisfy

$$-\frac{\partial f}{\partial x}\bigg|_{x=0}=\frac{\alpha g}{\omega}e^{i\omega t} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2.8)$$

with an appropriate choice of $\{a_m\}$. Then, the real part of it is what is sought for: the velocity potential of our problem. Substitution of (2.7) for the left-hand side of (2.8) results in

$$\sum_{m=0}^{\infty}a_m\mu_mY_m(\eta)=\frac{\alpha gh}{\omega}\,,\quad 0<\eta<1 \quad \cdots\cdots(2.9)$$

provided that the termwise differentiation is permissible. By virtue of the fact that $\int_0^1Y_m(\eta)Y_n(\eta)d\eta=0$ if $m\neq n$, a_m are determined as

$$a_m=\frac{\alpha gh}{\omega}\,\frac{\displaystyle\int_0^1Y_m(\eta)d\eta}{\displaystyle\mu_m\int_0^1\{Y_m(\eta)\}^2d\eta} \quad \cdots\cdots\cdots\cdots(2.10)$$

The hydrodynamic pressure is given as the time derivative of the velocity potential multiplied by W_0/g. Hence it is represented formally as

* Since no reflection occurs at the reservoir bottom in this case, it was supposed in 3) that the pressure level on the wall surface could be given by solving the Laplace equation which is derived by letting c_0 (the sound velocity of water) to be infinity.

$$\sigma_\beta=-\alpha W_0 h\times Im\left\{e^{i\omega t}\sum_{m=0}^{\infty}a_m'e^{-\mu_m\xi}\left(\cosh\lambda_m\eta+\frac{iq}{\lambda_m}\sinh\lambda_m\eta\right)\right\}\Bigg|_{\substack{\xi=x/h\\ \eta=(y+h)/h}}\quad\cdots\cdots\cdots(2.11)$$

$$\text{where}\quad a_m'=\frac{\int_0^1 Y_m(\eta)d\eta}{\mu_m\int_0^1\{Y_m(\eta)\}^2d\eta}\quad(m=1,2,\cdots)$$

3. DISTRIBUTION OF EIGENVALUES λ_m, AND THE CONVERGENCE OF THE EXPANSION SERIES

Eigenvalues λ_m are zero-points of the transcendental integral function in the left-hand side of (2.4). It is known in the theory of transcendental integral function that such functions have generally infinite zero-points in the complex plane, and that the point ∞ is the only accumulating point of them. Clearly, $\lambda=0$ is a root of (2.4). But it is discarded since the coefficients a and b in the corresponding general solution $Y(\eta)=a\eta+b$ vanish due to the boundary condition. Hence only roots $\lambda\neq0$ could become the eigenvalues. Suppose $\lambda=\lambda_m$ satisfies (2.4), then $\lambda=-\lambda_m$ also satisfies (2.4). Therefore, eigenvalues are located symmetrically with respect to the origin $\lambda=0$ in the λ-plane. Let $\lambda_m=u_m+iv_m$. The following identity is easily shown.

$$\int_0^1|Y_m(\eta)|^2d\eta=-\frac{q}{2u_mv_m}\quad\text{for all } m\,.$$

Since the left-hand side is nonnegative, u_mv_m must be negative, which implies that all of λ_m are located in the second and the fourth quadrants. Clearly $Y_m(\eta)$ is an even function in λ_m. Consequently those roots which lie in the second quadrant are sufficient to do with. Furthermore, it may be shown that λ_m satisfy the following properties. These properties are useful in determining numerical values of them.

i) λ_m are simple roots if $s\geq q+2$.

ii) Domain $D_m=\{\lambda|0\leq Im\lambda\leq(m-1/2)\pi i\}$ in the λ-plane contains exactly m eigenvalues $\lambda_0,\cdots,\lambda_{m-1}$, if m is sufficiently large.

iii) $\lim_{m\to\infty}u_mv_m=-q$, and $\lim_{m\to\infty}v_m\tan v_m=-s$.

iv) $\lambda_0\simeq-s$ if $s\gg q$.

Next, let us study the convergence of the expansion series of (2.7). The functions $Y_m(\eta)$ are eigenfunctions of the ordinary differential quation (2.3) subject to homogeneous boundary conditions with complex coefficients. (In our particular case, one of the coefficients is pure imaginary and the others are real.)

Suppose that a function $w(\eta)$ is given where $w(\eta)$ is integrable on $0\leq\eta\leq1$. Define the Fourier expansion series of it with $Y_m(\eta)$ by

$$\left.\begin{aligned}&w(\eta)\approx\sum_{m=0}^{\infty}c_mY_m(\eta)\,,\quad 0<\eta<1\\&\text{where}\quad c_m=\frac{\int_0^1 w(\eta)Y_m(\eta)d\eta}{\int_0^1\{Y_m(\eta)\}^2d\eta}\quad(m=0,1,2,\cdots)\end{aligned}\right\}\quad\cdots\cdots\cdots(3.1)$$

On the other hand, it is known that the Fourier cosine series of the same function is

$$\left.\begin{aligned}&w(\eta)\approx\sum_{m=0}^{\infty}c_m'\cos m\pi\eta\,,\quad 0<\eta<1\\&\text{where}\quad c_0'=\int_0^1 w(\eta)d\eta\\&\qquad c_m'=2\int_0^1 w(\eta)\cos m\pi\eta d\eta\,,\\&\qquad(m=1,2,\cdots)\end{aligned}\right\}\cdots(3.2)$$

Let $S_n(\eta)$ denote the sum of the first n terms in the right-hand side of (3.1), and let $S_n'(\eta)$ denote the sum of the first n terms in the right-hand side of (3.2). Then the following holds[5].

$$\lim_{n\to\infty}\{S_n(\eta)-S_n'(\eta)\}=0\ \text{uniformly on}\ 0\leq\eta\leq1.$$

In our problem (2.9), $w(\eta)$ is essentially a constant one (1). Clearly the coefficients of its Fourier cosine series are $c_0'=1$, $c_1'=c_2'=\cdots=0$. Consequently, the identity (2.9) with coefficients a_m given by (2.10) is valid in the sense that the left-hand side is uniformly convergent to the right-hand side on the closed interval $0\leq\eta\leq1$.

Since the series given by (2.9) has this property, $\exp(i\omega t)\sum_{m=0}^{\infty}a_m\mu_m\exp(-\mu_m\xi)Y_m(\eta)$ has also the same property. Therefore, the termwise integration with respect to ξ is possible, and the resultant series is again uniformly convergent on $0\leq\eta\leq1$. This implies that the uniform convergence of the right-hand side of (2.7) on $0\leq\eta\leq1$.

The rate of convergence may be estimated by the decreasing order of a_m as $m\to\infty$. A simple calculation shows that $a_m=O(m^{-3})$. This means that if the series is truncated by the N-th term, the discrepancy of the original series and the truncated one is of $O(N^{-2})$. $N=10\sim15$ seems to be sufficient for practical purposes; in Section 5, $N=20$ will be taken.

4. RELATION TO HATANO'S SOLUTION

By letting $\beta\to\infty$, the boundary condition (1.4)

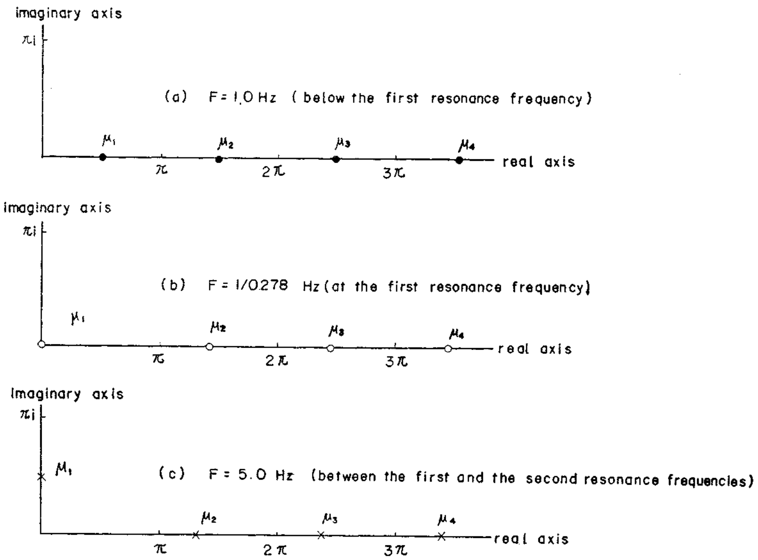

Fig. 1 Distribution of μ_m $(m=1, 2, \cdots)$ in the case of $\beta=\infty$.
(The gravity wave term $\mu_0 \approx \sqrt{h/g}\ 2\pi i/T$ is not shown in this figure.)

is reduced to (i) of (1.2). Therefore, it may be supposed that as $\beta \to \infty$ σ_β, which is the solution to (1.1) under boundary condition (ii) and (iii) of (1.2), and (1.4), coincides with σ in (1.3). This is found to be true by looking at the eigenvalues λ_m in the complex plane as functions of β as in the following.

When β tends to infinity, the characteristic equation (2.4) is reduced to

$$\lambda \tanh \lambda = s \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(4.1)$$

Equation (4.1) has a pair of real roots $\pm k_0 h$, and infinite pairs of pure imaginary roots $\pm i k_m' h$ $(m=1, 2, \cdots)$. By taking the sign into consideration, eigenvalues are $\lambda_0=-k_0 h$, and $\lambda_m = i k_m' h$ $(m=1, 2, \cdots)$. From this fact, the first term in σ_β coincides with the first term in σ, and the succeeding terms in σ_β and σ coincide each other when $\beta=\infty$. By the continuity in β, it is easily seen that for any finite value of β each term of σ_β corresponds to each term of σ.

Next, let us investigate the resonance inherent to (1.3) from the viewpoint of location of $\mu_m = \sqrt{-\Omega^2-\lambda_m^2}$ in the μ-plane. Let the first term μ_0 be set aside since it represents the gravity wave. When $\beta=\infty$, λ_m $(m=1, 2, \cdots)$ are pure imaginary. Hence, μ_m are either real or pure imaginary. In particular, if the frequency $F=\omega/2\pi$ is sufficiently small, all μ_m lie on the real axis as shown in Fig. 1(a). When the frequency increases, μ_m move toward the origin along the real axis. The frequency at which μ_1 reaches the origin (i.e., $1/\mu_1=\infty$) is what is called the first resonance frequency. (See, Fig. 1(b).) Beyond this frequency, μ_1 is transferred to the imaginary axis. (See, Fig. 1(c).) Gradually μ_1 moves on it, and μ_2 approaches the origin Generally, the frequency at which μ_r reaches the origin is the r-th resonance frequency, and around this frequency μ_r is transferred from the real axis to the imaginary axis. The formula σ of (1.3) represents the case where $\mu_1, \mu_2, \cdots, \mu_r$ are on the imaginary axis, and $\mu_{r+1}, \mu_{r+2}, \cdots$ are on the real axis.

Now, let us study the case when β is finite. In this case, μ_m $(m=1, 2, \cdots)$ are always located strictly inside the first quadrant of the complex plane, but never reach the origin. Therefore, σ_β contains no sharp resonance in the sense of $1/\mu_m=\infty$ in contrast to $\sigma=\sigma_{\beta=\infty}$. However, it is expected that around those frequencies at which each of μ_m passes by the vicinity of the origin a

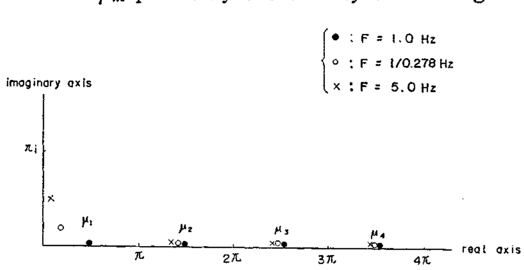

Fig. 2 Distribution of μ_m $(m=1, 2, \cdots)$ in the case of $\beta=5.0$.
(The gravity wave term $\mu_0 \approx \sqrt{h/g}\ 2\pi i/T$ is not shown in this figure.)

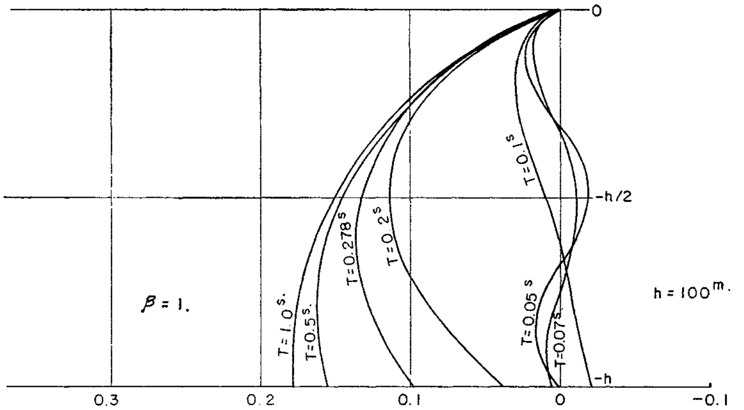

Fig. 3(a) Computed hydrodynamic pressure σ_β on the wall surface in the case of $\beta=1$, $h=100$ m.
Values shown are $\sigma_\beta/4\alpha W_0 h$ at the time point when the wall reaches the most downstream position.
The vibration periods are:
$T=1.0$, 0.5, 0.278, 0.2, 0.1, 0.07, and 0.05 seconds.

weak resonance could appear. Fig. 2 shows the train of μ_m which change the position on the complex plane according to increasing frequency in the case of $\beta=5$.

5. DISCUSSION BY NUMERICAL VALUES

In this section, we investigate the formula σ_β

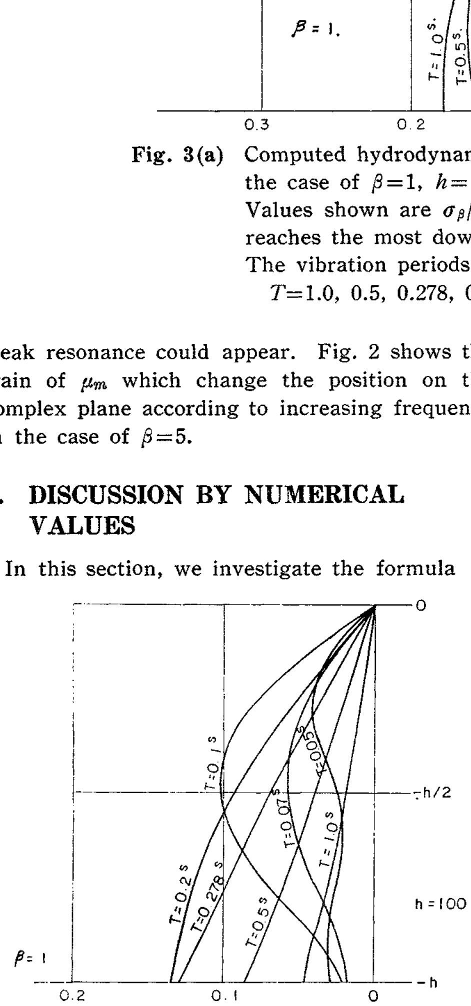

Fig. 3(b) Computed hydrodynamic pressure σ_β on the wall surface in the case of $\beta=1$, $h=100$ m.
Values shown are $\sigma_\beta/4\alpha W_0 h$ at the time point when the wall reaches the neutral position.
The vibration periods are:
$T=1.0$, 0.5, 0.278, 0.2, 0.1, 0.07, and 0.05 seconds.

of (2.11) numerically. Suppose that the rigid wall has a height of $h=100$ m. Fig. 3 shows the distribution of σ_β on the wall surface in the case of $\beta=1$, at two time points when the wall reaches the most downstream position (a); and then the neutral position (b). Fig. 4 is the case of $\beta=5$.

The curves given in Fig. 5 show the absolute maximum of σ_β over the wall surface and over one entire vibration period T, plotted as a function of T.

From these figures it turns out that if the vibration period is suitably larger than $T_1=0.278$ sec (T_1 is the period equivalent to the first resonance frequency, or the so-called cut-off frequency, associated with (1.3)), both the magnitude and the vertical distribution of σ_β with finite β are by no means different from those of $\sigma=\sigma_{\beta=\infty}$, or even from the solution of the Laplace equation subject to (1.2). At the first resonance period T_1, $|\sigma|$ becomes infinitely large. However, $|\sigma_{\beta=5}|$ at the same period is only twice as large as $|\sigma_{\beta=5}|$ at longer vibration period. This result agrees with the experiments made by one of the authors[3]. Further, the numerical computation indicates that the maximal pressure appears after the wall passes by the most downstream position by approximately $3\pi/10$ in terms of phase angle. This amount of phase difference also agrees with the above experiments. In the case of $\beta=5$, the reflection ratio is 4/6, hence the existence of the phase difference may be interpreted as what is one of inherent nature of a damping oscillation. The second resonance peak is much smaller than

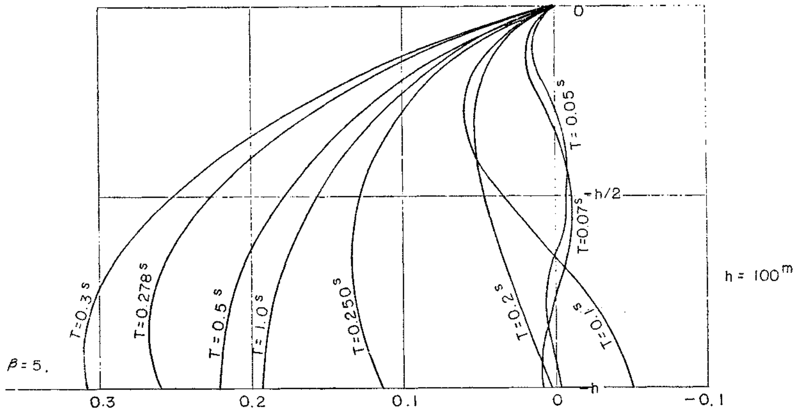

Fig. 4(a) Computed hydrodynamic pressure σ_β on the wall surface in the case of $\beta=5$, $h=100$ m.
Values shown are $\sigma_\beta/4\alpha W_0 h$ at the time point when the wall reaches the most downstream position.
The vibration periods are:
$T=1.0$, 0.5, 0.278, 0.2, 0.1, 0.07, and 0.05 seconds.

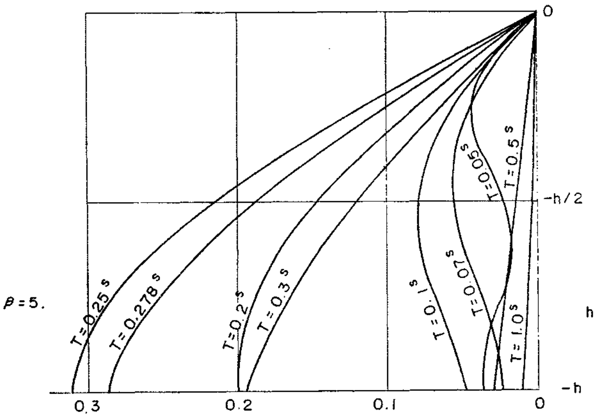

Fig. 4(b) Computed hydrodynamic pressure σ_β on the wall surface in the case of $\beta=5$, $h=100$ m.
Values shown are $\sigma_\beta/4\alpha W_0 h$ at the time point when the wall reaches the neutral position.
The vibration periods are:
$T=1.0$, 0.5, 0.278, 0.2, 0.1, 0.07, and 0.05 seconds.

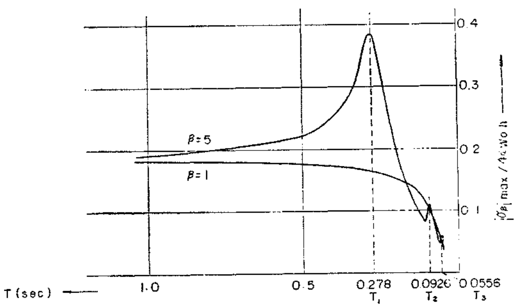

Fig. 5 Maximal pressure on the wall surface as a function of vibration period T. Values shown are the maximum of $|\sigma_\beta|/4\alpha W_0 h$ over $0\leqq t\leqq T$ and $-h\leqq y\leqq 0$ at $x=0$, in the cases of $\beta=1$ and 5, $h=100$ m.

the first one, and the pressure is diminishing as the vibration period tends to zero.

In the solution (1.3), the non-decaying plane waves represented by the second term $\sum_{m=1}^{r}$ dominate the entire phenomenon except at resonance frequencies. This implies that the vibration energy distributes over a wider area of reservoir at higher frequencies, which results in a decreasing pressure on the wall surface as the frequency rises. A similar interpretation may be applied to (2.11), this time for all vibration periods including resonance points as well. In the case of $\sigma_{\beta=1}$, the reflection vanishes at the reservoir bottom. Hence it is a natural consequence that $\sigma_{\beta=1}$ has no resonance peaks. Excluding the resonance peaks of $\sigma_{\beta=5}$, $\sigma_{\beta=1}$ and $\sigma_{\beta=5}$ have similar asymptotic characteristics with respect to vibration period T, that is, both approach the solution of the Laplace equation when T is larger than T_1, and both vanish as $T \to 0$.

To summarize the above discussions, the proposed analytical solution explains in a greater detail those fundamental characteristics of hydrodynamic pressure due to earthquakes which are observed in experiments. The numerical method for the seismic analysis of arch dam coupled with reservoir water, which has been proposed by the authors, may also be justified through this solution.

REFERENCES

1) Hatano, T.: Seismic Force Effect on Gravity-dams. (No. 2) (Accurate Solution of Dynamic Water Pressures on Dams during Earthquake), Proc. JSCE, 1949, pp. 174-183 (in Japanese).
2) Westergaard, H. M.: Water Pressures on Dams during Earthquakes, Proc. ASCE, 1933, pp. 418-472.
3) Hatano, T.: An Examination on the Resonance of Hydrodynamic Pressure during Earthquakes due to Elasticity of Water, Proc. JSCE, March 1966, pp. 1-5 (in Japanese).
4) Hatano, T. and T. Nakagawa: Seismic Analysis of Arch Dams—Coupled Vibration of Dam Body and Reservoir Water—, Proc. JSCE, November 1972, pp. 37-46 (in Japanese); Technical Report, Technical Laboratory No. 1, Central Research Institute of Electric Power Industry, November 1972, 25 pp. (in English).
5) Coddington, E. A. and N. Levinson: Theory of Ordinary Differential Equations, McGraw-Hill, 1955.

(*Received Feb. 25, 1974*)