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FUNCTIONALLY EQUIVALENT SUBSTITUTION OF RESERVOIR-STREAM NETWORKS

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1. INTRODUCTION

This paper is concerned with the rigorous conditions under which a complex reservoir-stream network can be substituted by a simpler one without losing the functional characteristics of the original network, where the functional characteristics of a reservoir system will be rigorously defined in Chapter 2. The theories associated with such conditions are called network substitution theories.

The writer is convinced that no matter what optimization techniques currently available may be used, some steps of system simplification are indispensable for practical computation of optimal control rules of complex reservoir-stream network systems. There are at least four types of technical counter actions to manage complexity. One is to consider the necessary accuracy of reservoir control problem in relation with the uncertainties involved in other components of the total system that is composed of various subsystems with reservoir system as a part. As long as various subsystems are operated in different levels of accuracy, the exceptionally high accuracy of one subsystem does not contribute to the better performance of the total system. Social behavioral patterns such as population growth, economic activities, reactions to unexpected environmental impacts and, above all, social preferences that constitute the planning objectives are the body of uncertainties that can be predicted and controlled only with less accuracy than the water-resource system behavior.

Other counter actions seek more direct technical improvements. They are developments of higher speed degital computers, more capable mathematical optimization procedures and application of reasonable system simplification or decomposition procedures. As for computers and optimization techniques, the state of arts is reported elsewhere. Among various types of system simplification. One is simplification of objective functions and constraints set, introduction of a priori sets of restrictions on phase space and control space, and similar procedures of various kinds. Linear or quadratic approximations to nonlinear curves and descretization of state and/or control variables are major procedures commonly excercised for this purpose. Another category of system simplification is to reduce the number of system components. Decomposition of large scale system into two or more subsystems is a standard procedure to make the computation practically feasible. Repetitive exchange of shadow prices between master program and subprogram is an example of that sort.

The type of system simplification that this paper is concerned is the remodeling of the original complex system into computationally manageable approximate system. For a reservoir-stream network system, it is often practiced that two or more reservoirs are combined into a single reservoir and considered to behave as a single component. The basic problem involved in this procedure is as follows: Although such approximations may be quite healthy from a practical point of view and defensible from the expediency basis, it must still be admitted that the approximations are usually not supported by any rigorous and consistent theories, with a resultant effect that the created networks are no longer guaranteed to function equivalently as the original networks. Therefore the solutions obtained for the approximate networks may not be what are wanted. The objective of this paper is in this very point, that is, to find a rule of network substitution that preserves the functional properties of the original system.

Investigations to lead a set of conditions to theoretically support such a network substitution theory involves three essential tasks. First, to

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establish a basis upon which the theory can be developed it is necessary to construct unambiguous definitions of the functional capability of a reservoir system, the functional equivalence between two reservoir systems and the functional superiority of a reservoir system over another. Second, theorems must be developed to show under what conditions two systems are functionally equivalent or one system is supeior to another. Thirdly, a procedure should be developed to decompose an operation rule which is determined for a substituted system back into the rule operational for the original system. The third task needs some elaboration. Suppose there is a multiple-component reservoir system which includes an n-component sub-reservoir system as a part. Suppose the control rule of this multiple-reservoir system was optimized by treating those n reservoirs as a consolidated single reservoir in a manner which satisfies a specific set of conditions on functional equivalency developed in the previous step. Let the control rule developed for the consolidated reservoir be $R(t)$, namely, the optimal amount of release from the n-component subsystem as a whole at time t is $R(t)$. The third task is concerned with the apportionment of the predetermined release $R(t)$ among n original reservoirs without losing overall optimality.

The literature relating to the subject of this paper was found none except for the third task. As an apportionment rule of releases from multiple component reservoir systems, so called "space rule" was introduced by Harvard Water Program about a decade ago (Bower et al., 1962). The space rule suggests, by assuming the existence of a deterministic drawdown-refill cycle of reservoirs, that the releases from parallel reservoirs should be made in such a way that, after the water has been withdrawn, the ratio of space available in each reservoir to that in all reservoirs equals, insofar as possible, the ratio of the predicted flow into each reservoir during the remainder of the drawdown-refill cycle to that into all reservoirs. This rule may or may not be proper depending on the characteristics of hydrological conditions and water resources activities associated with a system. It is important to note, however, that since this paper intends to establish (or at least initiate) a theoretical approach to system simplification, any procedure is out of interest regardless of its practical soundness as long as it is heavily based on intuitive judgement. The space rule is not an exception from this point of view.

2. DEFINITIONS

To facilitate the discussion the following notations are defined.

n : number of reservoirs in a system.

V_i : capacity of reservoir i.

V : total capacity of reservoirs in the system, i.e.,

$$V=\sum_{i=1}^{n} V_i .$$

$q_i(t)$: inflow into reservoir i at time t.

$Q(t)$: total inflow into the system at time t, i.e.,

$$Q(t)=\sum_{i=1}^{n} q_i(t) .$$

$r_i(t)$: release from reservoir i at time t.

$R(t)$: total release from the system at time t, i.e.,

$$R(t)=\sum_{i=1}^{n} r_i(t) .$$

$e_i(t)$: ratio of the empty space to the capacity of reservoir i at time t, referred to as the space-capacity ratio.

$E(t)$: total empty space of the system at time t, i.e.,

$$E(t)=\sum_{i=1}^{n} e_i(t) V_i .$$

$p_i(t)$: ratio of the storage to the capacity of reservoir i at time t, referred to as the storage-capacity ratio.

It follows that $e_i(t)+p_i(t)=1$ for all i at any t.

$P(t)$: total storage of the system at time t, i.e.,

$$P(t)=\sum_{i=1}^{n} p_i(t) V_i .$$

$T_i(t)$: refilling time of reservoir i at time t or the time necessary for reservoir i to become full under an assumption that no release will be made after time t.

For the case where reservoir i is not in series, the relationship between $e_i(t)$ and $T_i(t)$ is as follows:

$$e_i(t) V_i=\int_{t}^{t+T_i(t)} q_i(z)dz .$$

$\bar{T}(t)$: system refilling time at time t or the time necessary for all reservoirs in the system to become full under an assumption that no release will be made after time t, i.e.,

$$\bar{T}(t)=\max_{i} \{T_i(t)\} .$$

$\underline{T}(t)$: first refilling time of the system at time t or the time necessary for a reservoir or some reservoirs to become full first of all reservoirs in the system under an assumption that no release will be made after time t, i.e.,

$$\underline{T}(t)=\min\{T_i(t)\}.$$

Obviously, $\underline{T}(t)\leq T_i(t)\leq \bar{T}(t)$ for all i at any t.

$w_i(t, u)$: anticipated spill from reservoir i at time u $(u\geq t)$ under an assumption that no release will be made after time t.

For the case where reservoir i is not in series,

$$w_i(t, u)=\begin{cases} q_i(u) & \text{for } u>t+T_i(t)\,, \\ 0 & \text{for } t\leq u\leq t+T_i(t)\,. \end{cases}$$

$W(t, u)$: anticipated total spill from the system at time u $(u\geq t)$ under an assumption that no release will be made after time t.

In a parallel reservoir system,

$$W(t, u)=\sum_{i=1}^{n} w_i(t, u)\,.$$

It follows that $W(t, t)$ is the actual total spill from the system at time t.

$CW(t)$: anticipated cumulative total spill from the system during the system refilling time starting from time t, i.e.,

$$CW(t)=\int_t^{t+\bar{T}(t)} W(t, u)du\,.$$

Finally, when two systems are being compared, reference to a particular system will be established by assigning a superscript to each of the variables of that system; e.g. in comparing systems A and B, V^A and V^B will refer to total capacities of the respective systems.

In addition to the above definitions and notations, the following conditions are assumed to hold for systems under study.

1. Economically affected areas are located downstream of all confluences of those tributaries on which reservoirs belonging to the system are located. Even if economic activities exist in upstream reaches within a system network, they are not highly sensitive to internal allocations of resources.
2. Release capacities of reservoirs' outlets and channel capacities of streamlines are assumed unbounded. Furthermore it is assumed that no economic effects exist in association with flooding in the economically affected area.
3. Economic values associated with the system are independent of the unsteady nature of hydrology and can be expressed as a function of time-averaged releases and spills.
4. Amount of release specified by a release rule $R(t)$ pertains only to the reservoirs in the system and does not include the water which flows into the economically affected area without going through reservoirs in the system.
5. Any given release rule $R(t)$ is designed so as not to release more water than is available.
6. The future inflows are predictable.

Condition 1 implies that recreation benefits on upstream reservoirs can be measured in terms of total water stored and that the hydroelectric power generation is excluded from the activities associated with the reservoir systems in discussion. This is because the hydroelectric power is quite sensitive to the internal allocation of water storages. Condition 2 implies that the release capacity and channel capacity do not constrain the control of water storages in the system and that the effects of flooding are set aside from the discussion. These two conditions practically restrict the application of theorems developed in this paper exclusively to water supply systems. Condition 3 eliminates the need to consider responses to instantaneous flow conditions in this analysis. Condition 4 isolates the system under analysis from larger surrounding areas. Condition 5 separates the decision problem of the release rule $R(t)$ from the substitution problem of functionally equivalent networks. The last condition is the most drastic; it restricts the analysis to systems having deterministic futures. Obviously this condition never holds in practice, but some "deterministic equivalent" flows could be substituted for future inflows. One of the simplest examples of such deterministic equivalent flows would be a series of weekly or monthly mean flows. If a safety factor is considered necessary to be imposed on the flow searies, the searies might be shifted to the safer side by subtracting some portion of standard deviations.

Given these definitions and conditions, functional capability of a system, functional equivalence between two systems, and functional superiority of one system over another can be rigorously defined.

Definition: Functional Capability

Functional capability of a reservoir system (regardless of multi-unit or single-unit) at a certain instant t is defined by three elements:

(1) the amount of water that the system can supply at time t, namely $P(t)$;

(2) the amount of future inflow that the system can hold before all reservoirs in the system become full under an assumption that no release will be made after time t, i.e., $E(t)$; and

(3) the anticipated amount of water that the system would spill at every instant before all reservoirs in the system become full under an assumption that no release will be made after time t, i.e., $W(t, u)$ for $t \leq u \leq t+\bar{T}(t)$.

Definition: Functional Equivalence

Two reservoir systems, which receive the same amount of inflow and are required to release the same amount at every instant, are called functionally equivalent if and only if the three elements of functional capability remain equal at every instant. Using given notations, systems 1 and 2 satisfying the relation:

$$\left.\begin{aligned} &Q^1(t)=Q^2(t) \\ &R^1(t)=R^2(t) \quad \text{at any } t, \end{aligned}\right\} \cdots\cdots(1)$$

are called functionally equivalent if and only if the following relations hold at any t:

$$\left.\begin{aligned} &P^1(t)=P^2(t) \\ &E^1(t)=E^2(t) \\ &W^1(t,u)=W^2(t,u) \quad \text{for any } u. \end{aligned}\right\} \cdots(2)$$

Conditions (1) imply that this definition applies only for the systems whose their input-output characteristics are the same over time. Conditions (2) require that for functional equivalence between two systems the capacities of the systems must be equal and that their anticipated spills before both systems are filled be equal.

Definition: Functional Superiority

For two systems that have the same capacity, receive the same inflow, and are required to release the same amount at every instant, one system is said to be functionally superior to the other if and only if one system always has more storage to supply than the other, and always has less anticipated instantaneous spill than the other before either of systems become full. Using given notations, between systems 1 and 2 satisfying conditions:

$$\left.\begin{aligned} &Q^1(t)=Q^2(t) \\ &R^1(t)=R^2(t) \quad \text{at any } t, \\ &V^1=V^2 \end{aligned}\right\} \cdots\cdots(3)$$

system 1 is functionally superior if and only if the following conditions hold at any t:

$$\left.\begin{aligned} &P^1(t) \geq P^2(t) \\ &W^1(t,u) \leq W^2(t,u) \\ &\qquad \text{for } t \leq u \leq \min\{\bar{T}^1(t), \bar{T}^2(t)\} \end{aligned}\right\} \cdots(4)$$

where strict inequality holds at least for one value of u.

The idea behind on this particular interval specification for the second condition of (4) is as follows: A functionally superior system spills less so that fills sooner. After that the superior system automatically spills all the inflows it receives while the functionally inferior system may still have some empty space left and accordingly spills less. Thus the functional superiority reverses when either of two systems become full with respect to the second condition of (4). The above specification avoids this complexity and is considered reasonable since the flooding effects are out of consideration throughout this paper.

3. EQUIVALENT NETWORK SUBSTITUTION THEOREMS

Using these definitions it is possible to develop the several theorems and corollaries that follow.

Theorem 1

Two reservoir systems, which receive the same amount of inflow and are required to maintain the same release at every instant, whose total capacity and total initial storage are the same, are maintained functionally equivalent if and only if the third element of functional capability of these systems are kept equal at every instant.

Using given notations, if systems 1 and 2 satisfy the relationships

$$\left.\begin{aligned} &Q^1(t)=Q^2(t) \\ &R^1(t)=R^2(t) \\ &P^1(t_1)=P^2(t_1) \quad \text{at any } t, \\ &V^1=V^2 \end{aligned}\right\} \cdots\cdots(5)$$

where t_1 indicates a certain initial time such as $t \geq t_1$, they are functionally equivalent if and only if

$$W^1(t,u)=W^2(t,u) \quad \text{for any } u \text{ at any } t \cdots\cdots(6)$$

holds.

Proof

The total storage in the system at time t can be expressed as

$$P(t)=P(t_1)+\int_{t_1}^{t} Q(z)dz-\int_{t_1}^{t} R(z)dz$$
$$-\int_{t_1}^{t} W(z, z)dz \quad \cdots\cdots(7)$$

where the first term is the initial storage in the system at time t_1, the second is the cumulative inflow to the system between time t_1 and t, the third is the cumulative release, and the last is the cumulative spill.

From relation (5), the first three terms in the right hand side of (7) are equal for systems 1 and 2. If condition (6) holds,

$$W^1(t, t)=W^2(t, t) \quad \text{at any } t, \quad \cdots\cdots(8)$$

that is, the actual spill from both systems are equal at every instant. Therefore the forth term is also equal for two systems. Hence

$$P^1(t)=P^2(t) \quad \text{at any } t.$$

Furthermore, since $V^1=V^2$ as the last expression in relation (5) indicates,

$$E^1(t)=V^1-P^1(t)$$
$$=V^2-P^2(t)$$
$$=E^2(t) \quad \text{at any } t.$$

Therefore condition (6) is sufficient for functional equivalence. It is necessary as well according to the definition of functional equivalence.

Q.E.D.

Corollary to Theorem 1

If systems 1 and 2 satisfy relation (5) system 1 is functionally superior to system 2 if and only if

$$W^1(t, u)\leq W^2(t, u)$$
$$\text{for } t\leq u\leq t+\min\{\bar{T}^1(t), \bar{T}^2(t)\} \text{ at any } t, \quad \cdots\cdots(9)$$

where strict inequality holds at least for one value of u.

Proof

In expression (7) the first three terms on the right hand side are equal for both systems. If (9) holds,

$$W^1(t, t)\leq W^2(t, t) \quad \text{at any } t.$$

This implies

$$P^1(t)\geq P^2(t) \quad \text{at any } t.$$

Therefore, condition (9) is sufficient for system 1 to be functionally superior to system 2. It is necessary as well by definition. Q.E.D.

Theorem 2

A multiple n-component parallel reservoir system, say system A, can be substituted without losing functional equivalence by a single reservoir system, say system B, which has the same inflow, release requirement, total capacity and initial storage as system A, if and only if system A has the following property: After making required releases spaces can always be apportioned in such a manner that all reservoirs become full at the same time under an assumption that no release will be made after the current one. Using given notations, system A and B satisfying relation (5), are functionally equivalent if and only if

$$\underline{T}^A(t)=\bar{T}^A(t) \quad \text{at any } t \quad \cdots\cdots(10)$$

holds.

Proof

Since systems A and B are under relation (5), Theorem 1 is applicable to systems A and B. That is, they are functionally equivalent if and only if condition (6)

$$W^A(t, u)=W^B(t, u) \quad \text{for any } u \text{ at any } t$$

holds.

Because system B has only a single reservoir, after the system refilling time its spill equals the inflow so that

$$W^B(t, u)=\begin{cases} Q^B(u) & \text{for } u>t+\bar{T}^B(t) \\ 0 & \text{for } t\leq u\leq t+\bar{T}^B \text{ at any } t. \end{cases} \quad \cdots\cdots(11)$$

Similarly, after each reservoir in system A has filled, its spill is equal to its inflow so that

$$W^A(t, u)=\sum_{i=1}^{n} w_i^A(t, u)$$

where

$$w_i^A(t, u)=\begin{cases} q_i(u) & \text{for } u>t+T_i^A(t) \\ 0 & \text{for } t\leq u\leq t+T_i^A(t). \end{cases}$$

Since

$$\underline{T}^A(t)\leq T_i^A(t)\leq\bar{T}^A(t) \quad \text{for any } i,$$

$$W^A(t, u)=\begin{cases} Q^A(u) & \text{for } u>t+\bar{T}_i^A(t) \\ \text{between } (0, Q^A(u)) & \text{for } t+\underline{T}^A(t)<u\leq t+\bar{T}_i^A(t) \\ 0 & \text{for } t\leq u\leq t+\underline{T}^A(t) \text{ at any } t. \end{cases} \quad \cdots\cdots(12)$$

Here $Q^A(u)=Q^B(u)$. Therefore in order to maintain relation (6), it is necessary and sufficient that

$$\underline{T}^A(t)=\bar{T}^A(t) \quad \text{at any } t, \quad \cdots\cdots(13)$$

and

$$\bar{T}^A(t)=\bar{T}^B(t) \quad \text{at any } t. \quad \cdots\cdots(14)$$

Since initially the two systems have the same space, the same inflow, and are required the same release, the total storages and spaces are kept equal at least until actual spill occurs. If all reservoirs in system A always become full at the same time, the system refilling times of A and B are equal, i.e., $\bar{T}^A(t)=\bar{T}^B(t)$ at least until actual spill occurs. This implies that the actual spill occurs at the same time, and accordingly that they spill the same amount. Therefore the storages and spaces in both systems are kept equal regardless of the occurrence of the spill, and then

$$\bar{T}^A(t)=\bar{T}^B(t) \qquad \text{at any } t.$$

Therefore, if condition (13) holds, condition (14) is satisfied. Hence, condition (13), namely, condition (10) is a necessary and sufficient condition for the functional equivalence of systems A and B. Q.E.D.

It should be noted that condition (10) practically specifies the control rule to maintain functional equivalence, namely, that the space-capacity ratios should be selected for system A so that all reservoirs become full simultaneously. However, it can be shown rigorously, as well as being intuitively obvious, that such a condition cannot always be maintained.

Theorem 3

The consolidation of a multiple reservoir system 2 into a single reservoir system 1 satisfying condition (5) always results a functionally superior system or at least a functionally equivalent system.

Proof

Since the two system satisfy (5), corollary to Theorem 1 is applicable. Recall relationships (11) and (13)

$$W^1(t,u)=\begin{cases} Q^1(u) & \text{for } u>t+\bar{T}^1(t) \\ 0 & \text{for } t\leq u\leq t+\bar{T}^1(t) \text{ at any } t. \end{cases}$$

$$W^2(t,u)=\begin{cases} Q^2(u) & \text{for } u>t+\bar{T}^2(t) \\ \text{between } (0, Q^2(u)) & \text{for } t+\underline{T}^2(t)<u\leq t+\bar{T}^2(t) \\ 0 & \text{for } t\leq u<t+\underline{T}^2(t) \text{ at any } t. \end{cases}$$

Here $Q^2(u)=Q^1(u)$ for any u. Accordingly, it is obvious that

$$W^1(t,u)\leq W^2(t,u) \quad \text{for } t\leq u\leq t+\min\{\bar{T}^2(t), \bar{T}^1(t)\} \text{ at any } t. \qquad \cdots(15)$$

From Theorem 1 and its corollary, condition (15) is necessary and sufficient for system 1 to be functionally superior or at least equivalent to system 2. Q.E.D.

Theorem 4

Suppose that an n-component parallel reservoir system A and a single reservoir system B satisfy condition (5), and that n inflows in system A satisfy the following relation:[1]

$$a_1\frac{q_1(t)}{V_1}=a_2\frac{q_2(t)}{V_2}=\cdots=a_n\frac{q_n(t)}{V_n} \qquad \text{at any } t, \qquad \cdots(16)$$

where $a_i>0$ for all i, and a_i's are strictly constant over time. Then the functional equivalence of systems A and B holds if and only if the reservoirs in system A are controlled so as to keep the following relation:

$$a_1e_1(t)=a_2e_2(e)=\cdots=a_ne_n(t) \qquad \text{at any } t. \qquad \cdots(17)$$

Proof

By the definition of space-capacity ratio $e_i(t)$ of reservoir i in system A,

$$e_i(t)=\int_t^{t+T_i^A(t)}\frac{q_i(z)}{V_i}dz \qquad \text{for all } i. \quad \cdots(18)$$

From given condition (16),

$$\frac{q_i(z)}{V_i}=\frac{a_jq_j(z)}{a_iV_j} \qquad \text{for all } i \text{ and } j \text{ at any } z. \qquad \cdots(19)$$

By substituting (19) into (18), the following relationship is obtained:

$$e_i(t)=\frac{a_j}{a_i}\int_t^{t+T_i^A(t)}\frac{q_j(z)}{V_j}dz \qquad \text{for all } i \text{ and } j \text{ at any } t. \qquad \cdots(20)$$

Now, recall the necessary and sufficient condition for functional equivalence from Theorem 2:

$$\underline{T}^A(t)=T_i^A(t)=\bar{T}^A(t) \qquad \text{for all } i \text{ at any } t,$$

or

$$T_i^A(t)=T_j^A(t) \qquad \text{for all } i \text{ and } j \text{ at any } t.$$

If system A is functionally equivalent to system B, expression (20) becomes

$$e_i(t)=\frac{a_j}{a_i}\int_t^{t+T_j(t)}\frac{q_j(z)}{V_j}dz=\frac{a_j}{a_i}e_j(t)$$

[1] This condition can be interpreted as all inflows are perfectly correlated.

for all i and j at any t. ……(21)

Hence

$$a_i e_i(t) = a_j e_j(t) \quad \text{for all } i \text{ and } j \text{ at any } t. \qquad \cdots\cdots(22)$$

Thus, relation (17) is a necessary condition.

Conversely, if (17) holds, from (20) it follows that

$$\int_t^{t+T_i^A(t)} \frac{q_j(z)}{V_j} dz = e_i(t) \frac{a_i}{a_j}$$
$$= e_j(t)$$
$$= \int_t^{t+T_j^A(t)} \frac{q_j(z)}{V_j} dz$$

for all i and j at any t.

Therefore

$$T_i^A(t) = T_j^A(t) \quad \text{for all } i \text{ and } j \text{ at any } t. \qquad \cdots\cdots(23)$$

Thus, condition (17) is necessary and sufficient for functional equivalence. Q.E.D.

This theorem leads to another important corollary.

Corollary to Theorem 4

Suppose that an n-component parallel reservoir system A and a single reservoir system B satisfy condition (5), and that n inflows in system A satisfy the following relation:[2]

$$\frac{q_1(t)}{V_1} = \frac{q_2(t)}{V_2} = \cdots = \frac{q_n(t)}{V_n} \quad \text{at any } t. \qquad \cdots\cdots(24)$$

Then the functional equivalence of systems A and B holds if and only if the reservoirs in system A are controlled so as to keep the following relation:

$$e_1(t) = e_2(t) = \cdots = e_n(t) \quad \text{at any } t. \qquad \cdots\cdots(25)$$

Proof

This corollary can be immediately proved by setting all a_i's at unity in Theorem 4. Q.E.D.

Both conditions (17) and (25) specify the control rules to maintain functional equivalence. The importance of the preceding corollary is that control rule (25) can always be practicable as long as condition (24) holds while condition (17) can not always be maintained under condition (16). This implies that a parallel reservoir system can always be substituted by a single reservoir system if the perfect correlation among inflows and equal ratios of inflow magnitudes to reservoir capacities are satisfied, which are extremely strict conditions.

[2] This condition can be interpreted as all inflows are perfectly correlated and their magnitudes are proportional to corresponding reservoir capacities.

Theorem 5

A series reservoir system, say system C, and a single reservoir system B that satisfy condition (5) are functionally equivalent if and only if the downmost reservoir of the series system, say reservoir n, is operated so as to be the last reservoir to fill. Using given notations a series reservoir system C and a single reservoir system B that satisfy condition (5) are functionally equivalent if and only if

$$T_n^C(t) \geq T_i^C(t) \quad \text{for all } i \text{ at any } t. \quad \cdots(26)$$

Proof

Since the two systems satisfy condition (5), Theorem 1 applies. In accordance with conditions of the Theorem, the systems are functionally equivalent if and only if

$$W^C(t, u) = W^B(t, u) \quad \text{for all } u \text{ at any } t. \qquad \cdots\cdots(27)$$

It is obvious, however, as discussed in the proof of Theorem 2,

$$W^C(t, u) = \begin{cases} Q^C(u) & \text{for } u > t + \bar{T}^C(t) \\ \text{between } (0, Q^C(u)) & \text{for } t + T_n^C(t) < u \leq t + \bar{T}^C(t) \\ 0 & \text{for } t \leq u \leq t + T_n^C(t) \end{cases} \qquad \cdots\cdots(28)$$

and

$$W^B(t, u) = \begin{cases} Q^B(u) & \text{for } u > t + \bar{T}^B(t) \\ 0 & \text{for } t \leq u \leq t + \bar{T}^B(t) \end{cases} \qquad \cdots\cdots(29)$$

In order to satisfy condition (27), it is necessary to hold the following condition at any t:

$$T_n^C(t) = \bar{T}^C(t), \qquad \cdots\cdots(30)$$

or equivalently,

$$T_n^C(t) \geq T_i^C(t) \quad \text{for all } i. \qquad \cdots\cdots(31)$$

Conversely if condition (26) holds, condition (27) is obviously satisfied. Therefore condition (26) is necessary and sufficient for the two systems to be functionally equivalent. Q.E.D.

4. OPTIMAL APPORTIONMENT

As shown by Theorem 3, consolidation of multi-unit reservoirs always results a functionally superior or at least equivalent system both in series networks and in parallel networks. This implies that the optimal control of a multi-unit reservoir system is the control that guarantees the functional equivalence with a single reservoir system if ever it is possible. In other words, the optimal control rule of system A is to maintain relation (10) and that of system C is to maintain relation (31).

As mentioned previously, however, relation (10) or (31) is not necessarily maintainable in general. As a matter of fact, either of them holds only in extremely rare cases.

The very same problem is the theme of the third task, the optimal apportionment of the required total release determined for a consolidated reservoir system to the releases from the original component reservoirs. The implication of the fact that relation (10) or (26) is not operational in general is that the optimal apportionment is not necessarily feasible at least in the sense introduced in the definition of functional superiority and its corresponding consequence Theorem 3.

In order to break through this difficulty, the optimal apportionment is dfined independently of the functional superiority definition. Its replacement is the following:

Definition: Optimal Apportionment

Optimal apportionment of the required total release among two or more reservoirs in a system is an apportionment for which the empty spaces of reservoirs are allocated so as to make the total spill before all reservoirs become full minimum under an assumption that no release is made after the current one. Using the given notation, control rule 1 is optimal for apportionment of the required total release among multiple reservoirs if

$$CW^1(t) \leq CW^2(t) \quad \text{at any } t, \quad \cdots\cdots\cdots(32)$$

where superscript 2 indicates any control rule other than control rule 1.[3]

[3] The super-postscripts are used as indicaters of systems, but their usage is extended to control rules by supposing that there is a system which is called system 1 when it is controlled by control rule 1 and is called system 2 when it is controlled by any other control rule.

It is noteworthy that if the definition of functional superiority were applied for an optimal control, it would say that the anticipated spill under the assumption of no further release should always be smaller by the optimal control rule than by any other control rule; namely, if control rule 1 is optimal, then

$$W^1(t, u) \leq W^2(t, u) \quad \text{for } t \leq u \leq t + \min\{\bar{T}^1(t), \bar{T}^2(t)\} \text{ at any } t. \quad \cdots\cdots(33)$$

The definition chosen above does not require the instantaneous spill to be smallest of all, but requires its cumulative volume to be smallest. The underlying idea of this definition is as follows: The basic criterion of choosing the optimal apportionment definition is to specify such an apportionment (1) that is always practicable and (2) that is satisfactory in the light of proper subjective judgement. First, the definition selected provides the operational apportionment rules regardless of the status of storages and inflows as will be seen in Theorems 6 and 7. Secondly, the minimization of the total spill before all reservoirs in a system become full can be considered quite sound for the water supply systems that satisfy basic conditions 1 and 2 of Chapter 2, since the detention function of a reservoir system exists only up to the system refilling time and since the definition selected requires the wasting spill before that time minimized.

The definition of optimal apportionment provides the practical apportionment rule for a series reservoir system and for a parallel reservoir system as Theorems 6 and 7, respectively.

Theorem 6

The optimal apportionment of the required release among series reservoirs is to release water so as to make the center of gravity of the remaining water highest.

In practice, this theorem suggests to release water first from the lowest reservoir in the system until it is exhausted, next from the second lowest reservoir until it is also exhausted and repeat the same procedure for other reservoirs succesively from lower to upper until the requirement is met or the up-most reservoir becomes empty. In addition, no transfer should be made from an upper reservoir to a lower reservoir unless that transfer is necessary to make the requisit release from the system.

Proof

Theorem 6 can be proven by using following

Lemmas 1, 2 and 3.

Lemma 1

In a series reservoir system, the transfer of an arbitrary amount of water ΔQ from an upper reservoir to a lower reservoir means the increase or no change of the system refilling time.

Lemma 2

In a series reservoir system, the transfer of an arbitrary amount of water ΔQ from a lower reservoir to an upper reservoir means the shortening or no change of the system refilling time.

Lemma 3

The anticipated cumulative spill $CW(t)$ from a reservoir system before all reservoirs become full under the no release assumption is minimized when the system refilling time $\bar{T}(t)$ is minimized.

Lemmas 1 and 2 imply that the apportionment of the required release so as to make the center of gravity of the remaining water highest is the way to make the system refilling time shortest. Then Lemma 3 ensures that this apportionment produces the smallest anticipated cumulative spill. Therefore it is the optimal apportionment rule in accordance with the definition. Q.E.D.

Proof of Lemma 1

Let the upper reservoir and the lower reservoir be the i-th and j-th from the top in the searies. Let the last reservoir to become full under no further release assumption be denoted by i_c meaning a critical reservoir. Let the superscripts $\circ$ and $*$ indicate the systems before and after the transfer of water ΔQ. Finally let the system refilling time be denoted by T for simplisity. Then the effect of transferring water ΔQ from reservoir i to reservoir j on the position of the critical reservoir in the series can be classified into the following four cases.

Case 1: $i_c^\circ \in [1, j-1]$ and $i_c^* \in [1, j-1]$

Since by the transfer of water from reservoir i $(1 \leq i \leq j-1)$ to reservoir j, the refilling times of reservoirs 1 through $j-1$ are unexceptionally either prolonged or unchanged. Therefore as long as both i_c° and $i_c^* \in [1, j-1]$, T^* is not shorter than T°, i.e., $T^* \geq T^\circ$.

Case 2: $i_c^\circ \in [1, j-1]$ and $i_c^* \in [j, n]$

By the transfer of water from reservoir i to reservoir j, the refilling times of reservoirs 1 through $j-1$ are either prolonged or unchanged and those of reservoirs j through n are either shortened or unchanged. Therefore if ever the critical reservoir shifts from an upstrem reservoir to a downstream reservoir, T^* should equal T° at which both reservoirs i_c° and i_c^* of the new system are in fact simultaneously filled.

Case 3: $i_c^\circ \in [j, n]$ and $i_c^* \in [1, j-1]$

Suppose $T^* < T^\circ$, then by the time T^* all reservoirs 1 through $j-1$ of both the original system and the new system must be filled. Once the reservoirs 1 through $j-1$ of the new system become full, the effect of transferring water from reservoir i to reservoir j disappears and consequently both systems behave identically after time T^*. Hence the assumption $T^* < T^\circ$ is faulse. Accordingly $T^* \geq T^\circ$.

Case 4: $i_c^\circ \in [j, n]$ and $i_c^* \in [j, n]$

Since by transferring water the refilling times of reservoirs j through n are either shortened or unchanged, i.e., $T^* \leq T^\circ$. By the time T^* when all reservoirs 1 through $j-1$ of both the original system and the new system are filled, the effect of the transfer disappears and thereafter both systems behave identically. Therefore T^* should equal T°.

In any case, by the transfer of water ΔQ from an upper reservoir to a lower reservoir, the system refilling time is either prolonged or unchanged. Q.E.D.

Proof of Lemma 2

The same notations as used in the preceding proof will be used. The effect of transferring water ΔQ from reservoir j to reservoir i on the position of the critical reservoir in the searies can be classified into the following four cases.

Case 1: $i_c^\circ \in [1, j-1]$ and $i_c^* \in [1, j-1]$

By the transfer of water the refilling times of reservoirs 1 through $j-1$ are either shortened or unchanged. Therefore as long as both i_c° and $i_c^* \in [1, j-1]$, $T^* \leq T^\circ$.

Case 2: $i_c^\circ \in [1, j-1]$ and $i_c^* \in [j, n]$

Suppose $T^* > T^\circ$, then by the time T° when all reservoirs 1 through $j-1$ of both systems are filled, the effect of the transfer of water disappears and thereafter both the new system and the original system behave identically. Therefore the assumption is faulse, and $T^* \leq T^\circ$.

Case 3: $i_c^\circ \in [j, n]$ and $i_c^* \in [1, j-1]$

By transferring water from reservoir j to reservoir i, the refilling times of reservoirs j through n are unexceptionally either prolonged or unchanged, and those of reservoirs 1 through $j-1$ are either shortened or unchanged. Therefore if ever the critical reservoir shifts from a downstream reservoir to an upstream reservoir, the system refilling time should be unchanged at which both reservoirs i_c° and i_c^* of the new system are in fact simultaneously filled.

Case 4: $i_c^\circ \in [j, n]$ and $i_c^* \in [j, n]$

Obviously $T^* = T^\circ$.

In any case, by transferring water ΔQ from a lower reservoir to an upper reservoir the system refilling time is either shortened or unchanged.

Q.E.D.

Proof of Lemma 3

The anticipated cumulative spill $CW(t)$ can be expressed as follows:

$$CW(t)=\int_t^{t+\bar{T}(t)} Q(z)dz-E(t) \quad \cdots\cdots\cdots\cdots\cdots(34)$$

where the first term of the right-hand-side is the total inflow during the system refilling time $\bar{T}(t)$ starting at time t, and the second term is the space available at the beginning of time t.

Since $Q(z)$ and $E(t)$ are uncontrollable, minimizing $CW(t)$ is equivalent to minimizing the system refiilling time $\bar{T}(t)$. Q.E.D.

It should be noted that this lemma holds for a parallel reservoir system as well as a series reservoir system.

Theorem 7

The optimal apportionment of a required release among parallel reservoirs is as follows:

(1) Allocate spaces so as to equalize the refilling times of all reservoirs in so far as possible.

(2) Once it becomes impossible by some reservoirs being emptied before the required release is made, keep those reservoirs empty and allocate spaces of other reservoirs so as to equalize their refilling times.

Proof

According to Lemma 3, the optimal apportionment rule or a rule that minimizes the anticipated cumulative spill is equivalent to the one that minimizes the system refilling time.

Suppose a necessary total space to be allocated at time t after releasing required water is $E(t)$, where

$$0 \leq E(t) \leq V \quad \text{at any } t. \quad \cdots\cdots\cdots\cdots\cdots(35)$$

Using given notations, the optimal apportionment problem can be expressed as an optimization problem of finding a set of space-capacity ratios which minimizes the system refilling time; namely,

$$\min_{e_i(t)} \bar{T}(t) = \min_{e_i(t)} \max_i \{T_i(t)\} \quad \cdots\cdots\cdots\cdots\cdots(36)$$

subject to

$$\left.\begin{aligned} & e_i(t)=\int_t^{t+T_i(t)} \frac{q_i(z)}{V_i}dz \\ & 0 \leq e_i(t) \leq 1 \\ & \sum_{i=1}^n e_i(t)V_i = E(t)\,. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots(37)$$

Here the only changeable variables are $e_i(t)$ and the corresponding $T_i(t)$ and the rest are either constants or unchangeable. It is obvious from the third equality constraint of (37) that an increase of any $e_i(t)$ results a corresponding decrease in some $e_j(t)$, $j \neq i$. It follows from the first expression of (37) that an increase of some $T_i(t)$ results in a decrease of some $T_j(t)$, $j \neq i$. Accordingly if all the refilling times other than the longest are increased as much as possible, the longest refilling time, i.e., the system refilling time, is automatically decreased to its minimum. Therefore the minimal system refilling time can be attained by equalizing all the reservoir refilling times, or equivalently, by controlling all reservoirs to become full at the same time under the no release assumption. This completes the proof for the first half of the theorem.

As the second constraint of (37) shows, the biggest space-capacity ratio is bounded by 1. Therefore some refilling times may reach their highest values before the necessary total space is allocated. In that case it is obvious that the allocation of spaces so as to equalize the rest of refilling times, leaving the reservoirs that have already been emptied as they are, is the way to minimize the system refilling time. This completes the proof for the latter half of the theorem.

Q.E.D.

5. SUMMARY AND CONCLUSIONS

The objective of this paper was to develop a substitution theory for replacing a multiple component reservoir-stream network by a single component network with the functionally equivalent capability. First the functional capability of a reservoir system was rigorously defined in terms of the volume of water stored, the empty space and the anticipated instantaneous spill before all reservoirs become full under no further release assumption. Based on this fundamental definition, the functional equivalency of two systems was defined as having the identical functional capability and the functional superiority of a system over another as having equal or more stored water, and equal or less anticipated instantaneous spill throughout the time before the system becomes completely filled. These definitions then lead to a series of theorems, which included ones to specify the concrete procedure of equivalent network substitution.

Theorem 1 and its corollary show that as long as the exogenous conditions of two reservoir systems are identical, the functional equivalence and superiority can be determined by the anticipated instantaneous spill alone. Theorem 2 implies that a multiple component parallel reservoir system can be treated as if it were a consolidated single reservoir system as long as the exogenous conditions are kept unchanged and the all reservoirs in the system are controlled in such a way that they become full at the same time under no further release assumption. If the latter condition of equal reservoir refilling times does not hold as expected in real circumstances, Theorem 3 shows that the consolidation of multiple reservoirs into a single reservoir always results a functionally superior system. Theorem 4 and its corollary deal with the same condition under the presence of particular relation between the inflows and the reservoir capacities. In case that the inflows to parallel reservoirs are perfectly correlated, the equal refilling time condition holds if the empty spaces of reservoirs are kept proportional to their inflows. It is obvious, however, that this condition cannot always be satisfied since the empty space is bounded by the capacity of the reservoir while the inflow is unbounded in general. In case that the inflow magnitudes are not only perfectly correlated but also proportional to the respective reservoir capacities, the condition can always be met and the single reservoir substitution exists for this parallel reservoir system. As for a series reservoir system, Theorem 5 shows that the single reservoir substitution that preserves the functional equivalence is possible if the original system can be controlled in such a way that the downmost reservoir is always the last reservoir to become full under no further release assumption. This control is also not always possible in general.

Thus the conditions that guarantee the functionally equivalent single reservoir substitution are not necessarily always practicable. Therefore if a multiple reservoir system is converted into a single reservoir system, the resultant system is unexceptionally superior to the original system. If this situation is taken for granted and some kind of reservoir substitution is still considered inevitable in the system optimization procedure, the only way left is to minimize the difference of the functional capability between the original and the consolidated (and accordingly superior) systems. This objective can be pursued in the form of the optimal apportionment of required releases among two or more reservoirs. For this purpose the optimality of apportionment was first defined as an apportionment that minimizes the anticipated total spill before all reservoirs become full under no further release assumption. This definition is not a direct extension of the definition of functional superiority, but was selected because of its practicality. The optimal apportionment rules were then determined for a series reservoir system by Theorem 6 and for a parallel reservoir system by Theorem 7. In a series reservoir system, it is proved optimal to release water so as to make the center of the gravity of the remaining water highest. In a parallel reservoir system, on the other hand, it is optimal to release water so as to equalize the refilling times of all reservoirs in so far as possible. These rules are applicable regardless of the configulation of reservoir-stream network.

It should be noted that all the substitution rules and the apportionment rules developed in this paper are valid only within the realm of given definitions and basic postulates stated in Chapter 2. Although it is admitted that the definitions are subject to further discussions and that the theorems developed so far are far from practice, it may still be concluded that the very objective of this paper is successfully satisfied by arousing the theoretical attention to the inevitable part of system remodeling procedure.

REFERENCE

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