PROC. OF JSCE,
No. 223, MARCH 1974

BUCKLING OF COLUMNS UNDER AXIAL LOADS AND TORSIONS

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SYNOPSIS

The buckling of thin-walled open section columns due to axial forces and conservative torsions are presented in this paper.

Greenhill has studied first on the buckling of a column with a solid section under the same loads. Ziegler has pointed out that the buckling under consideration should be classified into two categories, namely the conservative and non-conservative bucklings. He also divided the conservative torsions into quasi-tangential, semi-tangential and pseudotangential torsions, and computed characteristic values of buckling for each cases.

In order to obtain stability equations of the columns with the thin-walled open sections, the total potential energy is formulated by using the concept of the initial stresses.

The differential equations governing buckling of the columns under the axial forces and the conservative torsions are derived from the total potential energy.

The obtained boundary conditions of bending moments coincide with the semitangential moments defined by Ziegler for the columns having two axes of symmetry.

The buckling of a channel section column is studied and the effects of warping and torsional deformations on the buckling are investigated.

1. INTRODUCTION

The buckling of columns due to constant axial loads and torsions applied at both ends of the column has been first investigated by Greenhill[1]. He has derived the differential equations for the columns with the double symmetric cross sections, namely Greenhill's equations. Ziegler[2] studied precisely boundary conditions relating to bending moments at the ends of the columns, which are produced by buckling deformations. He showed that the torsion applied to the columns should be classified into two categories, namely the conservative and non-conservative torsions, and corrected the Greenhill's results. The instability of the columns under the axial loads and the conservative torsions has been investigated by Beck[3]. These investigations have been for columns with solid sections.

For thin-walled open section columns, Kawai[4] pointed out that the buckling occurs generally a combination of bending and torsion. But he did not mention about mechanical boundary conditions.

No other studies seem to be available in literature regarding the buckling of thin-walled open section columns due to axial loads and conservative torsions.

The present study is intended to derive general stability equations of the columns with the thin-walled open cross sections under the axial forces and the conservative torsions, and clarify mechanical boundary conditions.

The variational method is applied effectively to derive the differential equations, together with the concept of the initial stresses[5].

Finally, the buckling of the columns having channel cross sections with simply supported ends are investigated and the effects of the warping function on the buckling are studied.

2. RELATIONS BETWEEN INITIAL STRESSES AND THEIR RESULTANTS

The coordinate system used in the present paper is that a z axis coincides with the centroidal axis of the columns and x and y axes are perpendicular to the z axis, which form the right handed system. The x and y coordinates of a point M on the profile line of the cross section are functions of an argument s. The distance from the profile line in the direction of the wall thickness

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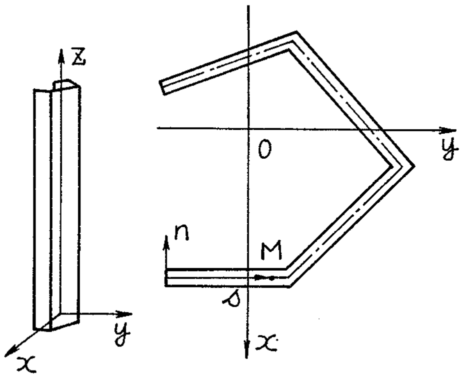

Fig. 1 Coordinates.

is measured by n as shown in Fig. 1.

Initial stresses in the column just prior to buckling due to the axial force and the conservative torsion are given by[6)]

$$\left.\begin{aligned}\sigma_z^{(0)}&=\frac{P_z^{(0)}}{A}+\frac{\omega}{I_\omega}M_\omega^{(0)}\\ \tau_{xz}^{(0)}&=\frac{\partial\Phi_0}{\partial y}-\frac{1}{I_\omega t}\left(\int_0^s\omega t\,ds\right)\frac{dM_\omega^{(0)}}{ds}\cos\alpha\\ \tau_{yz}^{(0)}&=-\frac{\partial\Phi_0}{\partial x}-\frac{1}{I_\omega t}\left(\int_0^s\omega t\,ds\right)\frac{dM_\omega^{(0)}}{ds}\sin\alpha\end{aligned}\right\}\quad\cdots\cdots(1)$$

where $\sigma_z^{(0)}$, $\tau_{xz}^{(0)}$ and $\tau_{yz}^{(0)}$ are the initial normal and shearing stresses, and $P_z^{(0)}$ and $M_\omega^{(0)}$ are the axial force and the warping moment just prior to buckling. Constants, A and I_ω, are the cross sectional area and the coefficient of the warping rigidity of the cross section. A function ω is the unit warping with respect to the shear center (x_s, y_s) of the section. The wall thickness is denoted by t and α in Eq. (1) is the angle that the tangent to the profile line at the point M makes with the x axis. Φ_0 is the St. Venant's stress function expressed for the thin-walled open section[9)] by

$$\Phi_0=G\left(\frac{t^2}{4}-n^2\right)\frac{d\theta_0}{dz}\quad\cdots\cdots(2)$$

The St. Venant torsion, $M_s^{(0)}$, is written by the stresses function as[7)]

$$M_s^{(0)}=2\int_A\Phi_0\,dA\quad\cdots\cdots(3)$$

3. DISPLACEMENTS AND STRAINS DUE TO BUCKLING

Displacements in the x, y and z directions at an arbitrary point of the columns due to buckling are

$$\left.\begin{aligned}U&=u-(y-y_s)\theta\\ V&=v+(x-x_s)\theta\\ W&=-x\frac{du}{dz}-y\frac{dv}{dz}+\omega\frac{d\theta}{dz}\end{aligned}\right\}\quad\cdots\cdots(4)$$

in which u and v are the displacements of the shear center in the direction of the x and y axes, and θ is the angle of rotation of the cross section.

Strains caused by buckling are

$$\left.\begin{aligned}\varepsilon_z=&-x\frac{d^2u}{dz^2}-y\frac{d^2v}{dz^2}+\omega\frac{d^2\theta}{dz^2}\\ &+\frac{1}{2}\left[\frac{du}{dz}-(y-y_s)\frac{d\theta}{dz}\right]^2\\ &+\frac{1}{2}\left[\frac{dv}{dz}+(x-x_s)\frac{d\theta}{dz}\right]^2\\ \varepsilon_{xz}=&-\left(y-y_s-\frac{\partial\omega}{\partial x}\right)\frac{d\theta}{dz}+\theta\left[\frac{dv}{dz}+(x-x_s)\frac{d\theta}{dz}\right]\\ &\underline{+\left(\frac{du}{dz}-\frac{\partial\omega}{\partial x}\frac{d\theta}{dz}\right)\left(x\frac{d^2u}{dz^2}+y\frac{d^2v}{dz^2}-\omega\frac{d^2\theta}{dz^2}\right)}\\ \varepsilon_{yz}=&\left(x-x_s+\frac{\partial\omega}{\partial y}\right)\frac{d\theta}{dz}-\theta\left[\frac{du}{dz}-(y-y_s)\frac{d\theta}{dz}\right]\\ &\underline{+\left(\frac{dv}{dz}-\frac{\partial\omega}{\partial y}\frac{d\theta}{dz}\right)\left(x\frac{d^2u}{dz^2}+y\frac{d^2v}{dz^2}-\omega\frac{d^2\theta}{dz^2}\right)}\end{aligned}\right\}\quad\cdots\cdots(5)$$

The underlined terms in Eq. (5) are usually neglected to derive the stability equations[6)].

4. THE PRINCIPLE OF VIRTUAL WORK

The concept of the initial stresses is introduced to write the principle of virtual work for the buckling of the columns[5)].

It is

$$\begin{aligned}&\int_V\left[\sigma_z\delta\left(-x\frac{d^2u}{dz^2}-y\frac{d^2v}{dz^2}+\omega\frac{d^2\theta}{dz^2}\right)\right.\\ &\quad-\tau_{xz}\left(y-y_s-\frac{\partial\omega}{\partial x}\right)\delta\left(\frac{d\theta}{dz}\right)\\ &\quad\left.+\tau_{yz}\left(x-x_s+\frac{\partial\omega}{\partial y}\right)\delta\left(\frac{d\theta}{dz}\right)\right]dV\\ &+\int_V\frac{1}{2}\sigma_z^{(0)}\delta\left[\left\{\frac{du}{dz}-(y-y_s)\frac{d\theta}{dz}\right\}^2\right.\\ &\quad\left.+\left\{\frac{dv}{dz}+(x-x_s)\frac{d\theta}{dz}\right\}^2\right]dV\\ &+\int_V\tau_{xz}^{(0)}\delta\left[\theta\left\{\frac{dv}{dz}+(x-x_s)\frac{d\theta}{dz}\right\}\right.\\ &\quad\left.+\left(\frac{du}{dz}-\frac{\partial\omega}{\partial x}\frac{d\theta}{dz}\right)\left(x\frac{d^2u}{dz^2}+y\frac{d^2v}{dz^2}-\omega\frac{d^2\theta}{dz^2}\right)\right]dV\\ &+\int_V\tau_{yz}^{(0)}\delta\left[-\theta\left\{\frac{du}{dz}-(y-y_s)\frac{d\theta}{dz}\right\}\right.\\ &\quad\left.+\left(\frac{dv}{dz}-\frac{\partial\omega}{\partial y}\frac{d\theta}{dz}\right)\left(x\frac{d^2u}{dz^2}+y\frac{d^2v}{dz^2}-\omega\frac{d^2\theta}{dz^2}\right)\right]dV\\ &=0\quad\cdots\cdots(6)\end{aligned}$$

where σ_z, τ_{xz} and τ_{yz} are the normal and shearing stresses caused by the buckling deformations.

Using Hook's law, the principle of virtual work is rewritten as follows:

$$\delta\pi=0 \quad \cdots\cdots(7)$$

where π is the total potential energy of the thin-walled section columns due to buckling and expressed by

$$\begin{aligned}
\pi=&\frac{E}{2}\int_0^l\left[I_x\left(\frac{d^2u}{dz^2}\right)^2+I_y\left(\frac{d^2v}{dz^2}\right)^2+I_\omega\left(\frac{d^2\theta}{dz^2}\right)^2\right]dz\\
&+\frac{G}{2}\int_0^l J\left(\frac{d\theta}{dz}\right)^2 dz\\
&+\frac{1}{2}\int_0^l\left[P_z^{(0)}\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2+\gamma^2\left(\frac{d\theta}{dz}\right)^2\right\}\right.\\
&\qquad\left.+\beta_\omega M_\omega^{(0)}\left(\frac{d\theta}{dz}\right)^2\right]dz\\
&+\int_0^l\left[P_z^{(0)}\left(y_s\frac{du}{dz}\frac{d\theta}{dz}-x_s\frac{dv}{dz}\frac{d\theta}{dz}\right)\right.\\
&\qquad\left.+\left(\int_A\tau_{xz}^{(0)}dA\right)\frac{dv}{dz}\theta-\left(\int_A\tau_{yz}^{(0)}dA\right)\frac{du}{dz}\theta\right]dz\\
&-\int_0^l\left[\left\{\int_A\left(\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right)x\,dA\right\}\frac{d^2u}{dz^2}\frac{d\theta}{dz}\right.\\
&\qquad+\left(\int_A\tau_{xz}^{(0)}\omega\,dA\right)\frac{du}{dz}\frac{d^2\theta}{dz^2}\\
&\qquad+\left\{\int_A\left(\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right)y\,dA\right\}\frac{d^2v}{dz^2}\frac{d\theta}{dz}\\
&\qquad+\left(\int_A\tau_{yz}^{(0)}\omega\,dA\right)\frac{dv}{dz}\frac{d^2\theta}{dz^2}\\
&\qquad-\left(\int_A\tau_{xz}^{(0)}y\,dA\right)\frac{du}{dz}\frac{d^2v}{dz^2}\\
&\qquad\left.-\left(\int_A\tau_{yz}^{(0)}x\,dA\right)\frac{dv}{dz}\frac{d^2u}{dz^2}\right]dz\\
&-\frac{1}{2}\int_0^l\left[\int_A\frac{\partial}{\partial z}\{\tau_{xz}^{(0)}(x-x_s)+\tau_{yz}^{(0)}(y-y_s)\}\,dA\cdot\theta^2\right.\\
&\qquad+\int_A\frac{\partial}{\partial z}\left(\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right)\omega\,dA\cdot\left(\frac{d\theta}{dz}\right)^2\\
&\qquad+\int_A\frac{\partial}{\partial z}(\tau_{xz}^{(0)}x)\,dA\cdot\left(\frac{du}{dz}\right)^2\\
&\qquad\left.+\int_A\frac{\partial}{\partial z}(\tau_{yz}^{(0)}y)\,dA\left(\frac{dv}{dz}\right)^2\right]dz\\
&+\frac{1}{2}\left|\int_A\{\tau_{xz}^{(0)}(x-x_s)+\tau_{yz}^{(0)}(y-y_s)\}\,dA\cdot\theta^2\right.\\
&\qquad+\int_A\left(\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right)\omega\,dA\cdot\left(\frac{d\theta}{dz}\right)^2\\
&\qquad+\int_A(\tau_{xz}^{(0)}x)\,dA\left(\frac{du}{dz}\right)^2\\
&\qquad\left.+\int_A(\tau_{yz}^{(0)}y)\,dA\left(\frac{dv}{dz}\right)^2\right|_0^l \quad\cdots\cdots(8)
\end{aligned}$$

in which $|\;|_0^l$ indicates values at the $z=0$ and $z=l$ sections, and I_x, I_y, J, γ^2 and β_ω are the geometrical characteristics of the cross section defined by

$$\left.\begin{aligned}
I_x&=\int_A x^2\,dA\,,\qquad I_y=\int_A y^2\,dA\\
J&=\int_A\left[\left(x-x_s+\frac{\partial\omega}{\partial y}\right)^2+\left(y-y_s-\frac{\partial\omega}{\partial x}\right)^2\right]dA\\
\gamma^2&=\frac{1}{A}(I_x+I_y)+x_s^2+y_s^2\\
\beta_\omega&=\frac{1}{I_\omega}\int_A\omega(x^2+y^2)\,dA\\
I_\omega&=\int_A\omega^2\,dA
\end{aligned}\right\} \quad\cdots\cdots(9)$$

Terms relating to the initial shearing stresses in Eq. (8) can be expressed in terms of the stress resultants[6].

$$\int_A\tau_{xz}^{(0)}\,dA=0\,,\qquad\int_A\tau_{yz}^{(0)}\,dA=0 \quad\cdots\cdots(10)$$

$$\begin{aligned}
\int_A\tau_{xz}^{(0)}y\,dA&=\int_A\left[\frac{\partial\Phi_0}{\partial y}-\frac{1}{I_\omega t}\left(\int_0^s\omega t\,ds\right)\frac{dM_\omega^{(0)}}{dz}\cos\alpha\right]dA\\
&=-\int_A\Phi_0\,dA-\frac{1}{I_\omega}R_{yc}\frac{dM_\omega^{(0)}}{dz}\\
&=-\frac{1}{2}M_s^{(0)}-\frac{1}{I_\omega}R_{yc}\frac{dM_\omega^{(0)}}{dz} \quad\cdots\cdots(11)
\end{aligned}$$

where R_{yc} is a new geometrical characteristic of the cross section defined by

$$R_{pq}=\int_A\frac{1}{t}\left(\int_0^s\omega t\,ds\right)pq\,dA \quad\cdots\cdots(12)$$

The subscript c of the constant, R_{yc}, denotes $\cos\alpha$.

$$\int_A\tau_{xz}^{(0)}\omega\,dA=\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{\omega c}\frac{dM_\omega^{(0)}}{dz} \quad\cdots\cdots(13)$$

in which K_{hs} is a new geometrical constant of the cross section defined by

$$K_{pq}=\int_A\left(\frac{t^2}{4}-n^2\right)pq\,dA \quad\cdots\cdots(14)$$

The subscript s of K_{hs} represents $\sin\alpha$, and the subscript h denotes the length of the perpendicular from the shear center to the tangent of the profile line at M.

$$h=(x-x_s)\sin\alpha-(y-y_s)\cos\alpha$$

Using the same procedure, the other terms of integration of the initial shearing stresses are transformed into the following formulas:

$$\left.\begin{aligned}\int_A \tau_{yz}^{(0)} x\,dA &= \frac{1}{2}M_s^{(0)} - \frac{1}{I_\omega}R_{xs}\frac{dM_\omega^{(0)}}{dz}\\ \int_A \tau_{yz}^{(0)}\omega\,dA &= -\frac{1}{J}K_{hc}M_s^{(0)} - \frac{1}{I_\omega}R_{\omega s}\frac{dM_\omega^{(0)}}{dz}\end{aligned}\right\} \quad\cdots\cdots(15)$$

$$\left.\begin{aligned}&\int_A\left(\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right)x\,dA\\ &\quad= -\frac{1}{J}K_{hs}M_s^{(0)} + \frac{1}{I_\omega}R_{xh}\frac{dM_\omega^{(0)}}{dz}\\ &\int_A\left(\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right)y\,dA\\ &\quad= \frac{1}{J}K_{hc}M_s^{(0)} + \frac{1}{I_\omega}R_{yh}\frac{dM_\omega^{(0)}}{dz}\end{aligned}\right\}\quad\cdots\cdots(16)$$

Since no lateral loads are acting on the columns, the following terms become zero.

$$\left.\begin{aligned}&\int_A \frac{\partial}{\partial z}\{\tau_{xz}^{(0)}(x-x_s)+\tau_{yz}^{(0)}(y-y_s)\}\,dA=0\\ &\int_A \frac{\partial}{\partial z}\left\{\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right\}\omega\,dA=0\\ &\int_A \frac{\partial}{\partial z}(\tau_{xz}^{(0)}x)\,dA=0,\quad \int_A\frac{\partial}{\partial z}(\tau_{yz}^{(0)}y)\,dA=0\\ &\left|\int_A\left(\tau_{xz}^{(0)}\frac{\partial\omega}{\partial x}+\tau_{yz}^{(0)}\frac{\partial\omega}{\partial y}\right)\omega\,dA\right|_0^l=0\\ &\left|\int_A\{\tau_{xz}^{(0)}(x-x_s)+\tau_{yz}^{(0)}(y-y_s)\}\,dA\right|_0^l=0\\ &\left|\int_A\tau_{xz}^{(0)}x\,dA\right|_0^l=0\\ &\left|\int_A\tau_{yz}^{(0)}y\,dA\right|_0^l=0\end{aligned}\right\}\quad\cdots\cdots(17)$$

Using Eqs. (10) through (17), the total potential energy can be finally expressed as

$$\begin{aligned}\pi=&\frac{E}{2}\int_0^l\left[I_x\left(\frac{d^2u}{dz^2}\right)^2+I_y\left(\frac{d^2v}{dz^2}\right)^2+I_\omega\left(\frac{d^2\theta}{dz^2}\right)^2\right]dz\\ &+\frac{G}{2}\int_0^l J\left(\frac{d\theta}{dz}\right)^2 dz\\ &+\frac{1}{2}\int_0^l\left[P_z^{(0)}\left\{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2+\gamma^2\left(\frac{d\theta}{dz}\right)^2\right\}\right.\\ &\left.+\beta_\omega M_\omega^{(0)}\left(\frac{d\theta}{dz}\right)^2\right]dz\\ &+\int_0^l\left[P_z^{(0)}\left(y_s\frac{du}{dz}-x_s\frac{dv}{dz}\frac{d\theta}{dz}\right)\right.\\ &+\left(\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{xh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d^2u}{dz^2}\frac{d\theta}{dz}\\ &-\left(\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{\omega c}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}\frac{d^2\theta}{dz^2}\\ &-\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{yh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d^2v}{dz^2}\frac{d\theta}{dz}\\ &+\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{\omega s}\frac{dM_\omega^{(0)}}{dz}\right)\frac{dv}{dz}\frac{d^2\theta}{dz^2}\\ &-\left(\frac{1}{2}M_s^{(0)}+\frac{1}{I_\omega}R_{yc}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}\frac{d^2v}{dz^2}\\ &\left.+\left(\frac{1}{2}M_s^{(0)}-\frac{1}{I_\omega}R_{xs}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d^2u}{dz^2}\frac{dv}{dz}\right]dz\end{aligned}\quad\cdots\cdots(18)$$

5. DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The differential equations and the corresponding mechanical boundary conditions governing buckling due to the axial loads and the torsions are obtained from the total potential energy in accordance with the principle of minimum potential energy.

Substituting Eq. (18) into Eq. (7) and integrating by parts, the following stability equations are derived:

$$\left.\begin{aligned}&EI_x\frac{d^4u}{dz^4}-P_z^{(0)}\left(\frac{d^2u}{dz^2}+y_s\frac{d^2\theta}{dz^2}\right)+M_z^{(0)}\frac{d^3v}{dz^3}\\ &\quad+\frac{d}{dz}\left[\left\{\frac{2}{J}K_{hs}M_s^{(0)}\right.\right.\\ &\quad\left.\left.-\frac{1}{I_\omega}(R_{xh}+R_{\omega c})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{d^2\theta}{dz^2}\right]\\ &\quad+\frac{d}{dz}\left[\frac{d}{dz}\left(\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{xh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d\theta}{dz}\right.\\ &\quad\left.+\frac{d}{dz}\left(\frac{1}{2}M_s^{(0)}+\frac{1}{I_\omega}R_{xs}\frac{dM_\omega^{(0)}}{dz}\right)\frac{dv}{dz}\right]=0\\ &EI_y\frac{d^4v}{dz^4}-P_z^{(0)}\left(\frac{d^2v}{dz^2}-x_s\frac{d^2\theta}{dz^2}\right)-M_z^{(0)}\frac{d^3u}{dz^3}\\ &\quad-\frac{d}{dz}\left[\left\{\frac{2}{J}K_{hc}M_s^{(0)}\right.\right.\\ &\quad\left.\left.+\frac{1}{I_\omega}(R_{yh}+R_{\omega s})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{d^2\theta}{dz^2}\right]\\ &\quad-\frac{d}{dz}\left[\frac{d}{dz}\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{yh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d\theta}{dz}\right.\\ &\quad\left.+\frac{d}{dz}\left(\frac{1}{2}M_s^{(0)}+\frac{1}{I_\omega}R_{yc}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}\right]=0\\ &EI_\omega\frac{d^4\theta}{dz^4}-GJ\frac{d^2\theta}{dz^2}\\ &\quad+P_z^{(0)}\left(x_s\frac{d^2v}{dz^2}-y_s\frac{d^2u}{dz^2}-\gamma^2\frac{d^2\theta}{dz^2}\right)\\ &\quad-\frac{d^2}{dz^2}\left[\left\{\frac{2}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}(R_{xh}+R_{\omega c})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{du}{dz}\right]\\ &\quad+\frac{d^2}{dz^2}\left[\left\{\frac{2}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}(R_{yh}+R_{\omega s})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{dv}{dz}\right]\\ &\quad-\frac{d}{dz}\left(\beta_\omega M_\omega^{(0)}\frac{d\theta}{dz}\right)\end{aligned}\right.$$

$$+\frac{d}{dz}\left[\frac{d}{dz}\left(\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{xh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}-\frac{d}{dz}\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{yh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{dv}{dz}\right]=0 \qquad \cdots(19)$$

where $M_z^{(0)}$ is the total torsional moment.

When the column is built in at the $z=0$ section and subjected to the axial load and the torsion at the other end, the mechanical boundary conditions at the loading section are

$$\left.\begin{aligned}
&EI_x\frac{d^2u}{dz^2}+\left(\frac{1}{2}M_s^{(0)}-\frac{1}{I_\omega}R_{xs}\frac{dM_\omega^{(0)}}{dz}\right)\frac{dv}{dz}+\left(\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{xh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d\theta}{dz}=0\\
&-EI_x\frac{d^3u}{dz^3}+P_z^{(0)}\left(\frac{du}{dz}+y_s\frac{d\theta}{dz}\right)-M_z^{(0)}\frac{d^2v}{dz^2}-\left\{\frac{2}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}(R_{xh}+R_{\omega c})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{d^2\theta}{dz^2}\\
&\quad-\frac{d}{dz}\left(\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{xh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d\theta}{dz}-\frac{d}{dz}\left(\frac{1}{2}M_s^{(0)}+\frac{1}{I_\omega}R_{yc}\frac{dM_\omega^{(0)}}{dz}\right)\frac{dv}{dz}=0\\
&EI_y\frac{d^2v}{dz^2}-\left(\frac{1}{2}M_s^{(0)}+\frac{1}{I_\omega}R_{yc}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}-\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{yh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d\theta}{dz}=0\\
&-EI_y\frac{d^3v}{dz^3}+P_z^{(0)}\left(\frac{dv}{dz}-x_s\frac{d\theta}{dz}\right)+M_z^{(0)}\frac{d^2u}{dz^2}+\left\{\frac{2}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}(R_{yh}+R_{\omega s})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{d^2\theta}{dz^2}\\
&\quad+\frac{d}{dz}\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{yh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{d\theta}{dz}+\frac{d}{dz}\left(\frac{1}{2}M_s^{(0)}-\frac{1}{I_\omega}R_{xs}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}=0\\
&EI_\omega\frac{d^2\theta}{dz^2}-\left(\frac{1}{J}K_{hs}M_\omega^{(0)}-\frac{1}{I_\omega}R_{\omega c}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}+\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{\omega s}\frac{dM_\omega^{(0)}}{dz}\right)\frac{dv}{dz}=0\\
&-EI_\omega\frac{d^3\theta}{dz^3}+GJ\frac{d\theta}{dz}+P_z^{(0)}\left(y_s\frac{du}{dz}-x_s\frac{dv}{dz}+\gamma^2\frac{d\theta}{dz}\right)\\
&\quad+\frac{d}{dz}\left[\left\{\frac{2}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}(R_{xh}+R_{\omega c})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{du}{dz}\right]-\frac{d}{dz}\left[\left\{\frac{2}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}(R_{yh}+R_{\omega s})\frac{dM_\omega^{(0)}}{dz}\right\}\frac{dv}{dz}\right]\\
&\quad-\frac{d}{dz}\left(\frac{1}{J}K_{hs}M_s^{(0)}-\frac{1}{I_\omega}R_{xh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{du}{dz}+\frac{d}{dz}\left(\frac{1}{J}K_{hc}M_s^{(0)}+\frac{1}{I_\omega}R_{yh}\frac{dM_\omega^{(0)}}{dz}\right)\frac{dv}{dz}+M_\omega^{(0)}\beta_\omega\frac{d\theta}{dz}=0
\end{aligned}\right\} \qquad \cdots(20)$$

Eqs. (19) and (20) are the general differential equations governing buckling due to the axial loads and the conservative torsions and the corresponding mechanical boundary conditions.

When the column has a thin-walled double symmetrical cross section, the geometrical characteristics relating to the warping function vanish except for the warping constant, I_ω.

Then,

$$\left.\begin{aligned}&K_{hs}=K_{hc}=R_{xs}=R_{xh}=R_{yc}=R_{yh}=R_{\omega s}=R_{\omega c}=0\\&x_s=y_s=0\end{aligned}\right\} \qquad \cdots(21)$$

The differential equations and the mechanical boundary conditions in this case become as

$$\left.\begin{aligned}
&EI_x\frac{d^4u}{dz^4}-P_z^{(0)}\frac{d^2u}{dz^2}+M_z^{(0)}\frac{d^3v}{dz^3}=0\\
&EI_y\frac{d^4v}{dz^4}-P_z^{(0)}\frac{d^2v}{dz^2}-M_z^{(0)}\frac{d^3u}{dz^3}=0\\
&EI_\omega\frac{d^4\theta}{dz^4}-GJ\frac{d^2\theta}{dz^2}-P_z^{(0)}\gamma^2\frac{d^2\theta}{dz^2}=0
\end{aligned}\right\} \cdots(22)$$

$$\left.\begin{aligned}
&EI_x\frac{d^2u}{dz^2}+\frac{1}{2}M_z^{(0)}\frac{dv}{dz}=0\\
&-EI_x\frac{d^3u}{dz^3}+P_z^{(0)}\frac{du}{dz}-M_z^{(0)}\frac{d^2v}{dz^2}=0\\
&EI_y\frac{d^2v}{dz^2}-\frac{1}{2}M_z^{(0)}\frac{du}{dz}=0\\
&-EI_y\frac{d^3v}{dz^3}+P_z^{(0)}\frac{dv}{dz}+M_z^{(0)}\frac{d^2u}{dz^2}=0\\
&EI_\omega\frac{d^2\theta}{dz^2}=0\\
&-EI_\omega\frac{d^3\theta}{dz^3}+GJ\frac{d\theta}{dz}+P_z^{(0)}\gamma^2\frac{d\theta}{dz}=0
\end{aligned}\right\} \cdots(23)$$

The first and second equations of Eq. (22) are the generalized Greenhill's equations. When the two flexural rigidities are identical, they are reduced to the Greenhill's equations. The third equation of Eq. (22) expresses the torsional buckling of the columns under the axial loads.

The bending moment equilibriums at the ends of the column are shown by the first and third equations of Eq. (23). They coincide with the semi-tangential moments classified by Ziegler, who has derived them by mechanical consideration.

6. BUCKLING OF A COLUMN WITH A CHANNEL CROSS SECTION

In this section, the buckling of column with simply supported ends about the x, y and z axes is considered. The warping moment $M_\omega^{(0)}$, therefore, vanishes and the torsional moments at the arbitrary cross section of the column are equal to the applied torsion $M_z^{(0)}$.

(1) Geometrical Constants of the Cross Section

The geometrical constants of the thin-walled open cross section can be obtained by the simplified method[8].

For instance, the warping constant, I_ω, can be computed by the following way.

Let the cross section consist of n thin plates, and the distribution of the unit warping be known as shown in Fig. 2.

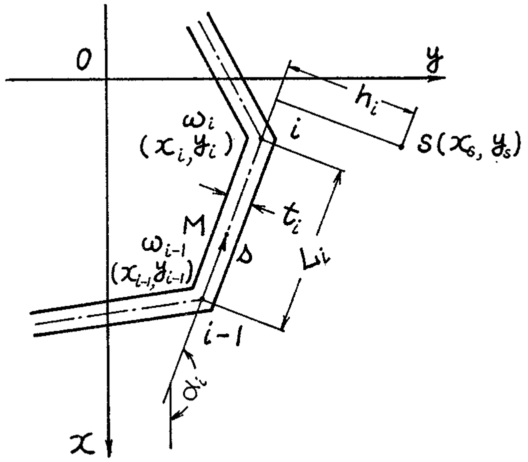

Fig. 2 Distribution of the Unit Warping.

The unit warping at the point M can be given by

$$\omega=\omega_{i-1}-h_i s \quad \cdots\cdots(24)$$

where ω and ω_{i-1} are values of the unit warping at the point M and the $(i-1)$th corner of the cross section. The quantity h_i is the length of the perpendicular from the shear center to the i-th plate element, and s is the distance measured from the $(i-1)$th corner to M along the profile line of the cross section.

The warping constant defined by Eq. (9) is expressed as

$$I_\omega=\int_A \omega^2\,dA=\sum_{i=1}^{n}\int_0^{L_i}(\omega_{i-1}-h_i s)^2 t_i\,ds$$

$$=\frac{1}{3}\sum_{i=1}^{n} t_i L_i(\omega_i^2+\omega_i\omega_{i-1}+\omega_{i-1}^2) \quad \cdots\cdots(25)$$

where t_i and L_i are the wall thickness and the length of the i-th plate element.

The new geometrical constants K_{hs} and K_{hc} are computed by the same procedure.

$$\left.\begin{aligned}
K_{hs}&=\int_A\left(\frac{t^2}{4}-n^2\right)h\sin\alpha\,dA\\
&=\sum_{i=1}^{n}\left[\int_{-t_i/2}^{t_i/2}\left(\frac{t^2}{4}-n^2\right)dn\right]\int_0^{L_i}h_i\sin\alpha_i\,ds\\
&=\frac{1}{6}\sum_{i=1}^{n}t_i^3 h_i(y_i-y_{i-1})\\
K_{hc}&=\frac{1}{6}\sum_{i=1}^{n}t_i^3 h_i(x_i-x_{i-1})
\end{aligned}\right\} \quad \cdots\cdots(26)$$

in which x_i, x_{i-1}, y_i and y_{i-1} are the coordinates of the i-th and $(i-1)$th corners, and α_i is the angle that the i-th plate element makes with the x axis.

The other geometrical characteristics are computed by the usual manner.

(2) Buckling of the Column of the Channel Section

To obtaine the critical torsions under the axial loads, the Galerkin method can be applied effectively.

The displacements u, v and θ are assumed by the series of the trigonometric functions as follows:

$$\left.\begin{aligned}
\frac{u}{a}&=\sum_{p=1}^{n_1}a_p\sin\frac{p\pi z}{l}=\sum_{p=1}^{n_1}a_p u_p\\
\frac{v}{a}&=\sum_{q=1}^{n_2}b_q\sin\frac{q\pi z}{l}=\sum_{q=1}^{n_2}b_q v_q\\
\theta&=\sum_{r=1}^{n_3}c_r\sin\frac{r\pi z}{l}=\sum_{r=1}^{n_3}c_r\theta_r
\end{aligned}\right\} \quad \cdots\cdots(27)$$

Using Eq. (27) and applying the Galerkin method to Eqs. (19) and (20), the characteristic equation of this buckling problem can be obtained as

$$\begin{vmatrix} A_{pP} & A_{qP} & A_{rP}\\ A_{pQ} & A_{qQ} & A_{rQ}\\ A_{pR} & A_{qR} & A_{rR}\end{vmatrix}=0, \quad \begin{pmatrix} p,\,P=1,\cdots,n_1\\ q,\,Q=1,\cdots,n_2\\ r,\,R=1,\cdots,n_3\end{pmatrix} \quad \cdots\cdots(28)$$

where A_{ij} are the elements of the matrix, Eq. (28), and given by the following formulas:

$$
\left.
\begin{aligned}
A_{pP}&=\bar{I}_x\int_0^1\frac{d^4u_p}{dL^4}u_P\,dL-\pi^2\bar{P}_z\int_0^1\frac{d^2u_p}{dL^2}u_P\,dL\\
&\quad+\left|\bar{I}_x\frac{d^2u_p}{dL^2}\frac{du_P}{dL}\right|_0^1\\
A_{qP}&=\lambda\pi^2\left[\int_0^1\frac{d^3v_q}{dL^3}u_P\,dL+\frac{1}{2}\left|\frac{dv_q}{dL}\frac{du_P}{dL}\right|_0^1\right]\\
A_{rP}&=-\pi^2\bar{y}_s\bar{P}_z\int_0^1\frac{d^2\theta_r}{dL^2}u_P\,dL\\
&\quad+\lambda\frac{\pi^2}{\bar{J}}\bar{K}_{hs}\left[2\int_0^1\frac{d^3\theta_r}{dL^3}u_P\,dL+\left|\frac{d\theta_r}{dL}\frac{du_P}{dL}\right|_0^1\right]\\
A_{pQ}&=-\lambda\pi^2\left[\int_0^1\frac{d^3u_p}{dL^3}v_Q\,dL+\frac{1}{2}\left|\frac{du_p}{dL}\frac{dv_Q}{dL}\right|_0^1\right]\\
A_{qQ}&=\bar{I}_y\int_0^1\frac{d^4v_q}{dL^4}v_Q\,dL-\pi^2\bar{P}_z\int_0^1\frac{d^2v_q}{dL^2}v_Q\,dL\\
&\quad+\left|\bar{I}_y\frac{d^2v_q}{dL^2}\frac{dv_Q}{dL}\right|_0^1\\
A_{rQ}&=\pi^2\bar{x}_s\bar{P}_z\int_0^1\frac{d^2\theta_r}{dL^2}v_Q\,dL\\
&\quad-\lambda\frac{\pi^2}{\bar{J}}\bar{K}_{hc}\left[2\int_0^1\frac{d^3\theta_r}{dL^3}v_Q\,dL+\left|\frac{d\theta_r}{dL}\frac{dv_Q}{dL}\right|_0^1\right]\\
A_{pR}&=-\pi^2\bar{y}_s\bar{P}_z\int_0^1\frac{d^2u_p}{dL^2}\theta_R\,dL\\
&\quad-\lambda\frac{\pi^2}{\bar{J}}\bar{K}_{hs}\left[2\int_0^1\frac{d^3u_p}{dL^3}\theta_R\,dL+\left|\frac{du_p}{dL}\frac{d\theta_R}{dL}\right|_0^1\right]\\
A_{qR}&=\pi^2\bar{x}_s\bar{P}_z\int_0^1\frac{d^2v_q}{dL^2}\theta_R\,dL\\
&\quad+\lambda\frac{\pi^2}{\bar{J}}\bar{K}_{hc}\left[2\int_0^1\frac{d^3v_q}{dL^3}\theta_R\,dL+\left|\frac{dv_q}{dL}\frac{d\theta_R}{dL}\right|_0^1\right]\\
A_{rR}&=\bar{I}_\omega\int_0^1\frac{d^4\theta_r}{dL^4}\theta_R\,dL-\bar{G}\bar{J}\frac{1}{\bar{a}^2}\int_0^1\frac{d^2\theta_r}{dL^2}\theta_R\,dL\\
&\quad-\pi^2\bar{\gamma}^2\bar{P}_z\int_0^1\frac{d^2\theta_r}{dL^2}\theta_R\,dL+\left|\bar{I}_\omega\frac{d^2\theta_r}{dL^2}\frac{d\theta_R}{dL}\right|_0^1
\end{aligned}
\right\}
\qquad\cdots\cdots(29)
$$

In Eq. (29), the following nondimensional constants are used

$$
\left.
\begin{aligned}
&\bar{I}_x=\frac{I_x}{I_y}, \quad \bar{I}_y=1, \quad \bar{J}=\frac{J}{I_y}, \quad \bar{I}_\omega=\frac{I_\omega}{I_ya^2}\\
&\bar{K}_{hs}=\frac{K_{hs}}{I_ya}, \quad \bar{K}_{hc}=\frac{K_{hc}}{I_ya}, \quad \bar{\gamma}^2=\frac{\gamma^2}{a^2}, \quad L=\frac{z}{l}\\
&\bar{x}_s=\frac{x_s}{a}, \quad \bar{y}_s=\frac{y_s}{a}, \quad \bar{a}=\frac{a}{l}, \quad \bar{G}=\frac{G}{E}
\end{aligned}
\right\}
\qquad\cdots\cdots(30)
$$

where a is a dimension representing the cross section.

$\bar{P}_z$ in Eq. (29) is the nondimensionalized axial force with respect to the critical thrust of pin-end columns, $P_{cr}=\pi^2EI_y/l^2$, defined by

$$\bar{P}_z=\frac{P_z^{(0)}}{P_{cr}} \qquad\cdots\cdots(31)$$

The characteristic value of buckling used in the present study is

$$\lambda=\frac{M_z^{(0)}}{P_{cr}l} \qquad\cdots\cdots(32)$$

The channel cross section under consideration is shown in Fig. 3, of which relative dimensions are chosen in analysis as follows:

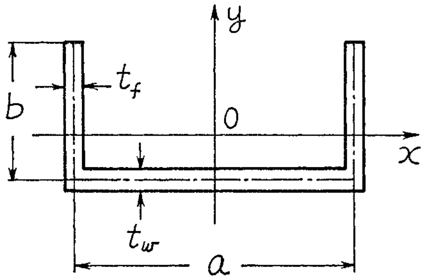

Fig. 3 Channel Section.

$$\frac{b}{a}=0.5, \quad \frac{t_f}{a}=\frac{t_w}{a}=0.1, \quad \frac{a}{l}=0.1$$

The values of the geometrical constants for this case are

$$\bar{I}_x=6.40, \quad \bar{I}_y=1.0, \quad \bar{J}=0.128$$
$$\bar{I}_\omega=0.175, \quad \bar{K}_{hs}=0, \quad \bar{K}_{hc}=-0.006$$
$$\bar{\gamma}^2=0.289, \quad \bar{x}_s=0, \quad \bar{y}_s=-0.313$$

The buckling characteristics computed from Eq. (28) are shown with the axial force coefficients in Fig. 4, the $\lambda\sim\bar{P}_z$ curve. The solid

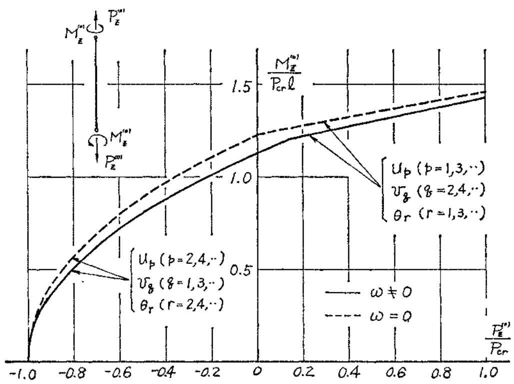

Fig. 4 Characteristic Values.

line in the figure shows results of the present study, which include the effects of warping and torsional deformations on the buckling of the columns. On the other hands, the critical characteristics computed by the use of the generalized

Greenhill equations are shown by the broken line in the same figure. The reduction of the critical characteristics due to the warping and torsional deformations is large, especially under the compressive axial loads.

The results, also, show that the columns buckle in two different shapes dependent on the axial forces. Under the axial tension larger than $\bar{P}_z=0.14$, the columns buckle in the odd modes of u and θ and the even mode of v. The buckling shapes of the columns are the even modes of u and θ and the odd mode of v under the smaller axial tensions than $\bar{P}_z=0.14$ and the axial thrusts. The axial loads, under which the buckling shapes change, differ from the results obtained by using the generalized Greenhill equations, as can be seen in Fig. 4.

7. CONCLUSIONS

The conclusions reached from the present study are as follows:

(1) The differential equations and the corresponding mechanical boundary conditions governing buckling of the thin-walled open section columns under the axial forces and the conservative torsions are derived in accordance with the variational principle.

(2) All nonlinear terms of the shearing strains should be used to obtained the stability equations.

(3) The buckling under the axial forces and torsions occurs usually a combination of the bending in the x and y directions and torsional deformations around the z axis. The columns with the double symmetric cross sections, as an exception, buckle in the bending modes and the torsional mode independently.

(4) The effects of the warping and torsional deformations on the characteristics of buckling can not be ignored, especially under the compressive axial forces.

(5) The boundary conditions for the bending moments coincide with the semi-tangential moments classified by Ziegler for the columns having two axes of symmetry.

The equations obtained can be used for the buckling analysis of the thin-walled open section columns with arbitrary boundary conditions under the axial loads and the conservative torsions.

ACKNOWLEDGEMENTS

The author expresses his thanks to Professor Tadahiko Kawai at University of Tokyo for his suggestions and encouragements.

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(Received June 18, 1973)