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STRESSES AND DISPLACEMENTS AROUND OPENINGS UNDER LONGITUDINAL SHEAR

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1. INTRODUCTION

It is of importance to obtain the fundamental data for the stress measurement around openings in an elastic medium and to estimate the fracture strength for the brittle materials, subjected to longitudinal shear. Therefore, some investigators have been carried out to analyze the elastic stresses near cracks in an isotropic body under longitudinal shear, that is, Sih[1] carried out the study about this problem by the use of the complex variable method with the conformal transformation, Barenblatt and Cherepanov[2], Yokobori et al[3] attempted to apply the concept of continuous distribution of infinitesimal dislocations, and Sih[4] has given the solution of an infinite sequence of parallel elastic cracks in an isotropic body. Sih[5] has been also solved the problems of the stress distribution in an isotropic or an anisotropic body with lines of discontinuities by the reduction of the Hilbert problem following the method of complex variables. However, it seems that the cases where the spacing is narrow is important in practice. With respect to the interaction of two elastic colinear cracks, stress concentration and fracture strength under longitudinal shear were obtained by Yokobori et al[3] and Tamate and Yamada[6], respectively for the case $b \approx h_0$.

In the present paper, the following problems are treated: (1) stresses and displacements around two or more openings with arbitrary cross sections in an isotropic body under longitudinal shear applied at infinity, and (2) stresses and displacements around two or more openings with circular or elliptical cross sections in an anisotropic elastic body under longitudinal shear at infinity. The method of analysis was used the successive approximation to be obtained by the point matching technique based on the exact solution of an isotropic or an anisotropic elastic body containing an opening.

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2. BASIC EQUATIONS OF ISOTROPIC OR ANISOTROPIC ELASTICITY

In the longitudinal shear problems, the fields of stress and displacement in a body with isotropy or plane anisotropy under consideration are such that

$$u=v=0,\ w=w(x,\ y),\ \sigma_x=\sigma_y=\sigma_z=\tau_{xy}=0\ , \quad \cdots\cdots(2.1)$$

where x, y, z are the Cartesian coordinates, u, v are the components of displacement along the x-, y-axes, and w the displacement parallel to the z-axis and the generators of the cylinder.

Let us consider the theory of the longitudinal shear of cylindrical bodies possessing rectangular anisotropy. As was formulated by Eq. (2.1) such that the anisotropy is charachterized by the presence of one plane of elastic symmetry normal to the generators, which are parallel to the z-axis, the theory is very much simplified. In this case

$$a_{14}=a_{15}=a_{24}=a_{25}=a_{34}=a_{46}=a_{56}=0\ . \quad \cdots(2.2)$$

From these considerations, the relation between stress and strain is therefore

$$\left.\begin{aligned}\gamma_{yz}&=\frac{\partial w}{\partial y}=a_{44}\tau_{yz}+a_{45}\tau_{xz}\ ,\\ \gamma_{xz}&=\frac{\partial w}{\partial x}=a_{45}\tau_{yz}+a_{55}\tau_{xz}\ .\end{aligned}\right\} \quad \cdots\cdots(2.3)$$

Introducing the stress function $\phi(x,\ y)$ to satisfy the equilibrium condition, and from the relation for the displacement w, the following equation can be obtained.

$$a_{44}\frac{\partial^2\phi}{\partial x^2}-2a_{45}\frac{\partial^2\phi}{\partial x\partial y}+a_{55}\frac{\partial^2\phi}{\partial y^2}=0\ . \quad \cdots\cdots(2.4)$$

Now, Eq. (2.4) may be written as

$$D_1D_2\phi=0\ , \quad \cdots\cdots(2.5)$$

in which D_k are the differential operators:

$$D_k=\frac{\partial}{\partial y}-\mu_k\frac{\partial}{\partial x}\,,\quad (k=1,\,2) \qquad \cdots(2.6)$$

From Eqs. (2.3) and (2.4), μ_k are the roots of the characteristic equation.

$$a_{55}\mu^2-2a_{45}\mu+a_{44}=0\,. \qquad \cdots(2.7)$$

Introducing the complex variable

$$z_1=x+\mu_1 y\,, \qquad \cdots(2.8)$$

the general solution of Eq. (2.5) may be obtained as

$$\phi=2Re[\phi_1(z_1)]\,. \qquad \cdots(2.9)$$

Then, the components τ_{xz}, τ_{yz} of anti-plane shear stress and displacement w in a plane anisotropic elastic body are expressed in term of the complex analytic function $\phi_1(z_1)$ and its derivative $\phi_1'(z_1)$ as follows.

$$\left.\begin{aligned}\tau_{xz}&=2Re[\mu_1\phi_1'(z_1)]\,,\\ \tau_{yz}&=-2Re[\phi_1'(z_1)]\,,\end{aligned}\right\} \quad \cdots(2.10a)$$

$$w=2Re\left[\left(a_{45}-\frac{a_{44}}{\mu_1}\right)\phi_1(z_1)\right]\,. \qquad \cdots(2.10b)$$

In the case of an isotropic body, the complex root μ_1 reduces to equal unit imaginary number, and the complex variable z_1 becomes

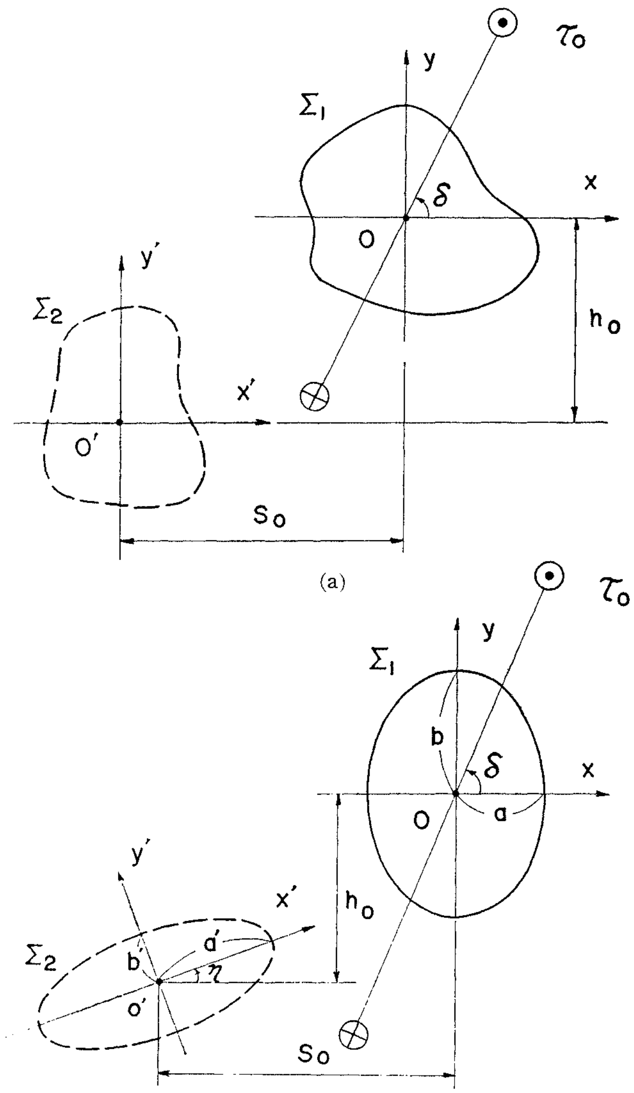
$$z_0=x+iy\,. \qquad \cdots(2.8)'$$

Then the stresses and displacements for the isotropic body can be expressed by the analytic function $\phi_0(z_0)$ and its derivative $\phi_0'(z_0)$

$$\left.\begin{aligned}\tau_{xz}&=-2I_m[\phi_0'(z_0)]\,,\\ \tau_{yz}&=-2Re[\phi_0'(z_0)]\,,\end{aligned}\right\} \quad \cdots(2.10a)'$$

$$w=-2I_m[a_{44}\phi_0(z_0)]\,. \qquad \cdots(2.10b)'$$

The complex stress functions $\phi_1(z_1)$ and $\phi_0(z_0)$ can be respectively determined as will be shown in the following section.



(b)

Fig. 1 Geometry of elastic body with cylindrical openings.

3. ANALYSES OF STRESSES AND DISPLACEMENTS IN ELASTIC BODY WITH ONE OPENING

(1) Boundary Condition

We refer the body under consideration to a rectangular Cartesian coordinate system $(x,\,y,\,z)$ where the origin lies in an elliptical hole of anisotropic body and in an arbitrary hole of isotropic body as shown in Figs. 1 (a) and (b) respectively. When we take the external stress along the contour of the hole as Z_n determined by the components τ_{xz}, τ_{yz} of shear stress, it can be represented as

$$Z_n=\tau_{xz}\cos(n,\,x_0)+\tau_{yz}\cos(n,\,y_0)\,, \qquad \cdots(3.1)$$

in which n is a unit vector directed to the inward normal to the contour of an opening, and x_0, y_0 are the components of coordinates for the contour of the opening. Using a tangential unit vector s for clockwise direction on the contour, we have

$$\cos(n,\,x_0)=-\frac{dy_0}{ds}\,,\quad \cos(n,\,y_0)=\frac{dx_0}{ds}\,. \qquad \cdots(3.2)$$

Thus by substituting from Eqs. (3.2), (2.10) and (2.10)′ into the right-hand sides of Eq. (3.1) and integrating with respect to the arc-length from an arbitrary initial point to a variable point s, the complex stress functions $\phi_1(z_1)$ and $\phi_0(z_0)$ appearing in Eqs. (2.10) and (2.10)′ respectively can be written as

$$2Re[\phi_1(z_1)]=-\int_0^s Z_n dS+C_1\ , \quad\cdots\cdots(3.3)$$

$$2Re[\phi_0(z_0)]=-\int_0^s Z_n dS+C_2\ . \quad\cdots\cdots(3.3)'$$

Where C_1, C_2 are constants which can be fixed arbitrary on the contour, since the region such as an infinite body containing a hole is to be considered; without loss of generality with respect to this case we can set these constants equal to zero.

Whereas, if the stress components equal and opposite signs to τ^0_{xz} and τ^0_{yz} applied at infinity act on the contour of the opening under consideration, then the boundary condition may be expressed as follow.

$$Z_n=-[\tau^0_{xz}\cos(n,\ x_0)+\tau^0_{yz}\cos(n,\ y_0)]\ . \quad\cdots\cdots(3.4)$$

In the case when the surface of the opening is not loaded and the stresses τ^0_{xz} and τ^0_{yz} are applied at infinity, we can solve the problem under consideration by superimposing the stress components τ_{xz}, τ_{yz} and displacement w by Eq. (2.10) or (2.10)′ to the stresses τ^0_{xz}, τ^0_{yz} and displacement w^0.

In the followings, explanations will conveniently be devided into two cases of isotropic body and anisotropic body.

(2) Case of Isotropic Body Containing an Opening With Arbitrary Cross Section

Let us consider an infinite isotropic body to a Cartesian coordinate system $(x,\ y,\ z)$ where the origin lies in arbitrary cross section of an opening and the z-axis coincides with the axial direction of the opening (see Fig. 1 (a)). And we consider the body with the opening, the contour of which is given by the following expression.

$$\left.\begin{aligned} x_0&=\alpha_0\cos\theta+\sum_{m=1}^{\nu}(\alpha_m\cos m\theta+\beta_m\sin m\theta)\ ,\\ y_0&=\alpha_0\sin\theta-\sum_{m=1}^{\nu}(\alpha_m\sin m\theta-\beta_m\cos m\theta)\ . \end{aligned}\right\} \quad\cdots\cdots(3.5)$$

In which θ is a parameter varing from 0 to 2π in a counterclockwise direction on the contour, and α_m, β_m represent the real constants to be decided by the cross section of an opening, and ν is a finite integer with plus sign. For example, it may be set $\alpha_m=\beta_m=0$ $(m=1,\ 2,\ \cdots,\ \nu)$ for a circular hole and $\beta_1=\alpha_m=\beta_m=0$ $(m=2,3,\ \cdots,\ \nu)$ for an elliptical one. Several investigators have calculated the values of the constants α_m, β_m in company with the variations of the rounded corner of an opening with several cross sections.

A complex plane $z_0(=x+iy)$ with the opening as given by Eq. (3.5) is conformally transformed onto the exterior of a unit circle $|\zeta|=1$ in the ζ-plane. The mapping function is expressed as

$$z_0\equiv w(\zeta)=\alpha_0\zeta+\sum_{m=1}^{\nu}(\alpha_m+i\beta_m)\zeta^{-m} \quad\cdots\cdots(3.6)$$

Substituting from Eq. (3.5) into Eq. (3.2) and then calculating Eq. (3.1), the integration of the right-hand side of Eq. (3.3)′ can easily be obtained. Therefore, if we assume that the complex analytic function $\phi_0(z_0)$ is given by

$$\phi_0(z_0)=\sum_{m=1}^{\infty}\bar{c}_m\zeta^{-m}\ , \quad\cdots\cdots(3.7)$$

complex coefficients $\bar{c}_m$ in above Eq. (3.7) may be represented by external stresses $^0_{xz}$, τ^0_{yz} applied at infinity and constants α_m, β_m defining the shape of an opening as:

$$\left.\begin{aligned} \bar{c}_1&=\frac{1}{2}\{\alpha_0(\tau^0_{yz}-i\tau^0_{xz})+\alpha_1(\tau^0_{yz}+i\tau^0_{xz})\\ &\qquad -\beta_1(\tau^0_{xz}-i\tau^0_{yz})\}\ ,\\ \bar{c}_m&=\frac{1}{2}\{\alpha_m(\tau^0_{yz}+i\tau^0_{xz})-\beta_m(\tau^0_{xz}-i\tau^0_{yz})\}\ ,\\ &\qquad (2\leqq m\leqq\nu)\\ \bar{c}_m&=0\ ,\qquad (\nu+1\leqq m) \end{aligned}\right\} \quad\cdots\cdots(3.8)$$

(3) Case of Anisotropic Body Containing an Opening with Elliptical Cross Section

In the case of anisotropic body dissimilar to the manner for isotropic body, we must perform the transformation (i.e. Eq. (2.8)) by using the complex root μ_1 of the characteristic equation (2.7) to obtain the complex analytic function $\phi_1(z_1)$. In such a case for $m\geqq 2$ of Eq. (3.5), however, z_1 is not a suitable function, because branch points of z_1-ζ_1 transformation occur when $|\zeta_1|>1$, and any function $\phi_1(z_1)$ of z_1 will not, in general, be single-valued throughout the region under consideration. The present paper will therefore treat only the problem of an anisotropic body having a circular or an elliptical hole (corresponding to $m=1$ in Eq. (3.5)) at which the complex function $\phi_1(z_1)$ is single-valued throughout the region, because z_1 has no zero on and outside of the unit circle $|\zeta|=1$.

Consequently the shape of an opening may be given as

$$x_0 = a\cos\theta,\ y_0 = b\sin\theta\ . \quad\cdots\cdots\cdots\cdots\cdots\cdots(3.9)$$

In which a, b are the semi-axes of an elliptical opening as shown in Fig. 1 (b), θ is the angle measured in a counter-clockwise direction from the x-axis. This is equivalent to set in Eq. (3.5) as

$$\nu = 1,\ \alpha_0 + \alpha_1 = a,\ \alpha_0 - \alpha_1 = b,\ \beta_1 = 0\ . \quad\cdots(3.10)$$

Therefore, a physical plane z_0 with the hole given by Eq. (3.9) may be conformally transformed onto the exterior of a unit circle $|\zeta| = 1$ in the ζ-plane as follow.

$$z_0 = \omega(\zeta) = \frac{1}{2}\{(a+b)\zeta + (a-b)\zeta^{-1}\}\ . \quad\cdots\cdots(3.6)'$$

And, we consider the solution of our problem by mapping the plane $z_1 (= x + \mu_1 y)$, as shown in Eq. (2.8), onto the exterior of a unit circle $|\zeta_1| = 1$ in the ζ_1-plane. Then the mapping function may be expressed as:

$$z_1 = \omega_1(\zeta_1) = \frac{1}{2}\{(a - i\mu_1 b)\zeta_1 + (a + i\mu_1 b)\zeta_1^{-1}\}\ . \quad\cdots\cdots\cdots\cdots\cdots\cdots(3.11)$$

Each of these functions (3.6)′ and (3.11) on the boundary of the cross section of an opening takes a value equal to $\zeta = \zeta_1 \equiv \sigma (= e^{i\eta})$.

Now, assuming that the function $\phi_1(z_1)$ is defined as the following form:

$$\phi_1(z_1) = \sum_{m=1}^{\infty} \bar{c}_m \zeta_1^{-m}\ , \quad\cdots\cdots\cdots\cdots\cdots\cdots(3.12)$$

then the complex coefficients $\bar{c}_m$ can be written by a, b, τ^0_{xz} and τ^0_{yz} as

$$\left.\begin{aligned} \bar{c}_1 &= \frac{1}{2}(a\tau^0_{yz} - ib\tau^0_{xz})\ , \\ \bar{c}_m &= 0\ . \quad (m \geqq 2) \end{aligned}\right\} \quad\cdots\cdots\cdots\cdots\cdots(3.13)$$

Let us use curvilinear coordinates (r, θ, z) in which r and θ define the point by means of two orthogonal intersecting curves. One of their closed curves consists of the contour of an opening under consideration. And let us specify the the stresses as τ_{rz}, the anti-plane shear component on a curve r=constant; $\tau_{\theta z}$, the antiplane shear component on a curve θ=constant. The following equations can be obtained from the fundamental relations of stresses and diplacements.

$$\left.\begin{aligned} \tau_{rz} &= \tau_{xz}\cos\hat{\theta} + \tau_{yz}\sin\hat{\theta}\ , \\ \tau_{\theta z} &= \tau_{yz}\cos\hat{\theta} - \tau_{xz}\sin\hat{\theta}\ , \\ w_z &= w\tan\hat{\theta},\ \hat{\theta} = -\tan^{-1}\left(\frac{dx}{dy}\right)\ . \end{aligned}\right\} \quad\cdots\cdots\cdots\cdots\cdots(3.14)$$

Thus by using Eq. (3.7) into the expressions of stresses and displacements, it can easily be solved the problem of the case such that the uniform shear stress at infinity acts an isotropic body having an opening with arbitrary shape. For example, Fig. 2 shows the distribution of the

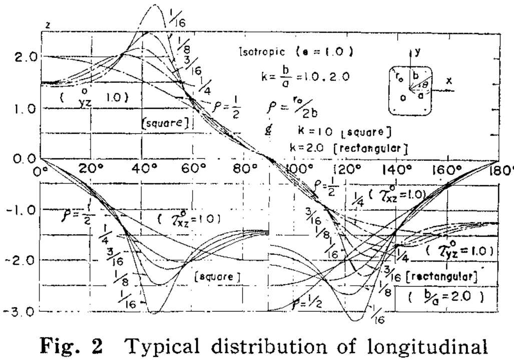


Fig. 2 Typical distribution of longitudinal shear stresses $\tau_{\theta z}$ along edge of (1) a square ($k = b/a = 1.0$) or (2) a rectangular ($k = 2.0$) opening in isotropic body subjected to uniform shear stress τ^0_{xz} or τ^0_{yz} at infinity.

longitudinal shear stress $\tau_{\theta z}$ on the contour of a square ($k = b/a = 1.0$) or a rectangular ($k = 2.0$) opening in an isotropic body, which is subjected to uniform shear stress τ^0_{xz} or τ^0_{yz} at infinity, for the variations of rounded corners ($\rho = r_0/2a$, r_0: radius of rounded corner) of openings as the parameters. Because of the symmetry or the antisymmetry of the stress distribution in this figure, we show only the range of θ from 0 to $\pi/2$ or from $\pi/2$ to π. In above example, we quoted the coefficients α_m, β_m ($m = 1, 2, \cdots, \nu$) of mapping functions for rectangular openings by the results of Heller and others[7]. The number of the terms being adopted in their mapping functions is $\nu = 7$.

As the similar manner to the case of isotropy, we can easily obtain the distribution of stresses and displacements for the case of plane-anisotropic body with an elliptical opening defined by Eq. (3.9). For example of this case, Fig. 3 shows the distribution of the shear stress $\tau_{\theta z}$ at the contour of a circular ($a = b$) hole which is perforated in a two-dimensional orthotropic body (G_{13}, G_{23}: principal shear moduli being mutually orthogonal, and perpedicular to the xy-plane), when the ratios $e (= G_{23}/G_{13})$ of the shear moduli and the angles φ measured by counter-clockwise direction from the x-axis to the plane of G_{13} take several typical values. For the sake of the symmetry of stress distribution for the cases of $\varphi = 0°$ (i.e. $G_x = G_{13}$, $G_y = G_{23}$) and $\varphi = 90°$ (i.e. $G_x = G_{23}$, $G_y = G_{13}$), the range of θ from 0 to $\pi/2$ and from $\pi/2$ to π were only adopted respec-

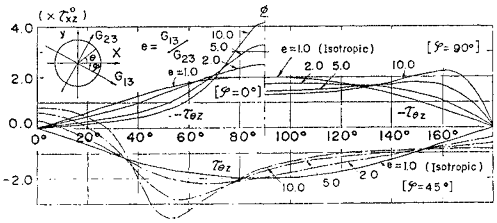


Fig. 3 Typical distribution of stresses $\tau_{\theta z}$ along edge of a circular opening in orthotropic body subjected to uniform stress τ^0_{xz} or τ^0_{yz} at infinity.

tively in the figure, and also for the sake of the antisymmetry for $\varphi=45°$, the range of θ from 0 to π was adopted.

Here, the elastic compliances a_{44}, a_{45} and a_{55} in Eq. (2.3) can be calculated by the shear moduli G_{13}, G_{23} and the angle φ as the following expressions.[8)]

$$\left.\begin{aligned} a_{44}&=\frac{\cos^2\varphi}{G_{23}}+\frac{\sin^2\varphi}{G_{13}} ,\\ a_{55}&=\frac{\sin^2\varphi}{G_{23}}+\frac{\cos^2\varphi}{G_{13}} ,\\ a_{45}&=\left(\frac{1}{G_{23}}-\frac{1}{G_{13}}\right)\cos\varphi\sin\varphi . \end{aligned}\right\} \quad \cdots\cdots(3.15)$$

4. ANALYSIS OF STRESSES AND DISPLACEMENTS IN ELASTIC BODY WITH TWO OR MORE OPENINGS

From the stresses and displacements around or near the opening in an infinite elastic body obtained in the previous section, we can solve the problem of multi-connected region such as an infinite body with several holes utilizing the point matching technique. For simplicity, we shall only describe the method of analysis for the problem of double-connected region such that an elastic body containing two arbitrary holes located with arbitrary spacing is applied the uniform longitudinal shear stress τ^0_{xz} or τ^0_{yz} at infinity. Substancially, quite the same method may be applicable for the case such that even if more than two holes exist in the body under consideration. The literatures [9] are to be refered for details relating to the solution of this problem of multi connected region by the use of the point matching method.

(1) Case of Isotropy

Let us take two rectangular Cartesian coordinate systems (x, y, z) and (x', y', z') whose origins are respectively the centers o and o' of two arbitrary non-intersecting openings $\sum_1$ and $\sum_2$, and whose centers are horizontal distance s_0 and vertical distance h_0 apart (see Fig. 1 (a)). The relation of these coordinates can be given by the expression:

$$x'=x+s_0,\ y'=y+h_0,\ z'=z \ \cdots\cdots\cdots\cdots\cdots(4.1)$$

Let the contour of an arbitrary hole $\sum_2$ be given by

$$\left.\begin{aligned} x_0'&=\alpha_0'\cos\theta' + \sum_{m=1}^{\nu}(\alpha_m'\cos m\theta'+\beta_m'\sin m\theta') ,\\ y_0'&=\alpha_0'\sin\theta' - \sum_{m=1}^{\nu}(\alpha_m'\sin m\theta'-\beta_m'\cos m\theta') . \end{aligned}\right\} \quad \cdots\cdots(4.2)$$

In which θ' is a parameter varing from 0 to 2π in a counterclockwise direction.

In the same manner as was mentioned in previous, section, we consider the problem by mapping conformally the complex plane $z_0'(=x'+iy)$ with the opening $\sum_2$ onto the exterior of a unit circle $|\zeta'|=1$. The relation between z_0' and ζ' is represented by

$$z_0'\equiv\hat{\omega}(\zeta')=\alpha_0'\zeta'+\sum_{n=1}^{\nu'}(\alpha_n'+i\beta_n')\zeta'^{-n} . \cdots(4.3)$$

On the other hand, for the complex variables z_0 and z_0', the relation may be deduced as

$$z_0'=z_0+(s_0+ih_0) . \ \cdots\cdots\cdots\cdots(4.4)$$

And then the relation between ζ and ζ' must satisfy the following algebraic equation:

$$\alpha_0\zeta^{\nu+1}+\left\{(s_0+ih_0)-\alpha_0'\zeta' - \sum_{n=1}^{\nu}(\alpha_n'+i\beta_n')\zeta'^{-n}\right\}\zeta^{\nu} + \sum_{m=1}^{\nu}(\alpha_m+i\beta_m)\zeta^{\nu-m}=0 , \quad \cdots\cdots(4.5a)$$

or

$$\alpha_0'\zeta'^{\nu'+1}-\left\{(s_0+ih_0)+\alpha_0\zeta + \sum_{m=1}^{\nu}(\alpha_m+i\beta_m)\zeta^{-m}\right\}\zeta'^{\nu'} + \sum_{n=1}^{\nu'}(\alpha_n'+i\beta_n')\zeta'^{\nu'-n}=0 . \cdots\cdots(4.5b)$$

Here, if we also seek the expression for the complex analytic function $\phi_0^*(z_0')$ corresponding to the body with an opening $\sum_2$ as the form:

$$\phi_0^*(z_0')=\sum_{k=1}^{\infty}\bar{C}_k'\zeta'^{-k} , \ \cdots\cdots\cdots\cdots(4.6)$$

the proceeding operation after this step will be applicable to the same manner as the previous paper[9)]. In which, the complex constants $\bar{C}_k'$ in

Eq. (4.6) are determined as the next equation by the use of the point matching approach.

$$-\int_0^s Z_n ds = C_0' + \sum_{k=1}^{\infty} (C_k' e^{ik\theta'} + \bar{C}_k' e^{-ik\theta'}) . \qquad \cdots\cdots(4.7)$$

Though we have to obtain the solutions of the algebraic equation of order $(\nu+1)$ or $(\nu'+1)$ as shown in Eq. (4.5) for the case of the body with two arbitrary shaped holes $\sum_1$, $\sum_2$, this equation can easily be solved by utilizing high speed electric computer.

(2) Case of Anisotropy

As was shown in Fig. 1 (b), let us take two coordinate systems (x, y, z) and (x', y', z') whose origins are respectively the centers o and o' of two arbitrary elliptical openings $\sum_1$ and $\sum_2$. The relation of these coordinates can be expressed as:

$$\left.\begin{aligned} x'\cos\eta - y'\sin\eta &= x + s_0 , \\ x'\sin\eta + y'\cos\eta &= y + h_0 , \\ z' &= z . \end{aligned}\right\} \qquad \cdots\cdots(4.8)$$

Let the boundary of an elliptical hole $\sum_2$ be given by

$$x_0' = a'\cos\theta', \ y_0' = b'\sin\theta' . \qquad \cdots\cdots(4.9)$$

If a complex plane $z_1'(=x'+\mu_1 y')$ is conformally mapped onto the exterior of a unit circle in the ζ_1'-plane, the following equation may be obtained as

$$z_1' = \frac{1}{2}\{(a'-i\mu_1 b')\zeta_1' + (a'+i\mu_1 b')\zeta_1'^{-1}\} . \qquad \cdots\cdots(4.10)$$

From the relation of complex variables z_1 and z_1', an algebraic equation of ζ_1 or ζ_1' may be formulated as

$$z_1' = \cos\eta\{z_1 + (s_0 + \mu_1 h_0)\} - \sin\eta \left\{\mu_1 z_1 + \frac{i}{2}(1+\mu_1^2)(\zeta_1 - \zeta_1^{-1}) - (h_0 - \mu_1 s_0)\right\} . \qquad \cdots\cdots(4.11)$$

Now, assuming the expression for the complex function $\phi_1^*(z_1')$ in the system (x', y', z') as

$$\phi_1^*(z_1') = \sum_{k=1}^{\infty} \bar{c}_k' \zeta_1'^{-k} , \qquad \cdots\cdots(4.12)$$

then the proceeding operation is also same to the case in above section 4.1.

5. NUMERICAL EXAMPLES

In order to check the accuracy of the results, the stress values $\tau_{\theta z}$ around two equal circular openings in an isotropic body under the applied shear stress τ^0_{xz} or τ^0_{yz} at infinity were compared the results obtained by the present approach with the results obtained by the method of bipolar coordinates. The results by the present method in which the angle of equal intervals $\Delta\theta$ sets equal to 2.5° (i.e. the number of selected boundary points is equal to 144), the number of terms in series (4.7) k equal to 20 and the number of repetitions i equal to 6, with the spacing $s_0=2.4a$, are in closed agreement with the results of Hirashima and Hisatake[10] as shown in Table 1.

Yokobori et al[3] and Tamate and Yamada[6] have been studied the interaction of two colinear cracks, and obtained the results of the coefficients of stress concentration at the crack tips (the case corresponding to $\eta=0$, $b=b'$, $a=a'=0$, $h_0=0$ in Fig. 1 (b)) under the case such that the isotropic body is applied the longitudinal shear stress τ^0_{xz} at infinity. In order to compare with these results, we calculated numerically the case under the same conditions with the ratio b/a equal to 10.0, 100.0 and 1000.0, and showed the results of the ratio of each value of s_0/b and $s_0/b=\infty$ (i.e. the case of one elliptical crack), that is, ratio of the coefficients of stress concentration in Table 2. From this table, it is observed that the larger the value of b/a, the closer the author's results become to the ones by others. In these calculations, we put $\Delta\theta=2.5°$ (i.e. the number of the selected points is equal to 144), $k=20$ and $i=4\sim7$, in spite of the variations of the ratio b/a. The difference in both results as above two examples is considerably small, and it may be concluded that the present method is sufficientlly accurate for practical usage.

Now, let us show some numerical examples under the loading conditions of external shear stress τ^0_{xz} or τ^0_{yz} applied at infinity. Fig. 4 shows the distribution of circumfential shear stresses $\tau_{\theta z}$ at the contour of the right side opening $\sum_1$ in an isotropic body containing two equal square openings with the spacing $s_0=3a$, for the variations of rounded corners ρ as the parameters. This corresponds to the left-half diagram of Fig. 2 which show the distribution of stresses $\tau_{\theta z}$ in the case with only a square hole in the body. With the spacing of distance of two origins as was illustrated here, the difference in these cases (inclease or decrease) is about 5~8%. However, especially in the case of $\tau^0_{xz}=1.0$ at the location about $\theta=130°$ on the contour of the opening $\sum_1$, the stress value $\tau_{\theta z}$ is observed to decreases quite remarkably. On the contrary, in the case of $\tau^0_{yz}=1.0$, the stress value $\tau_{\theta z}$ at the location about

Table 1 Comparison of the results obtained by this solution and bipolar coordinate solution, for the case of isotropic body containing two equal circular openings with the spacing $s_0=2.40a$ between the two origins o and o'.

θ	$\tau_{\theta z}$ $(\tau^0_{yz}=1.0)$		$\tau_{\theta z}$ $(\tau^0_{xz}=1.0)$	
	Bipolar Solution	$\Delta\theta=2.5°$ $k=20$ $i=6$	Bipolar Solution	$\Delta\theta=2.5°$ $k=20$ $i=6$
0°	2.227	2.227	0.000	−0.002
15°	2.161	2.161	−0.501	−0.501
30°	1.967	1.967	−0.966	−0.966
45°	1.660	1.660	−1.359	−1.359
60°	1.260	1.260	−1.649	−1.649
75°	0.793	0.793	−1.810	−1.811
90°	0.289	0.289	−1.825	−1.825
105°	−0.231	−0.234	−1.681	−1.681
120°	−0.765	−0.765	−1.378	−1.378
135°	−1.355	−1.356	−0.945	−0.945
150°	−2.104	−2.104	−0.473	−0.473
165°	−3.020	−3.019	−0.136	−0.136
180°	−3.535	−3.532	0.000	−0.008

Table 2 Comparison of the results obtained by some investigators, for the case of isotropic body containing two equal colinear cracks.

s_0/b			∞	2.00	1.00	0.50	0.25
Tamate & Yamada[6]			1.0000	0.9231	0.8604	—	—
Yokobori et al[3]			1.0000	0.9244	0.8549	0.7988	0.7622
Author	b/a	10.0	1.0000	0.9195	0.8532	0.8068	0.7974
		100.0	1.0000	0.9240	0.8547	0.7994	0.7686
		1000.0	1.0000	0.9244	0.8553	0.7989	0.7670

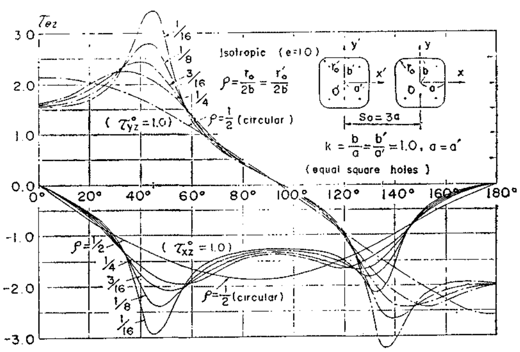


Fig. 4 Effects of the rounded corners on the stress concentrations around the right side opening $\sum_1$ in isotropic body containing two square openings.

$\theta=180°$ is observed to increase about 15∼20% compared with the one at about $\theta=0°$. When the stress τ^0_{xz} or τ^0_{yz} at infinity is applied to an isotropic body containing two equal square holes located on the x-axis (i.e. $h_0=0$), the distributions of stresses $\tau_{\theta z}$ and displacements w_z around the right side opening $\sum_1$ are shown in Fig. 5 (a) and (b) with varing parameters of $s_0/2a$. In this figure, the dotted lines are in the case with only a square opening (i.e. $s_0/2a=\infty$) under the same geometrical conditions. The state of stress concentrations changes with the variation for the spacing s_0 as shown in the figure. This shows a similar tendency to the case of the in-plane loading[11] (i.e. $\sigma_x{}^0$, $\sigma_y{}^0$ or τ^0_{xy} is applied at infinity).

As a more general example, Fig. 6 shows the stress distribution of $\tau_{\theta z}$, $\tau_{\theta z}'$ around the openings

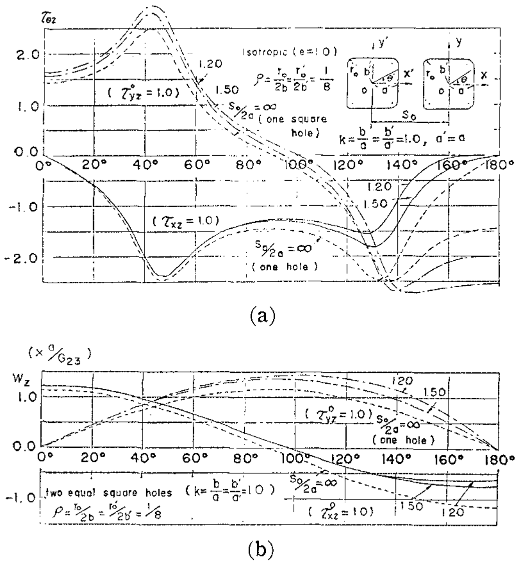


Fig. 5 Stresses $\tau_{\theta z}$ and displacements w_z around the opening $\sum_1$ in isotropic body containing two equal square openings.

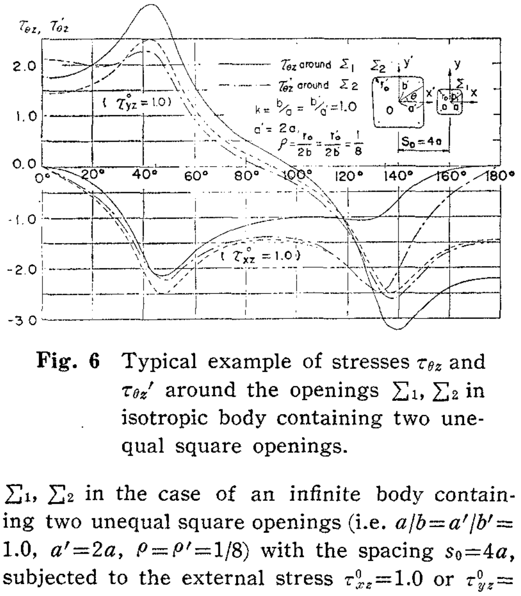


Fig. 6 Typical example of stresses $\tau_{\theta z}$ and $\tau_{\theta z}'$ around the openings $\sum_1$, $\sum_2$ in isotropic body containing two unequal square openings.

$\sum_1$, $\sum_2$ in the case of an infinite body containing two unequal square openings (i.e. $a/b=a'/b'=1.0$, $a'=2a$, $\rho=\rho'=1/8$) with the spacing $s_0=4a$, subjected to the external stress $\tau^0_{xz}=1.0$ or $\tau^0_{yz}=1.0$ at infinity. The dotted lines show the distribution in the case with only a square opening in the body, the solid lines show the distribution around the opening $\sum_1$ and the chain lines around the opening $\sum_2$. These show only the region from 0 to π from the reason due to the symmetries of the openings and the applied load. From this figure, the smaller opening $\sum_1$ is in the states that the distribution of stresses $\tau_{\theta z}$ around the opening $\sum_1$ varies with quite remarkable increase and decrease.

The examples in above mentions are in the case of an isotropic body containing two square openings, and in the followings, we shall show some of numerical examples of stresses $\tau_{\theta z}$ and displacements w_z around two arbitrary circular openings in a plane orthotropic elastic body. When the uniform stress τ^0_{xz} or τ^0_{yz} at infinity acts to a plane orthotropic body ($e=G_{13}/G_{23}=2.0$) having two equal circular openings located on the x-axis, the stress distributions of $\tau_{\theta z}$ around the right side opening $\sum_1$ are shown in Fig. 7

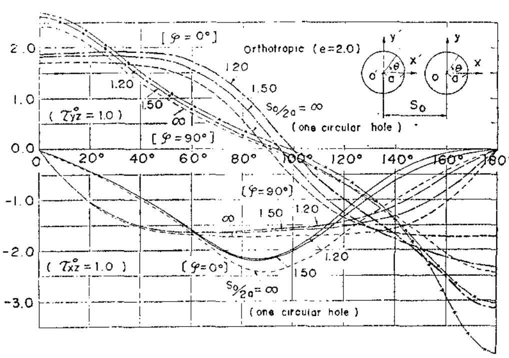


Fig. 7 Stresses $\tau_{\theta z}$ around the opening $\sum_1$ in orthotropic body ($e=G_{13}/G_{23}=2.0$) containing two equal circular openings.

with varing parameters of $s_0/2a$ and the angle φ, in which s_0 is the horizontal distance between the two origins of the openings, and φ the angle between the direction of the principal shear modulus G_{13} and the direction of the x-axis. Two cases of $\varphi=0°$ (i.e. $G_x=G_{13}$, $G_y=G_{23}$) and of $\varphi=90°$ (i.e. $G_x=G_{23}$, $G_y=G_{13}$) are given in this figure with the spacing $s_0/2a$ equal to ∞, 1.50 or 1.20. The dotted lines show the distribution of $\tau_{\theta z}$ in the case with only a circular opening in the orthotropic body, which are justly correspondence with the result to the case of $e=2.0$ in Fig. 3.

As an illustration of two unequal circular openings in an orthotropic body, let us calculate the stresses and the deformations in the body under the geometrical conditions such as $a=2a$, $s_0=4a$ and the loading condition such as the shear stress $\tau_0=1.0$ applied at infinity with an inclined angle $\delta=45°$ from the x-axis. Fig. 8 (a) and (b) show the distributions of stresses $\tau_{\theta z}$, $\tau_{\theta z}'$ and displacements w_z, w_z around the openings $\sum_1$, $\sum_2$ in orthotropic body for the varing parameter e,

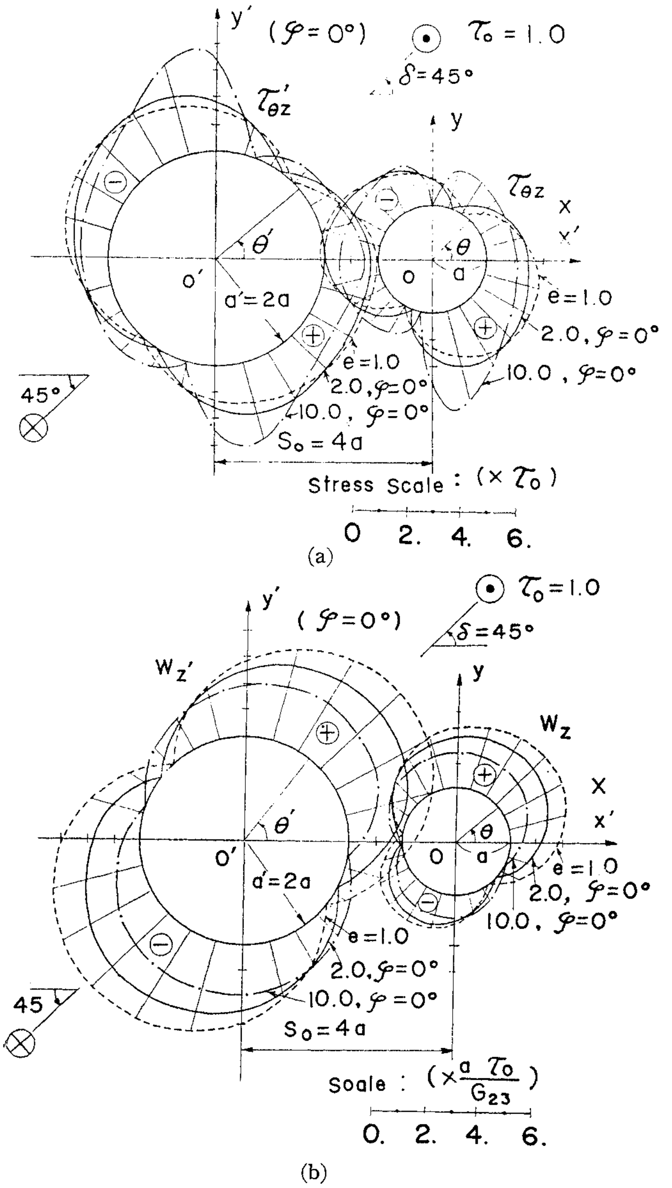


Fig. 8 Typical example of stresses $\tau_{\theta z}, \tau_{\theta z}'$ and displacements w_z, w_z' around the openings $\sum_1$, $\sum_2$ in isotropic or orthotropic body containing two unequal circular openings.

and the dotted lines in the figure are the case of an isotropic body (i.e. $e=1.0$).

The above examples showed only the cases with two openings in an isotropic or an orthotropic body. However, as being pointed out by the section 4, the present method can be applied to the problems with three or more openings.

We illustrate in the following the stress distributions obtained for several problems of an isotropic and an orthotropic bodies involving three openings. Fig. 9 shows the stress distributions of $\tau_{\theta z}$ and $\tau_{\theta z}'$ on the contour of the right side opening $\sum_1$ and the central opening $\sum_2$ in the case of three equal square openings (i.e. $a=a'=a''$, $\rho=\rho'=\rho''=1/8$) with the spacings $s_0/2a=s_1/2a=\infty$, 1.50 and 1.20 under the loading condition of $\tau_{xz}^0=1.0$ or $\tau_{yz}^0=1.0$. Though the distribution of $\tau_{\theta z}''$ around the left side opening $\sum_3$ is not shown in the figure, this reason dues to the fact such that the distribution of $\tau_{\theta z}''$ is the same result of $\tau_{\theta z}$ around the opening $\sum_1$ from the symmetries of the size and the shape of three openings and the applied stress. It is interesting to compare these results with the ones of Fig. 5 (a) in the case with two equal square openings under the same conditions. For example, as for the stresses $\tau_{\theta z}$ on the right side opening $\sum_1$, stress concentration factors in the case of Fig. 5 (a) are in the states such a little smaller in absolute values than in the case of Fig. 9 under the case of the loading condition $\tau_{yz}^0=1.0$. On the other hand, stress concentration factors in Fig. 5 (a) are in the states such a little larger in absolute values than in Fig. 9 under the case of $\tau_{xz}^0=1.0$. Fig. 10 is the stress distributions of $\tau_{\theta z}$, $\tau_{\theta z}'$ and $\tau_{\theta z}''$ around the three unequal circular openings $\sum_1$, $\sum_2$ and $\sum_3$ in an orthotropic body (i.e. $e=10.0$, $\varphi=0°$) under the loading condition $\tau_0=1.0$, $\delta=45°$. In this figure the stress distribution in the case of an isotropic body (i.e. $e=1.0$) is also plotted under the same conditions. About this illustration, comparing with the result of Fig. 8 (a), it can be discussed the difference in both cases such as with two openings and with three openings, but we omit here by reasoning of the limited space of the paper.

In above all calculations, we assumed that the angle of equal intervals with respect to the origins $\Delta\theta$ is equal to 2.5° (i.e. the number of the selected points on the contours of openings is equal to

144), the number of terms in series (4.6) or (4.12) k equal to 20 and the number of repetitions i equal to 4~8.

6. CONCLUDING REMARKS

A method of successive approximations utilizing the complex variable method has been used to obtain the distribution of stresses and displacements around two or more openings in an infinite isotropic or anisotropic body, when the longitudinal shear stress applies at infinity. The results obtained by the present approach were compared with the results by others in which isotropic body with two circular openings or with two colinear cracks was treated, and their results are in quite agreement. It can be concluded that the present method is very applicable to solve the problems of multi-connected regions such as an infinite isotropic or anisotropic body containing several openings.

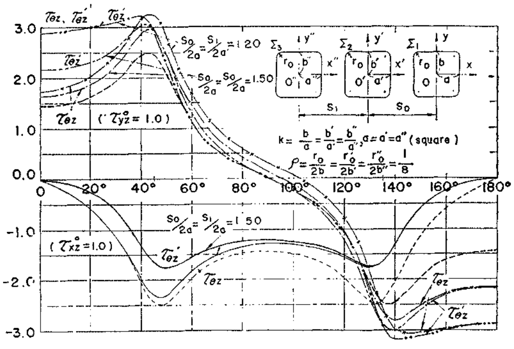


Fig. 9 Stresses $\tau_{\theta z}$ and $\tau_{\theta z}'$ around the openings $\sum_1$ and $\sum_2$ in isotropic body containing three equal square openings.

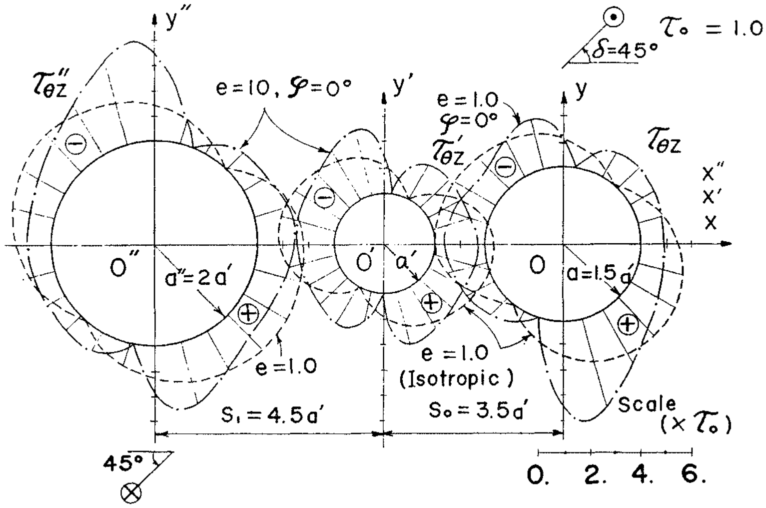


Fig. 10 Stresses $\tau_{\theta z}$, $\tau_{\theta z}'$ and $\tau_{\theta z}''$ around the openings $\sum_1$, $\sum_2$ and $\sum_3$ in isotropic or orthotropic body containing three unequal circular openings.

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