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THE LOCAL POTENTIAL APPROACH TO FINITE ELEMENT METHOD IN UNSTEADY VISCOUS INCOMPRESSIBLE FLUID FLOW

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1. INTRODUCTION

Applications of the finite element method to problems of unsteady fluid flow was presented for a restricted class of problems in potential flow by Visser.[1] This was suggested as a treatment for general transient problems and was elaborated by Zienkiewicz and Cheung.[2] Attempts to formulate a more general class of the problems have recently been presented; for viscous incompressible fluid flow[3)~7)]; for general compressible Newtonian fluid flow[8)~10)]. Applications of the method to a general class of field equations was recently presented from a view point of applied mathematics[11)].

The above papers may be classified into three main procedures from a view point of methodology of the finite element formulation; via the Ritz method[1),2)] which is used only when a classical extremal principle exists; via a kind of virtual work principle[3),4)]; via the method of weighted residuals, the Galerkin's method[5)~11)]. These recent papers which especially come under the last procedure demonstrate that the formulation of the finite element can be based on the governing differential equations directly or on the global form of the law of conservation of energy indirectly via the method, when an appropriate variational principle does not exist in the classical sense.

On the other hand, new methods of formulating variational principles based on the concept of the local potential was originated by Glansdorff and Prigogine[12)]. This technics has been applied in various fields of hydrodynamics by Glansdorff and Prigogine[13)], Schechter and Himmelblau[14)], Takagi[15)] and others.

An important application of the variational principle is as a means of developing methods to obtain approximate solutions. The extremal principles based on the concept of the local potential, of course, can be used as the basis for developing approximation methods. One is the finite-difference method[16)] and the other is the direct methods. Schechter[17)] has suggested that all of the direct methods, that is, the Ritz method, solution by partial integration, eigenvalues and eigenfunctions and others which are used for classical extremal principles are applicable to the local potential and pointed out that the direct methods using the local potential involve the Galerkin's method with the added advantage of treating the boundary conditions directly, though they do not give clear upper or lower bounds of the solution. However, these direct methods have been applied to the local potential for the whole region of interest. Therefore, it seems difficult to apply the methods to the problems which involve complex geometries and boundary conditions. The finite element method is one procedure to overcome such a difficulty.

Unfortunately, however, the concept of the local potential has been considered[18)] to play no role in finite element formulations by reason that the concept falls outside of the range of validity applicable to the principle of minimum entropy production in the classical sense.

Recently, formulation of a variational principle for the flow of a viscous incompressible fluid which includes the convective term and covers both time-independent and time-dependent phenomena was proposed by Lemieux et al.[19)]. This is based on the concept of the local potential. It is intended herein to discuss the formulation proposed by them, to rewrite it in a modified form and to apply it to a finite element formulation

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for the unsteady incompressible viscous fluid flow. To test the validity of the proposed approach, a numerical example for the transient flow through a rectangular channel is presented and the approximate solutions are compared with the theoretical ones. Unfortunately, the numerical example does not include the convective term and prescribes the pressure gradient as a time dependent function. In principle, it is possible to apply the procedure proposed here to a numerical analysis of the wake flow.

2. VARIATIONAL PRINCIPLE FOR UNSTEADY VISCOUS INCOMPRESSIBLE FLUID FLOW

2.1 Restricted Variational Principle

In advance of arguments on derivation of a modified variational formulation, the technique presented by Lemieux et al.[19] will be summarized and discussed.

The fundamental equations of an incompressible viscous fluid are the equation of continuity and the Navier-Stokes equation;

$$0=\frac{\partial u_j}{\partial x_j} \quad \cdots\cdots (1)$$

$$\rho\frac{\partial u_i}{\partial t}=-\rho u_j\frac{\partial u_i}{\partial x_j}+\rho X_i-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}}{\partial x_j} \quad \cdots (2)$$

where ρ=the fluid density; u_i=a velocity component; t=time; x_i=the coordinate system; τ_{ij}= the shear stress tensor; p=the pressure; X_i=the body force per unit mass. In the authors paper, the body force X_i is replaced by the gradient of potential, that is, by $-\partial\phi/\partial x_i$. Outlines of derivation of the functional, the local potential presented by them are in the following; Multiply Eq. (2) by a small velocity variation $-\delta u_i$ and integrate over the volume V. Next, use the Gauss theorem for the volume integrals, replace u_i by $u_i^0+\delta u_i$ and neglect terms of higher order. Thus they derived the following variational formulation:

$$\begin{aligned}&-\frac{\partial}{\partial t}\int_V \rho\frac{\delta u_i\delta u_i}{2}dV\\&=\delta\left\{\int_V\left(\rho\frac{\partial u_i^0}{\partial t}u_i-\rho u_i^0u_j^0\frac{\partial u_i}{\partial x_j}+\tau_{ij}^0\frac{\partial u_i}{\partial x_j}\right)dV\right.\\&\left.+\int[\rho u_i^0u_j^0u_i+(p+\rho\phi)^0u_j-\tau_{ij}^0u_i]n_jdA\right\}\\&\equiv\delta F_I \quad \cdots\cdots (3)\end{aligned}$$

In Eq. (3), it is noted that the term $\partial u_j/\partial x_j$ disappears as the equation of continuity is used. The superscript zero refers to quantities evaluated at the stationary (actual) state and these quantities are not subjected to variations. Only quantities without that superscript are subjected to variations. Let Δt be a small time interval, then the inequality

$$\begin{aligned}&-\int_{\Delta t}\frac{\partial}{\partial t}\int_V\rho\frac{\delta u_i\delta u_i}{2}dVdt=\frac{1}{2}\int_V\rho\delta u_i\delta u_idV\\&=\int_{\Delta t}\delta F_Idt\geqq 0 \quad \cdots\cdots (4)\end{aligned}$$

is given. Therefore, the local potential F_I takes a minimum when $\delta F_I=0$ with the subsidiary condition $u_i^0=u_i$. The derivation of Eq. (3) and the certification of Eq. (4) were given by the authors. Eq. (3) was applied to the Stoke's first problem in which the fluid obeys non-Newtonian model.

Now, for the sake of another development we will discuss on the variational formulation, Eq. (3).

Firstly, it is noted that the extremal

$$\frac{\delta F_I}{\delta u_i}=0 \quad \cdots\cdots (5)$$

does not inversely give the Navier-Stokes equation, Eq. (2) in which the body force X_i is replaced by $-\partial\phi/\partial x_i$, much less give the equation of continuity, Eq. (1). This is clear from the fact that the equation of continuity is already used in the process of deriving Eq. (3). In other words, the equation of continuity is satisfied implicitly. Generally, it seems difficult to find the velocity distribution u_i which satisfies the equation of continuity over the whole region V of interest. Moreover, it is worth recalling that a variational formulation can be shown to be correct by insuring that the Euler-Lagrange equations are identical with the appropriate forms of the balance equations (the Navier-Stokes equation and the equation of continuity in this case). In this meaning, Eq. (3) seems to be a restricted form.

Secondly, the surface integral in the right hand side of Eq. (3) is rewrited as

$$\begin{aligned}&\int_A[\rho u_i^0u_j^0u_i+(p+\rho\phi)^0u_j-\tau_{ij}^0u_i]n_jdA\\&=\int_A(\rho u_i^0u_j^0-\sigma_{ij}^0)n_ju_idA \quad \cdots\cdots (6)\end{aligned}$$

where

$$\sigma_{ij}^0=-(p+\rho\phi)^0\delta_{ij}+\tau_{ij}^0 \quad \cdots\cdots (7)$$

and δ_{ij} is the Kronecker delta. σ_{ij}^0 is the stress tensor including the effect of prescribed potential ϕ. Eq. (6) reveals that the momentum flux $(\rho u_i^0u_j^0-\sigma_{ij}^0)n_j$ or the velocity u_i should be specified on the boundary surface. It is noted that the momentum flux is expressed as a total flow including both the convective and the diffusive part. For the problems in which the stress or the velocity is specified on the boundary surface, however, the surface integral of Eq. (6) will not be a convenient form. For instance, a laminar flow in a duct which is filled with an incompressible viscous fluid corresponds to such a problem, if a prescribed pressure gradient determines the velocity distributions on the cross sectional areas perpendicular to flow. In the following, let us restrict ourselves to problems with the boundary surface on which the stress or the velocity is specified, as the numerous problems in which the boundary condition is expressed as Eq. (6) are not yet treated in this paper.

2.2 Modified Variational Principle

In this article, developing the technique presented by Lemieux et al. a modified variational formulation will be presented.

Multiplying Eq. (1) by a small variation $-\delta p$ and Eq. (2) by a small variation $-\delta u_i$, adding the resultant expression and integrating over the region V of interest, this gives

$$-\int_V \rho\frac{\partial u_i}{\partial t}\delta u_i dV=\int_V \rho u_j\frac{\partial u_i}{\partial x_j}\delta u_i dV -\int_V \rho X_i\delta u_i dV+\int_V \frac{\partial p}{\partial x_i}\delta u_i dV -\int_V \delta p\frac{\partial u_j}{\partial x_j}dV-\int_V \frac{\partial \tau_{ij}}{\partial x_j}\delta u_i dV \quad \cdots\cdots\cdots(8)$$

By using the Gauss theorem, the second and the third term in the right hand side of Eq. (8) are rewritten as

$$\int_V \frac{\partial p}{\partial x_i}\delta u_i dV=\int_A p\delta u_j n_j dA-\int_V p\delta\frac{\partial u_j}{\partial x_j}dV \quad \cdots\cdots\cdots(9)$$

and

$$\int_V \frac{\partial \tau_{ij}}{\partial x_j}\delta u_i dV=\int_A \tau_{ij}\delta u_i n_j dA-\int_V \tau_{ij}\delta\frac{\partial u_i}{\partial x_j}dV \quad \cdots\cdots\cdots(10)$$

Substitute Eq. (9) and (10) into Eq. (8), write $u_i=u_i^0+\delta u_i$ and $p=p^0+\delta p$ and neglect terms of higher order, then this gives

$$-\frac{\partial}{\partial t}\int_V \rho\frac{\delta u_i\delta u_i}{2}dV=\delta\left\{\int_V\left(\rho\frac{\partial u_i^0}{\partial t}u_i +\rho u_j^0\frac{\partial u_i^0}{\partial x_j}u_i-\rho X_i^0 u_i -p^0\frac{\partial u_j}{\partial x_j}-p\frac{\partial u_j^0}{\partial x_j}+\tau_{ij}^0\frac{\partial u_i}{\partial x_j}\right)dV +\int_A (p^0 u_j-\tau_{ij}^0 u_i)n_j dA\right\} \quad \cdots\cdots\cdots(11)$$

In the derivation of Eq. (11), it is noted that the equation of continuity, Eq. (1) is not yet used. Moreover, by using the variational equation

$$-\delta\left(p\frac{\partial u_j}{\partial x_j}\right) =-\delta p\frac{\partial u_j}{\partial x_j}-p\delta\frac{\partial u_j}{\partial x_j} =\delta\left(-p\frac{\partial u_j^0}{\partial x_j}-p^0\frac{\partial u_j}{\partial x_j}\right) \quad \cdots\cdots\cdots(12)$$

Eq. (11) is rewriten as

$$-\frac{\partial}{\partial t}\int_V \rho\frac{\delta u_i\delta u_i}{2}dV=\delta\left\{\int_V\left(\rho\frac{\partial u_i^0}{\partial t}u_i +\rho u_j^0\frac{\partial u_i^0}{\partial x_j}u_i-\rho X_i^0 u_i-p\frac{\partial u_j}{\partial x_j}+\tau_{ij}^0\frac{\partial u_i}{\partial x_j}\right)dV +\int_A (p^0 u_j-\tau_{ij}^0 u_i)n_j dA\right\}=\delta F_{II} \quad \cdots\cdots\cdots(13)$$

The left hand side of Eq. (11) or (13) is the same as that of Eq. (3). Therefore the inequality, Eq. (4) is still held by replacing F_I by F_{II}.

If we inversely use the Gauss theorem, Eq. (9) and (10), it is easily shown that the extremal

$$\frac{\delta F_{II}}{\delta u_i}=0 \quad \cdots\cdots\cdots(14)$$

gives Eq. (2) and the extremal

$$\frac{\delta F_{II}}{\delta p}=0 \quad \cdots\cdots\cdots(15)$$

gives Eq. (1) with the subsidiary conditions $u_i^0=u_i$ and $p^0=p$. By using Eq. (7), the surface integral of Eq. (13) is rewritten as

$$\int_A (p^0 u_j-\tau_{ij}^0 u_i)n_j dA=-\int_A \sigma_{ij}^0 n_j u_i dA \quad \cdots\cdots\cdots(16)$$

Eq. (16) reveals that the stress or the velocity should be specified on the boundary surface (in this case the stress does not include the potential ϕ).

In Eq. (13), it is clear that other arrangements of the terms are possible; if we apply the Gauss theorem to the convective term in the right hand side Eq. (8), the surface integral represented of Eq. (6) may be yielded and the convective term in the volum integral of Eq. (13) becomes a more complex form. Thus, no claim of uniqueness or superiority can be laid for Eq. (13). Eq. (13) simply represents a convenient stopping point in the analysis, for we restricted ourselves to the problems having the boundary surface on which the stress or the velocity should be specified.

3. MATRIX EXPRESSION OF THE LOCAL POTENTIAL

To acknowledge similarities and differences between fluid dynamics and solid dynamics and to carry out computations systematically, it is convenient to express the local potential presented in the previous article in a matrix form.

Let us use the next vectors

$$\{u\}=\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \cdots(17)$$

$$\{X\}=\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \qquad \cdots(18)$$

$$\{n\}=\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \qquad \cdots(19)$$

$$\{\tau\}=\begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{Bmatrix} \qquad \cdots(20)$$

$$\{e\}=\begin{Bmatrix} \partial u_1/\partial x_1 \\ \partial u_2/\partial x_2 \\ \partial u_3/\partial x_3 \\ \partial u_1/\partial x_2+\partial u_2/\partial x_1 \\ \partial u_2/\partial x_3+\partial u_3/\partial x_2 \\ \partial u_3/\partial x_1+\partial u_1/\partial x_3 \end{Bmatrix}=\begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{23} \\ e_{31} \end{Bmatrix} \qquad \cdots(21)$$

where $\{u\}=$ the fluid velocity vector; $\{X\}=$the body force vector; $\{n\}=$ the unit vector normal to boundary; $\{\tau\}=$ the shear stress vector; $\{e\}$ $=$ the total strain rate vector.

Expanding each term of the local potential F_{II} of Eq. (13) to a summed form and rearranging (see APPENDIX I.), the following expression is obtained:

$$F_{II}=\int_V\Big[\rho\{u\}^T\{\dot{u}^0\}+\rho\{u\}^T\Big(\sum_{i=1}^{3}\frac{\partial\{u^0\}}{\partial x_i}[I_i]\Big)\times\{u^0\}-\rho\{u\}^T\{X^0\}-\{e\}^T\{J\}p+\{e\}^T\{\tau^0\}\Big]dV+\int_A(\{u\}^T\{n\}p^0-\{u\}^T[S]\{\tau^0\})dA \qquad \cdots(22)$$

in which the next matrices are introduced:

$$\left.\begin{aligned} [I_1]&=[1\ \ 0\ \ 0] \\ [I_2]&=[0\ \ 1\ \ 0] \\ [I_3]&=[0\ \ 0\ \ 1] \end{aligned}\right\} \qquad \cdots(23a)$$

$$\{J\}=\{1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\}^T \qquad \cdots(23b)$$

$$[S]=\begin{bmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{bmatrix} \qquad \cdots(23c)$$

4. APPLICATION TO THE FINITE ELEMENT METHOD

The purpose of this article is to demonstrate the applicability of the local potential to finite element formulations. The method discussed here is, of course, not perfect as the numerical applications which simultaneously include both the local and the convective term of acceleration are yet unsolved.

4.1 Generality

General ideas of the finite element method in fluid dynamics are given by Oden[20]. The some basic points are in the following; The whole region of interest is separated by imaginary lines or surfaces into a finite number of subregions, elements as shown in Fig. 1. The elements are fixed in space and interconnected at a discrete number of nodal points suituated on their boundary surface. Fig. 2 shows a typical element with

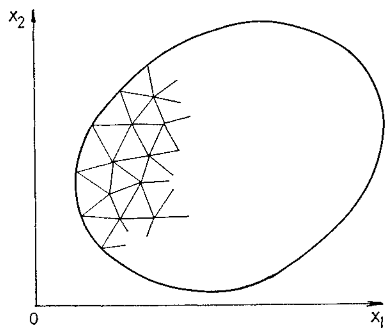

Fig. 1 A whole region divided into finite elements.

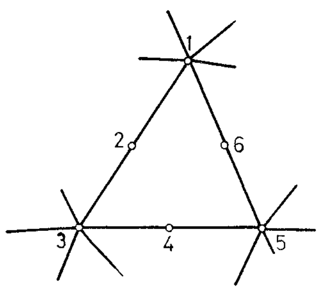

Fig. 2 Triangular element with six nodal points.

six nodes on the boundary surface. Generally, the fluid velocity and the pressure or their derivative values of these nodal points will be unknown parameters of the problem. The velocity and the pressure distribution within the element are approximated by different polynomial expansions, respectively. The order of approximation for the velocity and for the pressure uniquely corresponds to the number of nodes L_e for the velocity and M_e for the pressure, respectively. In Fig. 2, for instance, L_e may be six $(1, 2, \ldots, 6)$ and M_e may be three $(1, 3, 5)$. The coefficients of each term of the expansions are defined by the coordinates of nodes and the unknown parameters. Thus the velocity and the pressure within the element are represented as

$$u_i = \phi_L(x_1,\ x_2,\ x_3)\cdot u_i^L(t) \quad \cdots\cdots(24)$$

and

$$p = \varphi_M(x_1,\ x_2,\ x_3)\cdot p^M(t) \quad \cdots\cdots(25)$$

Here the repeated indices L and M are summed from 1 to L_e and from 1 to M_e respectively. $\phi_L(x_1, x_2, x_3)$ is interpolation function for the velocity and $\varphi_M(x_1, x_2, x_3)$ is that for the pressure. u_i^L denotes the unknown velocity in the direction of x_i axis at node L and p^M denotes the unknown pressure at node M. For the sake of simplicity, the derivative values of these unknown parameters are not considered.

Above basic ideas are applicable to the local potential approach, but the subsequent process to construct finite element formulations is cosiderably different from that developed by Oden using the Galerkin's method.

4.2 Ritz method for Local Potential

Eq. (24) and (25) are rewritten in a general form as

$$\{u\} = [N]\{v\}^e \cdots\cdots(26)$$

$$p = [C]\{p\}^e \cdots\cdots(27)$$

The components of $[N]$ and of $[C]$ consist of known interpolation functions. $\{v\}^e$ and $\{p\}^e$ consist of unknown nodal parameters. The strain rate vector $\{e\}$ in Eq. (22) is represented as

$$\{e\} = [B]\{v\}^e \cdots\cdots(28)$$

in which $[B]$ is derived from $[N]$ using the definition of Eq. (21). Let us consider an incompressible Newtonian fluid. The constitutive equation is expressed as

$$\{\tau\} = [D]\{e\} = [D][B]\{v\}^e \cdots\cdots(29)$$

in which

$$[D] = \mu\begin{bmatrix} 2 & & & & & 0 \\ & 2 & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{bmatrix} \cdots\cdots(30)$$

and μ= the shear viscosity. Let us use the self-consistent approximations

$$\left.\begin{aligned} \{u^0\} &= [N][v^0]^e \\ \{\dot{u}^0\} &= [N]\{\dot{v}^0\}^e \\ \{e^0\} &= [B]\{v^0\}^e \\ \{\tau^0\} &= [D][B]\{v^0\}^e \end{aligned}\right\} \cdots\cdots(31)$$

For the sake of simplicity, superscript e is abbreviated hereafter. Substitution of Eq. (26), (27), (28), (29) and (31) into Eq. (22) yields

$$\begin{aligned} F_{II} = \{v\}^T\int_V\Big[&\rho[N]^T[N]\{\dot{v}^0\} \\ &+\rho[N]^T\Big(\sum_{i=1}^{3}\frac{\partial[N]}{\partial x_i}\cdot\{v^0\}[I_i]\Big)[N]\{v^0\} \\ &-\rho[N]^T\{X^0\}-[B]^T\{J\}[C]\{p\} \\ &+[B]^T[D][B]\{v^0\}\Big]dV \\ &+\{v\}^T\int_A(p^0[N]^T\{n\}-[N]^T[S]\{\tau^0\})dA \end{aligned} \cdots\cdots(32)$$

In Eq. (32), it is noted that F_{II} now means the local potential which is contributed to the local potential for the whole region of interest by an element. V and A are its small integration volume and boundary surface respectively. Let

$$\frac{\delta F_{II}}{\delta\{v\}} = 0 \cdots\cdots(33)$$

and impose the subsidiary conditions $\{v^0\}=\{v\}$ and $\{p^0\}=\{p\}$, then this gives the following ordinary differential equation of motion:

$$\begin{aligned} &\int_V\rho[N]^T[N]dV\cdot\{\dot{v}\} \\ &+\int_V\rho[N]^T\Big(\sum_{i=1}^{3}\frac{\partial[N]}{\partial x_i}\{v\}[I_i]\Big)[N]dV\{v\} \\ &-\int_V[B]^T\{J\}[C]dV\{p\} \\ &+\int_V[B]^T[D][B]dV\{v\} \\ &=\Big(-\int_A p[N]^T\{n\}dA+\int_A[N]^T[S]\{\tau\}dA\Big) \\ &+\int_V\rho[N]^T\{X\}dV \end{aligned} \cdots\cdots(34)$$

Let

$$\frac{\delta F_{II}}{\delta\{p\}} = 0 \cdots\cdots(35)$$

and impose the same subsidiary conditions, then this gives

$$\int_V[C]^T\{J\}^T[B]dV\{v\} = \{0\} \cdots\cdots(36)$$

Eq. (36) expresses the equation of continuity in integrated form. Eq. (34) and (36) are basic finite element equations to determine an incompressible Newtonian fluid flow. In the left hand side of Eq. (34), the first and the second term are the mass matrix and the convective mass matrix respectively. The third term is the compressibility matrix, but Eq. (36) explicitly assures the incompressibility of the fluid (the continuity equation) in an average sense within the domain of finite element. Therefore the unknown nodal pressure $\{p\}$ may be interpreted as a Lagrangian multiplier. The fourth term is the viscous damping matrix. The terms in parenthesis and the third term in the right hand side of Eq. (34) compose the internal nodal force vector by the stress on the boundary surface of the element and by the prescrived body force acting within the element.

Thus, in the finite element formulations, the Ritz method is not applied to the local potential for the whole region of interest, but applied to that for each subdivided region, element. The fundamental equations, Eq. (34) and (36) established for all elements are assembled and interconnected, and the interconnected equations system is simultaneously solved for the unknown parameters.

In the case of a wake flow analysis, it is convenient to use nondimensional quantities as

$$\left.\begin{aligned} x_i&=x_i'D \\ \{v\}&=\{v'\}U \\ p&=p'\rho U^2 \\ \{\tau\}&=\{\tau'\}\rho U^2 \\ t&=t'D/U \\ R_e&=UD/\nu \end{aligned}\right\} \quad\cdots\cdots(37)$$

where D=diameter of obstacle; U=fluid velocity at infinite distance place; ν=kinematic viscosity; R_e=Reynolds number. Assuming the prescribed body force $\{X\}$=zero, Eq. (34) and (36) are represented in nondimensional form, respectively:

$$\begin{aligned} &\int_{V'}[N']^T[N']dV'\cdot\{\dot{v}'\} \\ &+\int_{V}[N']^T\left(\sum_{i=1}^{3}\frac{\partial[N']}{\partial x_i'}\{v'\}[I_i]\right)[N']dV'\cdot\{v'\} \\ &-\int_{V'}[B']^T\{J\}[C']dV'\cdot\{p'\} \\ &+\frac{1}{R_e}\int_{V'}[B']^T[D'][B']dV' \\ =&-\int_{A'}p'[N']^T\{n\}dA'\cdot\{v'\} \\ &+\int_{A'}[N']^T[S]\{\tau'\}dA' \quad\cdots\cdots(38) \end{aligned}$$

$$\int_{V'}[C']^T\{J\}^T[B']dV'\cdot\{v'\}=\{0\} \quad\cdots\cdots(39)$$

where a dash as superscript means the nondimensional quantities.

It is again noted that Eq. (34) and (36) or Eq. (38) and (39) are applicable to an incompressible Newtonian fluid flow through a fixed control volume in space. For an incompressible non-Newtonian fluid flow, the constitutive equation (Eq. 29) has to be changed appropriately.

5. TRANSIENT FLOW THROUGH A RECTANGULAR CHANNEL

In order to compare approximate solutions obtained by the finite element method with theoretical ones, let us consider a transient flow through a rectangular channel as shown in Fig. 3. This problem was treated by Oden[21] for a steady flow. The pressure gradient in the direction of x_1 axis is now prescribed as a time-dependet function. Let us consider Newtonian fluid and assume the

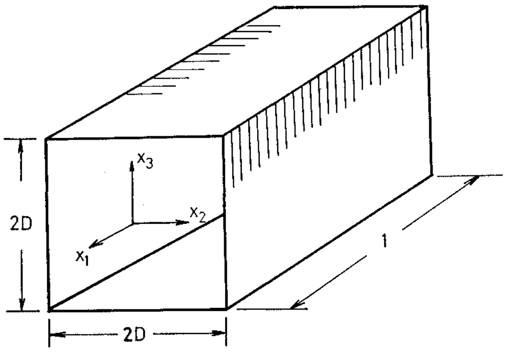

Fig. 3 Square channel with imposed coordinate system.

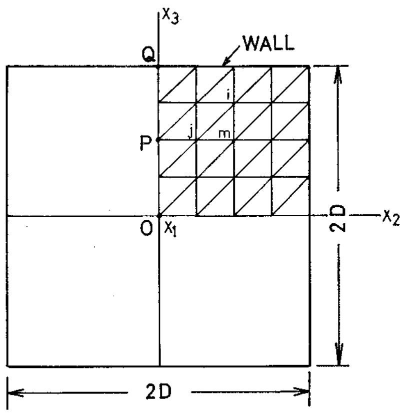

Fig. 4 Cross section of square channel.

fluid velocity to be purely axial everywhere, that is, assume the laminar flow. Then, the convective term of Eq. (34) disappers and the equation of continuity (Eq. 1 or 36) is automatically satisfied. For this two dimensional flow, Eq. (34) is reduced to the following:

$$\int_V \rho[N]^T[N]dV\cdot\{\dot{v}\}+\int_V [B]^T[D][B]dV\cdot\{v\}$$
$$=-\int_A p[N[^T n_1 dA+\int_A [N]^T[S]\{\tau\}dA \quad (40)$$

where

$$[D]=\mu\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \cdots\cdots(41a)$$
$$[S]=[\ n_2 \quad n_3\ ] \quad \cdots\cdots(41b)$$
$$\{\tau\}=\begin{Bmatrix}\tau_{21}\\ \tau_{31}\end{Bmatrix} \quad \cdots\cdots(41c)$$

In Eq. (40), it is noted the unknown parameters are only nodal velocity $\{v\}$ as the pressure gradient is prescribed. The triangular element with three nodes in which the fluid velocity varies linearly is used. The whole region (2D×2D×unit length) is subdivided to 32 element per a quater of the region as shown in Fig. 4. The unknown nodal velocity vector $\{v\}$ and the interpolation function $[N]$ in Eq. (40) are in the following:

$$\{v\}=\begin{Bmatrix}v_i\\ v_j\\ v_m\end{Bmatrix} \quad \cdots\cdots(42)$$

$$[N]=\frac{1}{2\Delta}\begin{bmatrix}a_i+b_ix_2+c_ix_3\\ a_j+b_jx_2+c_jx_3\\ a_m+b_mx_2+c_mx_3\end{bmatrix}^T \quad \cdots\cdots(43)$$

where v_i, v_j and v_m are the unknown velocities in the direction of x_1 axis at three nodes (i, j, m) and denoting the coordinate x_n of node i by x_{ni},

$$a_i=x_{2j}x_{3m}-x_{2m}x_{3j}$$
$$b_i=x_{3j}-x_{3m}$$
$$c_i=x_{2m}-x_{2j} \quad \text{etc.} \quad \cdots\cdots(44)$$

Δ=area of a triangular element (i, j, m)

The matrix $[B]$ is derived from $[N]$ using the definition of Eq. (21). Substitution of Eq. (43) into Eq. (40) yields

$$[M]\{\dot{v}\}+[K]\{v\}=\{P\} \quad \cdots\cdots(45)$$

where

$$[M]=\int_V \rho[N]^T[N]dV=\frac{\rho\Delta}{12}\begin{bmatrix}2 & 1 & 1\\ & 2 & 1\\ \text{SYM.} & & 2\end{bmatrix}$$
$$\cdots\cdots(46a)$$

$$[K]=\int_V [B]^T[D][B]dV$$
$$=\frac{\mu}{4\Delta}\begin{bmatrix}b_i^2+c_i^2 & b_ib_j+c_ic_j & b_ib_m+c_ic_m\\ & b_j^2+c_j^2 & b_jb_m+c_jc_m\\ \text{SYM.} & & b_m^2+c_m^2\end{bmatrix}$$
$$\cdots\cdots(46b)$$

$$\{P\}=-\frac{\partial p(t)}{\partial x_1}\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}\cdot\frac{\Delta}{3} \quad \cdots\cdots(46c)$$

Eq. (45) is in agreement with the equation analogue, governed by a quasiharmonic differential equation. This is obvious because for this two-dimensional flow the Navier-Stokes equation is reduced to the quasiharmonic differential equation.

Let us consider the case in which the pressure gradient in the x_1 direction suddenly occurs as shown in Fig. 5. The pressure gradient of Eq. (46c) is expressed as

$$\frac{\partial p(t)}{\partial x_1}=S(t)\frac{dp}{dx_1} \quad \cdots\cdots(47)$$

where $S(t)$=unit step function. Eq. (45) may be computed by the finite-difference method with respect to time. Herein, the step by step method developed by Wilson and Clough[22] was used. Table 1, 2 and 3 show a comparison between the approximate solutions obtained by the method and the theoretical ones derived using separation of variables, assuming the ratio $\nu/D^2=0.2504$ (sec^{-1}). The time increment Δt in the method was 0.001 (sec) and the computed values converged into significant five figures. Hence, the percentage of error in these Tables does not include influences brought about by the time increment Δt. In these Tables points O, P and Q are shown in Fig. 4 as typical points.

The errors of fluid velocity (Table 1) and of shear stress (Table 2) are generally larger than those of discharge G obtained by integrating the fluid velocity over the area ($2D\times2D$) and those of viscous drag force R obtained by integrating the viscous shear stress acting on the wall, respectively. This arises from the fact that the

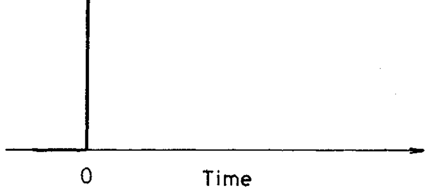

Fig. 5 Pressure variation with time.

Table 1 Fluid velocity variation with time at point O and P in a square channel ($\nu/D^2=0.2504$ sec)

Time sec	Velocity u_0 $\times\frac{D^2}{\mu}\left\|\frac{dp}{dx_1}\right\|$	u_0 exact	Error %	Velocity u_p $\times\frac{D^2}{\mu}\left\|\frac{dp}{dx_1}\right\|$	u_p exact	Error %
0.04	0.1007×10^{-1}	0.1001×10^{-1}	0.60	0.9985×10^{-2}	0.1000×10^{-1}	0.15
0.16	0.4007×10^{-1}	0.4005×10^{-1}	0.05	0.4090×10^{-1}	0.3915×10^{-1}	4.47
0.36	0.9133×10^{-1}	0.8876×10^{-1}	2.90	0.8348×10^{-1}	0.8050×10^{-1}	3.70
0.64	0.1526	0.1465	4.16	0.1273	0.1240	2.66
1.00	0.2074	0.1993	4.06	0.1649	0.1618	1.92
1.21	0.2297	0.2210	3.94	0.1800	0.1772	1.58
1.44	0.2482	0.2393	3.72	0.1925	0.1901	1.26
1.69	0.2630	0.2540	3.54	0.2025	0.2006	0.95
1.96	0.2745	0.2655	3.39	0.2103	0.2087	0.77
2.25	0.2832	0.2743	3.24	0.2162	0.2149	0.60
∞	0.3028	0.2947	2.75	0.2295	0.2293	0.09
	(0.2949)	(〃)	(0.07)	(0.2289)	(〃)	(0.17)

Table 2 Viscous shear stress variation with time at point P and Q in square channel ($\nu/D^2=0.2504$ sec)

Time sec	Viscous stress τ_p $\times D\left\|\frac{dp}{dx_1}\right\|$	τ_p exact	Error %	Viscous stress τ_q $\times D\left\|\frac{dp}{dx_1}\right\|$	τ_q exact	Error %
0.04	0.3778×10^{-4}	0.1459×10^{-4}	158.94	0.1099	0.1125	2.31
0.16	0.1067×10^{-1}	0.8790×10^{-2}	21.39	0.2104	0.2255	6.70
0.36	0.5880×10^{-1}	0.4922×10^{-1}	19.46	0.3164	0.3369	6.08
0.64	0.1227	0.1091	12.47	0.4156	0.4396	5.46
1.00	0.1813	0.1669	8.62	0.4984	0.5248	5.03
1.21	0.2052	0.1910	7.43	0.5318	0.5592	4.90
1.44	0.2250	0.2112	6.53	0.5594	0.5878	4.83
1.69	0.2410	0.2275	5.93	0.5817	0.6110	4.80
1.96	0.2533	0.2404	5.37	0.5989	0.6291	4.80
2.25	0.2627	0.2501	5.04	0.6119	0.6429	4.82
∞	0.2837	0.2727	4.03	0.6411	0.6749	5.01
	(0.2744)	(〃)	(0.62)	(0.6719)	(〃)	(0.44)

Table 3 Viscous drag force and discharge variation with time in a square channel ($\nu/D^2=0.2504$ sec)

Time sec	Viscous drag force R $\times D^2\left\|\frac{dp}{dx_1}\right\|$	R exact	Error %	Discharge G $\times\frac{D^4}{\mu}\left\|\frac{dp}{dx_1}\right\|$	G exact	Error %
0.04	0.9181	0.8514	7.83	0.3249×10^{-1}	0.3428×10^{-1}	5.22
0.16	1.605	1.602	0.19	0.1125	0.1161	3.10
0.36	2.234	2.250	0.71	0.2134	0.2184	2.29
0.64	2.774	2.796	0.79	0.3141	0.3203	1.94
1.00	3.212	3.234	0.68	0.3997	0.4075	1.91
1.21	3.387	3.409	0.65	0.4342	0.4429	1.96
1.44	3.533	3.555	0.62	0.4628	0.4724	2.03
1.69	3.649	3.673	0.65	0.4858	0.4963	2.12
1.96	3.740	3.766	0.69	0.5036	0.5150	2.21
2.25	3.808	3.836	0.47	0.5170	0.5293	2.32
∞	3.962	3.999	0.93	0.5472	0.5623	2.69
	(3.933)	(〃)	(0.15)	(0.5622)	(〃)	(0.02)

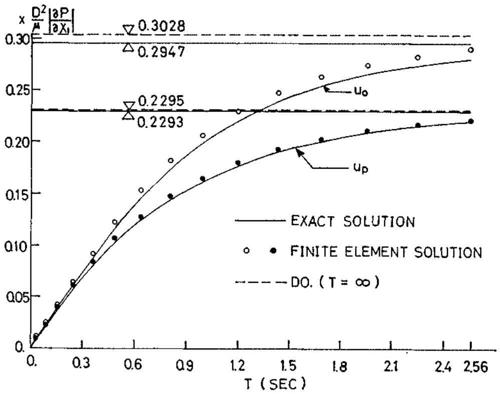

Fig. 6 Fluid velocity variation with time at point 0 and P in a square channel ($\nu/D^2=0.2504$ sec)

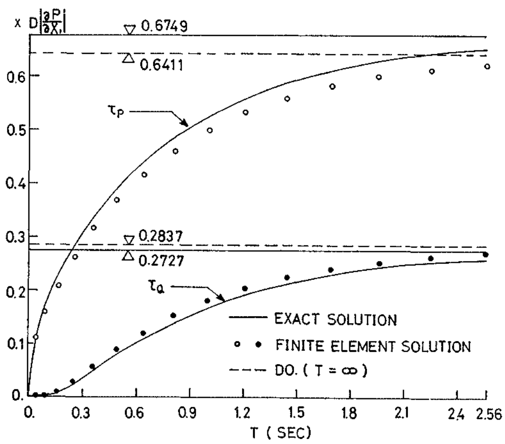

Fig. 7 Viscous shear stress variation with time at point P and Q in a square channel ($\nu/D^2=0.2504$ sec)

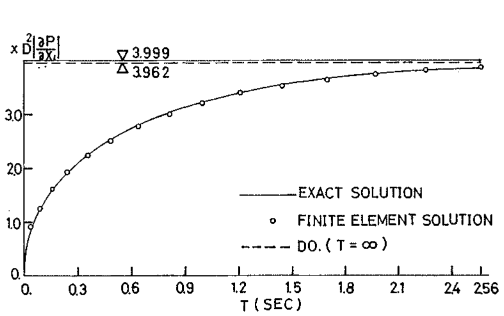

Fig. 8 Viscous drag force variation with time in a square channel ($\nu/D^2=0.2504$ sec)

integrating processes result in an averaging of the fluid velocity within the integration area and of the viscous shear stress on the integration

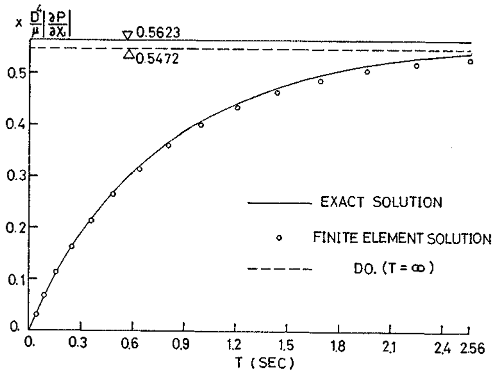

Fig. 9 Discharge variation with time in a square channel ($\nu/D^2=0.2504$ sec)

surface, the wall of the channel. In these Tables, the values in parentheses at $t=\infty$ are another approximate solutions presented by Sparrow and Siegel[23] for the steady flow. It is noted these values were obtained by approximating the velocity distribution within the entier region ($2D \times 2D$) with an eighth order polynomial expansion. Fig. 6, 7, 8 and 9 show the results of these Tables in illustration form.

6. SUMMARY AND CONCLUSIONS

Developing the technique presented by Lemieux et al.,[19] a modified variational formulation is proposed. The variational formulation can be shown to be correct by insuring that the Euler-Lagrangian equations are identical with the balance equations, that is, the Navier-Stokes equation and the contuinity equation and may be applied to the problems in which the stress or the fluid velocity is specified on the boundary surface.

To acknowledge similalities and differences between fluid dynamics and solid dynamics and to carry out computations systematically, the local potential is expressed in a matrix form.

In the finite element method, the entire region of interest is subdivided into a finite number of subregions, that is, elements. The fluid velocity or its derivative values of the nodal points of the elements is the unknown parameter of the problem. If the pressure is not prescribed except for a part of the boundary surface of the whole region, the pressure or its derivative value will be also the unknown parameter of the problem. Generally, the fluid velocity and the pressure within an element are approximated by different polynomial expansions, respectively and the parameters of these expansions are uniquely determined by the coordinates of nodes of the element and the unknown nodal parameters. The finite element equations are derived by minimizing the local potential with respect to the unknown nodal parameters. In other words, the Ritz method is applied to the local potential for the element. The equations of motion and the continuity equations obtained for all elements are assembled and interconnected, and the interconnected entire equations system is solved for the unknown parameters.

Applications of the Ritz method to the local potential are, of course, not new as a direct approximation method. By its nature, the conventional Ritz process is limited to relatively simple geometrical shapes of the whole region. On the contrary, in the finite element method, the whole region is regarded as the assembly of simple element shapes. The local potential for the whole region is also regarded as the sum of the contributions of each element and the Ritz method is applied to the local potentials for each element. Thus, the finite element method functions best in the problems involving complex geometrical shapes of the whole region.

In order to compare the approximate solution obtained by using the finite element method with the theoretical one, the problem of the determination of the fluid velocity and the shear stress distribution associated with the transient flow of a Newtonian fluid through a square channel was studied. In this problem, only the velocities of nodes of elements were unknown parameters as the pressure gradient was prescrived as the time-dependent function, that is, the unit step function. This brief study indicated the remarkable accuracy of the approximate solutions. It can not be concluded whether the approximate solutions presented are more accurate than those which could be obtained by using other approximation methods or not. However, at least, it can be said that the finite element method opens a new applicability of the Ritz method to the variational principle based on the concept of the local potential. Applications to the problems which include the convective term and in which the nodal pressures are also unknown parameters are now in progress.

The finite element equations presented in this paper were not compared with those given by Oden[10] and others using the Galerkin's method. As set forth by Schechter[17] generally, also in the finite element method, it could be shown that the equations presented are equivalent to those given by the Galerkin's method. The local potential approach has, however, the additional advantage of deriving the equations systematically by using the extremal conditions on the unknown parameters.

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APPENDIX 1. MATRIX EXPRESSION OF LOCAL POTENTIAL

In Eq. (22), the convective term in the volume integral is especially explained in the following.

The convective term of Eq. (13) is expanded as

$$\rho u_j^0 \frac{\partial u_i^0}{\partial x_j} u_i$$
$$=\left(u_1\frac{\partial u_1^0}{\partial x_1}+u_2\frac{\partial u_2^0}{\partial x_1}+u_3\frac{\partial u_3^0}{\partial x_1}\right)u_1^0$$
$$+\left(u_1\frac{\partial u_1^0}{\partial x_2}+u_2\frac{\partial u_2^0}{\partial x_2}+u_3\frac{\partial u_3^0}{\partial x_2}\right)u_2^0$$
$$+\left(u_1\frac{\partial u_1^0}{\partial x_3}+u_2\frac{\partial u_2^0}{\partial x_3}+u_3\frac{\partial u_3^0}{\partial x_3}\right)u_3^0$$

Using the velocity vector $\{u\}$ and the matrix $[I_i]$ deffined by Eq. (17) and (23a) respectively, this

$$=\{u\}^T\cdot\frac{\partial\{u^0\}}{\partial x_1}u_1^0+\{u\}^T\cdot\frac{\partial\{u^0\}}{\partial x_2}u_2^0$$
$$+\{u\}^T\cdot\frac{\partial\{u^0\}}{\partial x_3}\cdot u_3^0$$
$$=\{u\}^T\cdot\left(\sum_{i=1}^{3}\frac{\partial\{u^0\}}{\partial x_i}\cdot u_i^0\right)$$
$$=\{u\}^T\cdot\left(\sum_{i=1}^{3}\frac{\partial\{u^0\}}{\partial x_i}[I_i]\right)\{u^0\} \quad\cdots\cdots\cdots\cdots\cdots(48)$$

Eq. (48) is in agreement with the second term of Eq. (22). Another expressions of the term is possible, but Eq. (48) is the most simple expression. Other terms of Eq. (22) are analogously derived.

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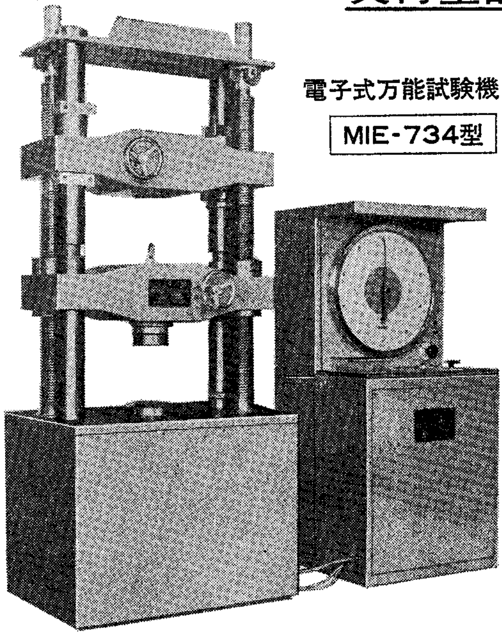

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