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THE ULTIMATE SHEAR STRENGTH OF THE BRACKETED PARTS OF REINFORCED CONCRETE BEAMS

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SYNOPSIS

The beams with brackets at their ends usually fail as a result of shear compression. The ultimate strength of this mode of failure of reinforced concrete beams was investigated. The beam-truss structure was proposed as the model to express the relation between the shear force acting on the reentrant corner of the bracketed parts of beams and the force in inclined steel. Equations of ultimate strength are derived. The ultimate strength is evaluated with regard to three cases, depending on the crushing of concrete or yield of steel. The beams were tested under moving load applied at various points along the beam until failure occured to simulate actual field conditions. The theoretical values are presented with the comparison of the test results, and show reasonably good agreement. Design approach is also described.

1. INTRODUCTION

There is some confusion in the terminology used for describing the different modes of shear failure in the extensive literature on shear. However, in a broad way, shear failure of uniform section beams may be classified into 4 modes; (a) shear compression failure, (b) direct shear, (c) shear bond failure, (d) the others, such as web crushing failure. For the beams with bracket at their end, it may fail in the modes of (b), (c), and (d) as uniform section beams, but failure is more common by the crushing of concrete above the upper end of the "corner crack", which is developed from the reentrant corner of the bracketed part of the concrete beam at an a/ds ratio of about 2.5 [8]. This mode of failure may be classified as the mode of (a), this paper is to deal with the ultimate strength of this mode of failure of the beams with bracket.

The internal structural system at ultimate state of bracketed parts of a beam's end are complex. The simplified model which is founded on the experimental studies of crack patterns is considered to simulate this internal structural system. Basing on this analogy, the approximate equation expressing the force in inclined steel is proposed.

The ultimate strength equations are derived using the equation obtained from the model analogy together with the three equilibrium equations.

A number of recent papers[10)~13)] on this subject have researched various aspects of this problem. However, a definitive general solution remains to be attained.

2. TEST SPECIMENS

Beam dimensions and loading arrangements are shown in Figs. 1, 2, 3, and 4. The specimens identified by the letter RC are reinforced without prestressing. The letter T denotes T section specimens. The letter D and E denote the specimens which were tested by the two points load. The letter A denotes the specimens which were tested by the moving load, applying the load from A load position to G load position at each increment of loading until failure occurred. Material properties are shown in Table 1.

3. FAILURE MECHANISMS

The beams with brackets at their end subjected to the moving load have the differential mechanism of failure from the usual two points loading. When the beams are subjected to approximatly 50% of design load at the point which is about twice the distance of the effective depth of brackets from the point of support, the corner crack,

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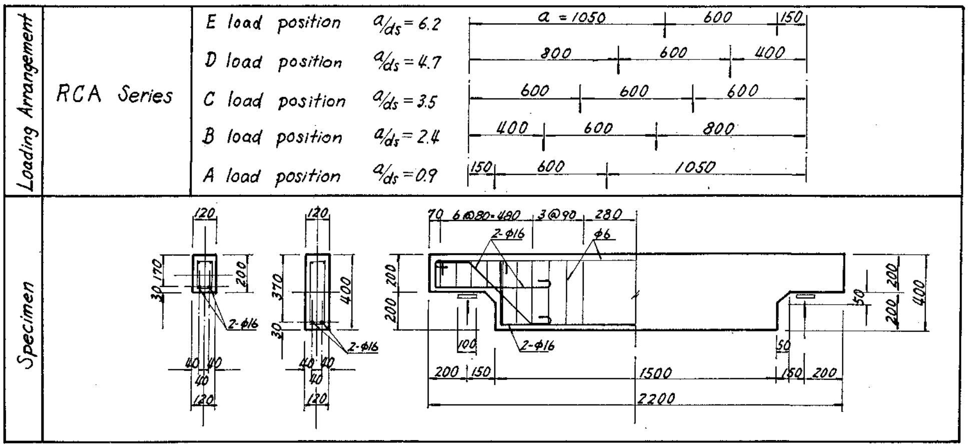


Fig. 1 Specimens and loading arrangement of reinforced concrete rectangular beams (moving load test).

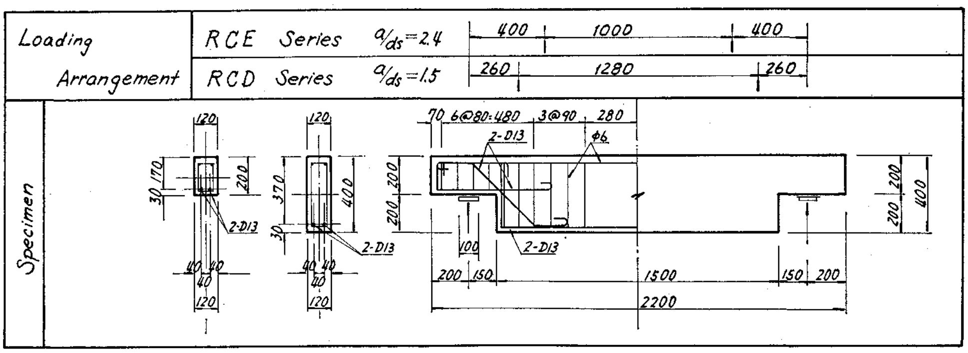


Fig. 2 Specimens and loading arrangement of reinforced concrete rectangular beams (two points loading test).

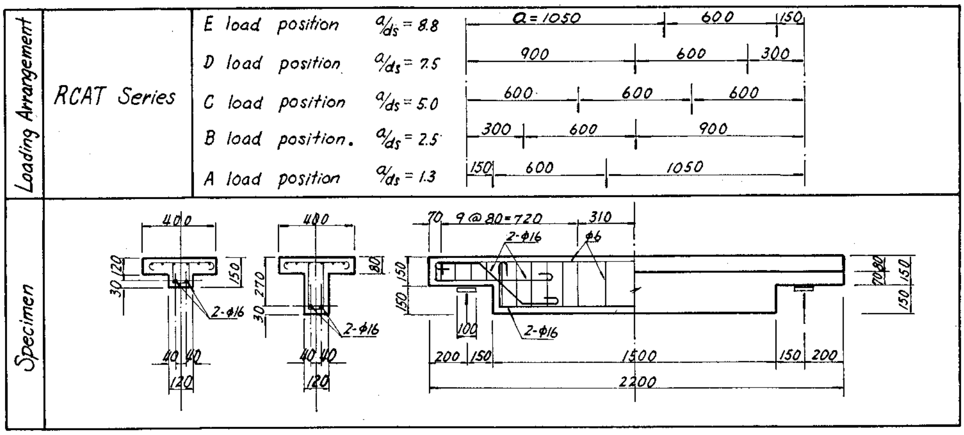


Fig. 3 Specimens and loading arrangement of reinforced concrete T beams (moving load test).

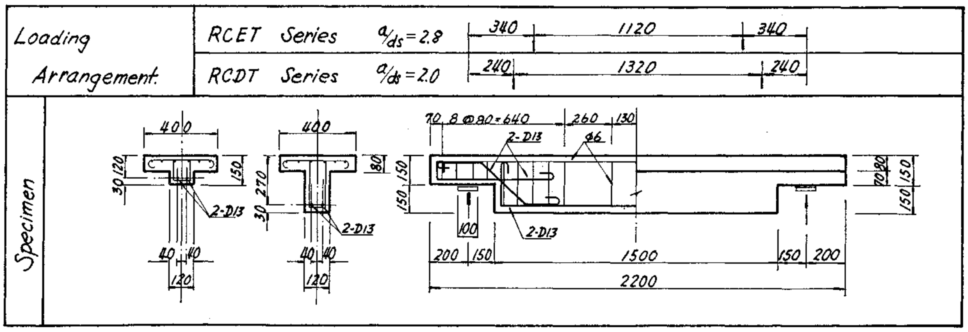


Fig. 4 Specimens and loading arrangement of reinforced concrete T beams (two points loading test).

Table 1 Experimental and calculated analysis of results

Specimen Mark	Experimental Value								Ultimate Load	
	$\frac{a}{ds}$	l_H (cm)	l_V (cm)	l_D (cm)	θ_D (deg.)	σ_{SY} (kg/cm²)	σ_{ck} (kg/cm²)	Mode of Failure	Pu (tons)	means (tons)
RCA-1 RCA-2	$\frac{40}{17}=2.4$	26	22	32	45	3 200	358	Shear	17.38 18.00	17.69
RCD-1 RCD-2	$\frac{26}{17}=1.5$	8	8	18	45	2 930	412	Shear	21.25 22.00	21.63
RCE-1 RCE-2	$\frac{40}{17}=2.4$	21	22	28	45	2 930	412	Flexual	19.50 19.85	19.68
RCAT-1 RCAT-2	$\frac{30}{12}=2.5$	12	12	19	45	3 200	458	Flexual	15.28 14.14	14.71
RCDT-1 RCDT-2	$\frac{24}{12}=2.0$	6	6	14	45	2 930	323	Shear	16.70 19.85	18.28
RCET-1 RCET-2	$\frac{34}{12}=2.8$	15	16	21	45	2 930	323	Flexual	13.00 14.15	13.58

Specimen Mark	Calculated Value: Shear Failure Case A (tons)	Shear Failure Case B (tons)	Shear Failure Case C (tons)	Flexual Failure Eq. of ACI (tons)	$\frac{\text{Calculated}}{\text{Experimental}}$
RCA-1 RCA-2	17.07	34.24	—	18.54	$\frac{17.07}{17.69}=0.96$
RCD-1 RCD-2	20.93	22.90	19.93	20.67	$\frac{19.93}{21.63}=0.92$
RCE-1 RCE-2	—	—	21.42	13.44	$\frac{13.44}{19.68}=0.68$
RCAT-1 RCAT-2	24.33	26.59	22.53	17.10	$\frac{17.10}{14.71}=1.16$
RCDT-1 RCDT-2	15.54	19.03	15.73	16.53	$\frac{15.54}{18.28}=0.85$
RCET-1 RCET-2	19.28	20.77	15.96	11.64	$\frac{11.64}{13.58}=0.86$

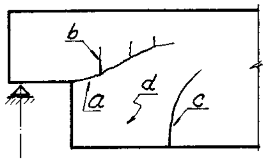


Fig. 5 Idealized pattern of corner crack developed by moving loads.

a, usually develops from the reentrant corner of the brackets[5),8)] as shown in Fig. 5.

After the moving load is increased and applied at various points along the beam from one side to the another over and over, the beam may develop the flexual crack from midspan. Subsequently the corner cracks and flexual cracks increase their length and the area of the compression zone decreases.

When the moving load is applied again on the top of the beam of corner cracked part, the secondary cracks b (Fig. 5) will branch out from the previously developed corner crack and extend upward directly toward the load point. These secondary crack are considered to occur as a result of bending of the bracket together with the entire tapered block which is above the corner crack. At this step, the beams have lost their special properties of abrupt change in depth of cross section and have been transformed into a statical indeterminate structure as regard to inner forces with high order, the compression zone of the cracked beam may be considered as the member to subject bending, shear and axial force, the concrete d (Fig. 5) between corner crack a and flexual-shear crack c (Fig. 5) may be considered as inclined members, and the steel reinforcement may be considered as tension members.

The failure takes place usually when the moving load approaches to the point about 2.5 times of effective depth of bracket from the support. The failure occurred initially in two ways: (1) crushing of concrete in the compression zone (2) yielding of steel. However, in both cases the beams reach maximum load when concrete in the compression zone crushes. Final failures in both cases appear somewhat similar.

4. MODEL ANALOGY

The model shown in Fig. 6 is adopted to represent the internal structural system of ultimate state of brackted part of concrete beam end. This model belongs to the beam-truss structure and is statically indeterminate to the first degree. The notations of each model member obtain as shown in Fig. 6.

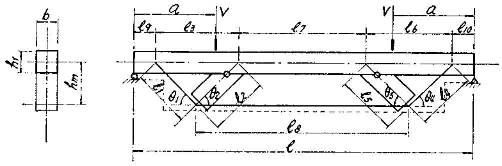


Fig. 6 Mechanical model of the beam with bracket.

(1) Assumption

1. members 3, 6, 7, 9, 10, behave as the beam, members 2, 5, behave as inclined members to take compression and the members 1, 4, 8, behave as the tension members to take tension only
2. material property of each member of model is equal to the portion of the ordinal beam that model member represents.

(2) Derivation of Equations

The model is subjected to the loads 2 V, and it is on the equilibrium state, the work done by the model may be written as

$$W=\int\frac{N^2}{2EA}dx+\int\frac{M^2}{2EI}dx \quad \cdots\cdots\cdots\cdots\cdots(1)$$

Setting the axial force D of member 1 as the statically indeterminate quantity, the force D can be found from Eq. (2)

$$\frac{\partial W}{\partial D}=\int\frac{\partial N}{\partial D}\frac{N}{EA}dx+\int\frac{\partial M}{\partial D}\frac{M}{EI}dx=0 \quad \cdots\cdots\cdots\cdots\cdots(2)$$

The first term of equation (2) is the first degree of D and the second term is the first degree of V and D, with a rearrangement, these take the form

$$D=\frac{\eta_2}{\eta_1+\eta_3}V \quad \cdots\cdots\cdots\cdots\cdots(3)$$

where η_1, η_2, and η_3 are coefficient which are determined from the sharp, material properties of members and the loading condition, these are as follow:

$$\eta_1=\frac{l_1}{A_1E_s\sin^2\theta_1}+\frac{l_2}{A_2E_c\sin^2\theta_2}+\frac{l_3}{A_3E_c\tan^2\theta_1}+\frac{l_4}{A_4E_s\sin^2\theta_4}\frac{H^2}{H'^2}+\frac{l_5}{A_5E_c\sin^2\theta_5}\frac{H^2}{H'^2}+\frac{l_6}{A_6E_c\tan^2\theta_4}\frac{H^2}{H'^2}+\frac{l_7H^2}{A_7E_c}+\frac{l_8H^2}{A_8E_s}$$

for $l_9<a<(l_9+l_3)$

$$\left.\begin{aligned}\eta_2&=\frac{3al_3^2-(a-l_9)^3}{6E_cI_3\sin\theta_1}+\frac{3al_6^2-(a-l_{10})^3}{6E_cI_6\sin\theta_1}\frac{H}{H'}\\&\quad+\frac{al_6l_7}{E_cI_7\sin\theta_1}\frac{H}{H'}\\\eta_3&=\frac{l_3^3}{3E_cI_3}+\frac{l_6^3}{3E_cI_6}\frac{H^2}{H'^2}+\frac{l_6^2l_7}{E_cI_7}\frac{H^2}{H'^2}\end{aligned}\right\}$$

for $(l_9+l_3)<a<l/2$

$$\begin{aligned}\eta_2&=\frac{l_3^2(2l_3+3l_9)}{6E_cI_3\sin\theta_1}+\frac{l_6^2(2l_6+3l_{10})}{6E_cI_6\sin\theta_1}\frac{H}{H'}\\&\quad+\frac{l_3(a+l_3+l_9)(a-l_3-l_9)}{2E_cI_7\sin\theta_1}\\&\quad+\frac{l_6(a+l_6+l_{10})(a-l_6-l_{10})+2al_6(l-2a)}{2E_cI_7\sin\theta_1}\frac{H}{H'}\\\eta_3&=\frac{l_3^3}{3E_cI_3}+\frac{l_6^3}{3E_cI_6}\frac{H^2}{H'^2}+\frac{l_3^2(a-l_3-l_9)}{E_cI_7}\\&\quad+\frac{l_6^2(l-l_6-l_{10}-a)}{E_cI_7}\frac{H^2}{H'^2}\end{aligned}\qquad\cdots\cdots(4)$$

(3) Simplified Equations

Giving equations more acceptable for design purposes, the following assumption are made to reduce the number of parameters.

1. Considering the crack pattern of the beam, the model is assumed to be symmetrical, and in uniform sections.
2. The work done by axial force in model is neglected, that is $\eta_1=0$.
3. Length of bracket is short enough to neglect, $l_9=0$.
4. Considering the ordinal proportion of beam depth span ratio, and the shear span effective depth ratio which would usually cause the shear compression failure, the terms which are less effective to the results are neglected.

From the assumptions 1 to 3 the Eq. (4) may simplified as follow:

$(\cot\theta_1+\cot\theta_2)hm\geqq a$

$$\frac{\eta_2}{\eta_3}=\left[\frac{3(\lambda-H)}{(3\lambda-4H)H\sin\theta_1}\right]\left(\frac{a}{hm}\right)-\left[\frac{1}{(3\lambda-4H)H^2\sin\theta_1}\right]\left(\frac{a}{hm}\right)^3$$

$(\cot\theta_1+\cot\theta_2)hm<a<l/2$

$$\frac{\eta_2}{\eta_3}=\left[\frac{3\lambda}{(3\lambda-4H)H\sin\theta_1}\right]\left(\frac{a}{hm}\right)-\left[\frac{3}{(3\lambda-4H)H\sin\theta_1}\right]\left(\frac{a}{hm}\right)^2-\left[\frac{H}{(3\lambda-4H)\sin\theta_1}\right]$$

and from the assumption 4, the simplified equation may be

$$D=\frac{\sin\theta_2}{2\sin(\theta_1+\theta_2)}\frac{a}{ds}V\qquad\cdots\cdots(5)$$

Assuming the inclined compression members of

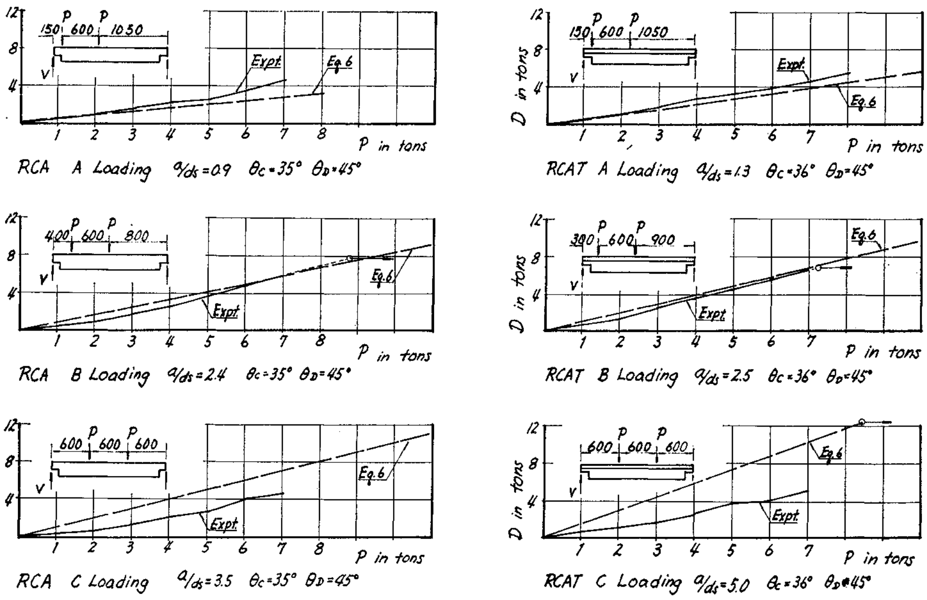


Fig. 7 Relationships between the applied load and tension force in inclined steel (moving load test).

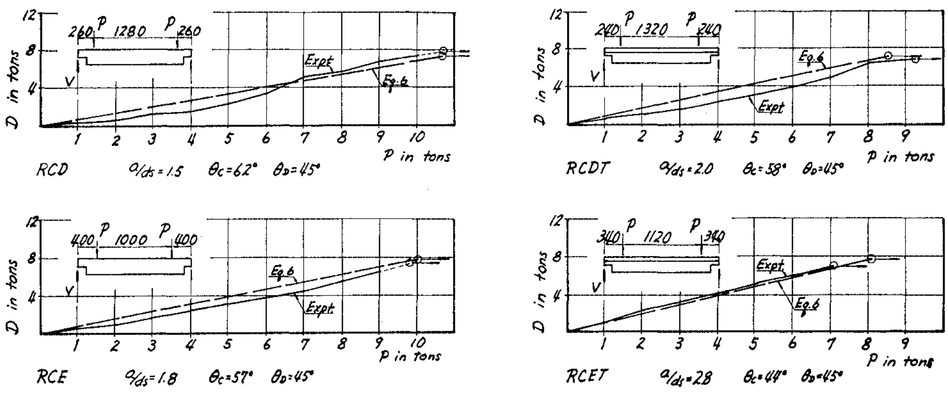


Fig. 8 Relationships between the applied load and tension force in inclined steel (two points load test).

the model are parallel to the corner cracks in the relation between the shear force acted on the reentrant corner of the bracketed parts of the beams and the force in inclined steel bar, an approximate equation may be written as follows

$$\sigma_{SD}A_{SD}=\frac{\sin\theta_c}{2\sin(\theta_c+\theta_D)}\frac{a}{ds}V \qquad \cdots\cdots\cdots(6)$$

$$\sigma_{SD}\leqq\sigma_{SYD}$$

The computed results using Eq. (6) are compared with the experimental result as shown in Figs. 7 and 8.

5. ULTIMATE STRENGTH

(1) Assumptions

The failure surface considered is as shown in Fig. 9 and it is on the equilibrium of forces of this surface that the analysis is based. The other assumptions considered are as follows:

1. Any shear transfer across the crack caused either by aggregate interlock or dowel action are neglected.
2. Local effect, such as caused by reactions and loads, are neglected.
3. The stress in vertical steel bar is equal to that of inclined steel bar.
4. The bracket should be provided with the inclined steel.

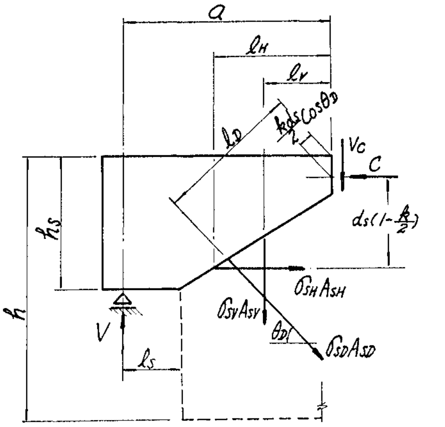


Fig. 9 Free body diagram of beam end with bracket.

(2) Derivation of Equations

Considering the state of stress at failure as given in Fig. 9, the following equations of equilibrium for the cut section may be written.

$\sum H=0$

$$bkds\alpha\sigma_{ck}=\sigma_{SH}A_{SH}+\sigma_{SD}A_{SD}\cos\theta_D \qquad \cdots\cdots(7)$$

$\sum V=0$

$$V_u=\sigma_{SV}A_{SV}+\sigma_{SD}A_{SD}\sin\theta_D+V_c \qquad \cdots\cdots(8)$$

$\sum M=0$

$$V_u a=\sigma_{SH}A_{SH}ds\left(1-\frac{k}{2}\right)+\sigma_{SV}A_{SV}l_V+\sigma_{SD}A_{SD}\left(l_D-\frac{kds}{2}\cos\theta_D\right) \qquad \cdots\cdots(9)$$

The above three equations are insufficient in number to determine all unknown quantities, as the additional conditions or equations for computing the pertinent unknowns, the aforementioned equations of model analogy were applied.

From the conditions of geometry,

$$\cot\theta_c=\frac{l_H}{(1-k)ds} \qquad \cdots\cdots(10)$$

substituting Eq. (10) into Eq. (6) yields

$$\sigma_{SW}A_{SD}=\frac{(1-k)a}{2[(1-k)ds\cos\theta_D+l_H\sin\theta_D]}V_u \quad \cdots\cdots(11)$$

where A_{SH}, A_{SV}, A_{SD}, and l_H are given by assuming the failure of the brackets occurred along a diagonal plane between the load and the re-entrant corner of the bracket.

Three cases triggering the failure, which depend on the yielding of steel and crushing of concrete are considered here. These as follows:

Case A Crushing of concrete.

Case B Yielding of horizontal steel bar.

Case C Yielding of inclined steel bar and vertical steel bar simultaneously.

The ultimate shear strength V_u is calculated by the following equations. The computed results used by these equations are compared with the experimental result as shown in Table 1.

Case A;

According to the other assumption of V_c neglected in this case, the four remaining unknowns are σ_{SH}, σ_{SW}, V_u, and k, these unknowns are determined from solving Eqs. (7), (8), (9), and (11), then the results of the solution are given by Eqs. (12a), (13a), (14a) and (15a)

$$k=1-\frac{2l_H}{a}\left[\frac{\sin\theta_D}{\frac{A_{SV}}{A_{SD}}+\sin\theta_D-\frac{2ds}{a}\cos\theta_D}\right] \quad \cdots\cdots(12a)$$

$$1>k>0$$

$$\sigma_{SH}=\frac{bkds\alpha\sigma_{ck}}{A_{SH}}\times\left[\frac{A_{SV}(a-l_V)+A_{SD}(a\sin\theta_D-l_D)}{A_{SV}(a-l_V)+A_{SD}(a\sin\theta_D+ds\cos\theta_D-l_D)}\right] \quad \cdots\cdots(13a)$$

$$\sigma_{SYH}\geqq\sigma_{SH}$$

$$\sigma_{SW}=\frac{bkds^2\alpha\sigma_{ck}\left(1-\frac{k}{2}\right)}{A_{SV}(a-l_V)+A_{SD}(a\sin\theta_D+ds\cos\theta_D-l_D)} \quad \cdots\cdots(14a)$$

$$\sigma_{SYW}\geqq\sigma_{SW}$$

$$V_u=\frac{bds^2\alpha\sigma_{ck}k(1-k)}{A_{SV}(a-l_V)+A_{SD}\left[a\sin\theta_D\frac{\times(A_{SV}+A_{SD}\cos\theta_D)}{+ds\left(1-\frac{k}{2}\right)\cos\theta_D-l_D}\right]} \quad \cdots\cdots(15a)$$

$$\sigma_{SYW}\geqq\sigma_{SW}\,,\qquad \sigma_{SYH}\geqq\sigma_{SH}$$

Case B;

The similar equations can be drived from Eqs. (7), (8), (9), and (11) from the failure standard, horizontal steel being yield.

$$\Psi_a k^2-[\Psi_a+q_H\Psi_b-\Psi_c]k+q_H(\Psi_b-\Psi_c)=0 \quad \cdots\cdots(12b)$$

$$1>k>0$$

$$\sigma_{SH}=\sigma_{SYH} \quad \cdots\cdots(13b)$$

$$\sigma_{SW}=\frac{bkds\alpha\sigma_{ck}-\sigma_{SYH}A_{SH}}{A_{SD}\cos\theta_D} \quad \cdots\cdots(14b)$$

$$\sigma_{SYW}\geqq\sigma_{SW}$$

$$V_u=\frac{\sigma_{SYH}A_{SH}ds}{a}\left(1-\frac{k}{2}\right)+\frac{bkds\alpha\sigma_{ck}-\sigma_{SYH}A_{SH}}{aA_{SD}\cos\theta_D}\times\left[A_{SV}l_V+A_{SD}\left(l_D-\frac{kds}{2}\cos\theta_D\right)\right] \quad \cdots\cdots(15b)$$

$$\sigma_{SYW}\geqq\sigma_{SW}\,,\qquad \sigma_{SYH}=\sigma_{SH}$$

where

$$\Psi_a=\frac{A_{SV}}{A_{SD}}\frac{l_V}{ds}+\frac{l_D}{ds}-2\cos\theta_D$$

$$\Psi_b=\frac{A_{SV}}{A_{SD}}\frac{l_V}{ds}+\frac{l_D}{ds}-3\cos\theta_D$$

$$\Psi_c=\frac{2l_H}{ds}\sin\theta_D$$

$$q_H=\frac{A_{SH}}{bds}\frac{\sigma_{SYH}}{\alpha\sigma_{ck}}$$

Case C;

In this case, the ultimate shear strength is determined by the assumption that vertical and inclined steel reached the yield strength. The derived equations which were simplified are as follow.

$$[3+q_D\cos\theta_D]k^2+[2q_D\Psi_b-q_D\cos\theta_D-2]k-2q_D[\Psi_b-\Psi_c]=0 \quad \cdots\cdots(12c)$$

$$1>k>0$$

$$\sigma_{SH}=\frac{bkds\alpha\sigma_{ck}-\sigma_{SYD}A_{SD}\cos\theta_D}{A_{SH}} \quad \cdots\cdots(13c)$$

$$\sigma_{SYH}\geqq\sigma_{SH}$$

$$\sigma_{SW}=\sigma_{SYW} \quad \cdots\cdots(14c)$$

$$V_u=\frac{(bkds\alpha\sigma_{ck}-\sigma_{SYD}A_{SD}\cos\theta_D)ds}{a}\left(1-\frac{k}{2}\right)+\frac{\sigma_{SYD}}{a}\left[A_{SV}l_V+A_{SD}\left(l_D-\frac{kds\cos\theta_D}{2}\right)\right] \quad \cdots\cdots(15c)$$

$$\sigma_{SYH}\geqq\sigma_{SH}\,,\qquad \sigma_{SYW}=\sigma_{SW}$$

where

$$q_D=\frac{A_{SD}}{bds}\frac{\sigma_{SYD}}{\alpha\sigma_{ck}}$$

6. DESIGN FORMULAS

For design, σ_{SH}, σ_{SD}, and σ_{SV} in Eqs. (7), (8), (9), and (11), may be assumed to be equal to the yield strength and σ_{ck} is known, then the design formulas are derived as follow.

$$[2+\Psi_x]k^2-[2(\Psi_x+\Psi_y)+\Psi_z]k+2\Psi_x\Psi_y+\Psi_z=0 \quad \cdots\cdots(16)$$

where

$$\Psi_x=\frac{l_H}{ds}\tan\theta_D+1$$

$$\Psi_y=\frac{(a-l_V)\phi V_u}{bds^2\alpha\sigma_{ck}}$$

$$\Psi_z=\left[\frac{l_V\sin\theta_D-l_D}{ds\cos\theta_D}+1\right]\frac{\phi V_u a}{bds^2\alpha\sigma_{ck}}$$

and

$$A_{SD}=\frac{(a-l_V)\phi V_u-bkds^2\alpha\sigma_{ck}\left(1-\frac{k}{2}\right)}{\sigma_{SY}[l_D-ds\cos\theta_D-l_V\sin\theta_D]} \quad \cdots\cdots(17)$$

$$\frac{A_{SV}}{A_{SD}}=\frac{\phi V_u[l_D-ds\cos\theta_D-l_V\sin\theta_D]}{(a-l_V)\phi V_u-bkds^2\alpha\sigma_{ck}\left(1-\frac{k}{2}\right)}-\sin\theta_D \quad \cdots\cdots(18)$$

$$\frac{A_{SH}}{A_{SD}}=\frac{bkds\alpha\sigma_{ck}[l_D-ds\cos\theta_D-l_V\sin\theta_D]}{(a-l_V)\phi V_u-bkds^2\alpha\sigma_{ck}\left(1-\frac{k}{2}\right)}-\cos\theta_D \quad \cdots\cdots(19)$$

In the practical design, the dimensions of the section are first assumed considering the following points.

(a) the ratio of the over all depth of the beam to the span length should be selected so as to get the most economical design.

(b) the over all depth of the bracket should be selected so that there is no sudden break in the outline of the structure at this portion.

(c) the length of the bracket should be long enough to equip the bearing plate, but shorter are the better.

(d) the width of the bracket should be large enough to take care of the shears.

then

(e) if the loading condition are unknown, assume $a=2.5ds$ as the critical loading condition and the failure plain as a diagonal plane between the load and the reentrant corner of the bracket.

(f) the arrangement of reinforcement, l_H, l_V, l_D, are assumed from previous experience or by crude approximation.

(g) the location of the neutral axis can be calculated from Eq. (16) or obtained from Charts 1-4.

(h) the amount of reinforcement can be calculated from Eqs. (17), (18), and (19).

for the numeral example, the specimen of RCA

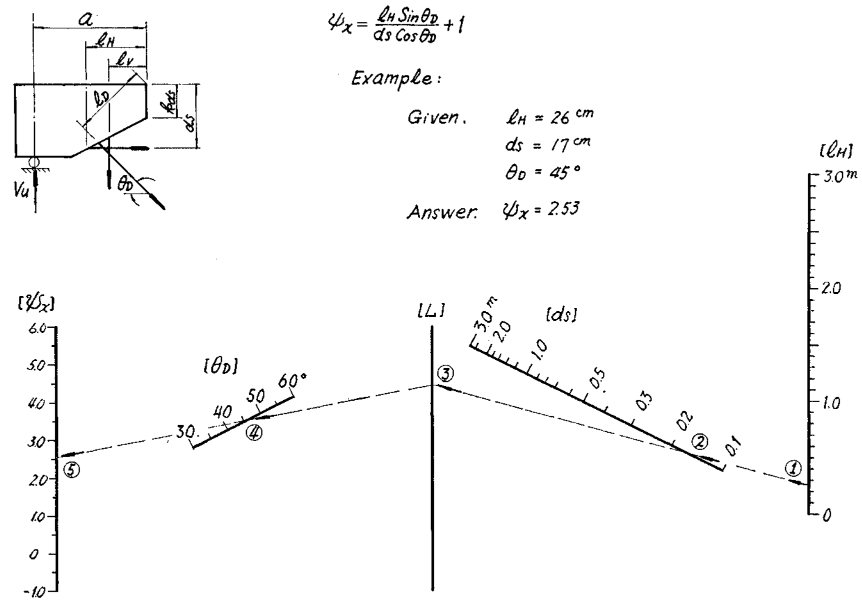


Chart 1 Nomograhy for Ψ_x of Chat 4.

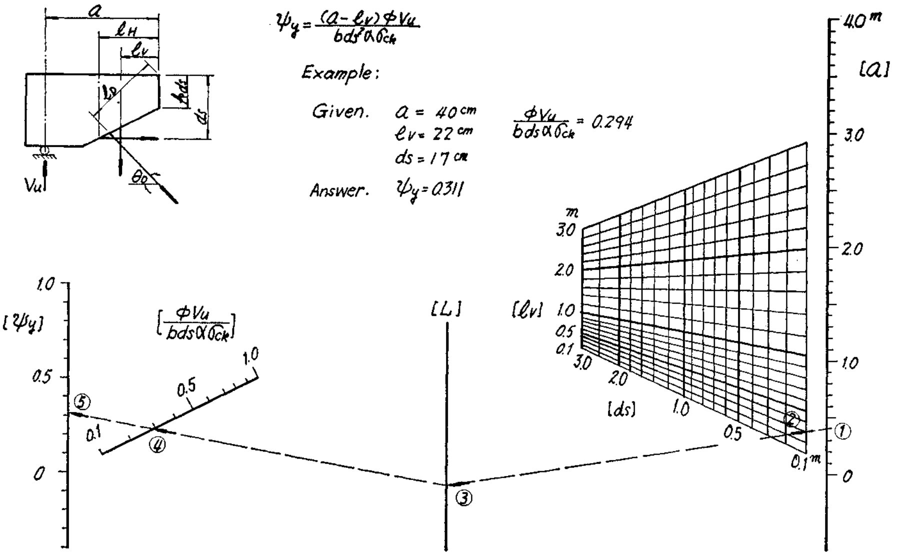


Chart 2 Nomograhy for Ψ_y of Chart 4.

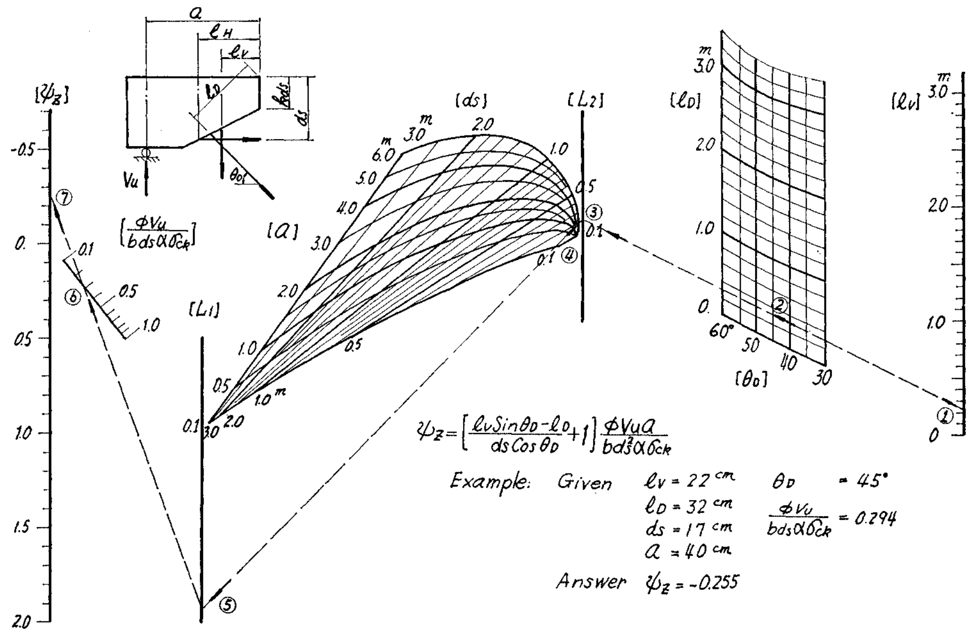
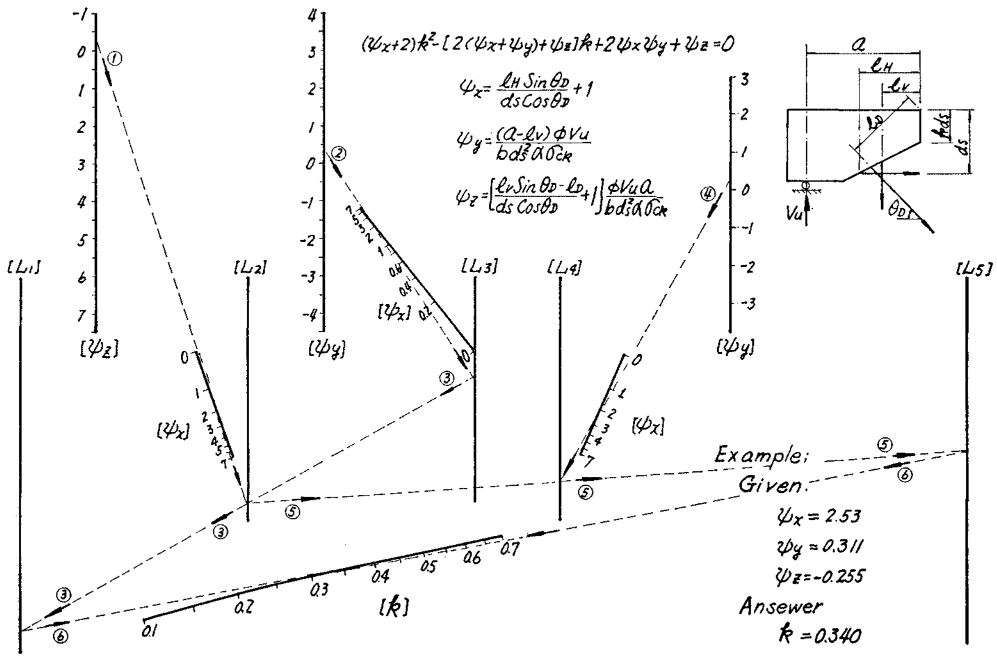


Chart 3 Nomograhy for Ψ_z of Chart 4.



Chat 4 Nomograhy for location of neutral axis.

which is subjected to $\phi V_u=18$ tons at $a=40$ cm and $\sigma_{SY}=3\,200$ kg/cm², $\alpha\sigma_{ck}=300$ kg/cm² is shown in Charts 1-4.

7. DISCUSSION

(1) Some of the simplifying assumptions which are employed may be noted as being arbitrary. However, it may be recognized as adequate considering the sensibility of using simple model to express macroscopically the internal structural system and the accuracy of the final result.

(2) Referencing Figs. 7, and 8, the measured internal force in diagonal steel is above the calculated value at low values of a/ds and under at high values of a/ds. However, as a whole the agreement is acceptable since the calculated values have a tendency to agree best with the experimental values at the ultimate load on the a/ds at which the shear compression failure usually takes place.

(3) The values k obtained from Eqs. (12a), (12b), and (12c) must satisfied all of the following conditions according to the engineering meaning, that is,

(a) $0<k<1$

(b) to give $\sigma_{SH}>0$, $\sigma_{SW}>0$ in Eqs. (13a), (14a), (13b), (14b), (13c), and (14c).

If Eqs. (12a), (12b), and (12c) do not give the values of k which can satisfy the above condition, this mean that the bracketed part of the beam end does not fail in shear compression failure at this loading condition. It may fail in other modes of failure which are beyond the scope of this paper.

(4) The factors influenced to behavior and strength of bracketed parts are numerous, for examples; the proportions and location of bracket, the amount and arrangement of reinforcement, type and location of the load, the properties of the concrete and steel, etc. The case A failures occur when the bracketed parts of the beam end are over-reinforced, the case B failures occur initially by excessive elongation of horizontal steel and the case C failures occur when the bracketed parts of the beam end are poor in web reinforcement.

(5) There are some limitations as stated in conclusion (3) for applying the formulas here developed. These limitations are imposed by the assumptions on which the formulas are based, and are undecided mathematically.

(6) The inclined steel bar should be principally provided in the reentrant corner of the bracket by the standpoint described here. In known examples from practice, the inclined steel is usually used together with the vertical and

horizontal steel with the range of about $(100A_{SV})/(b_{SV})=1\sim4$, $(100A_{SD})/(b_{SD}\sin\theta_D)=1\sim7$, $(100A_{SH})/(bds)=0.3\sim7$. These reinforcements are used to resist not only the stresses due to the loads but also the other causes, such as the nonuniform drying shrinkage or nonuniform temperature distribution during the hardening of concrete, etc. The eccentric loading or torsional loading, the tension produced by contraction of beams, and caused by imperfect action of the expansion bearings, etc., should also be considered in designing.

8. CONCLUSIONS

On the basis of the experimental results and within the limitations of this research the following conclusions may be drawn.

(1) The approximate value of the force in inclined steel at the reentrant corner of the bracket of reinforced concrete beams may be evaluated if the model analogue is adopted. This approximate value may be calculated from the equation

$$\sigma_{SD}A_{SD}=\frac{\sin\theta_c}{2\sin(\theta_c+\theta_D)}\frac{a}{ds}V$$

(2) The ultimated shear strength of reinforced concrete beams with bracket at its end is given by the minimum value of V_u among the calculated results by means of the cases A, B, and C.

(3) The derived equations are applicable under the following limitations

(a) the equations can not applicable to the bracket without failing as the result of shear compression failure induced by the corner crack.

(b) the equations are applicable to beams whose shear span effective bracket depth ratio is between 1.5 and 3.5.

(c) the length of bracket of ordinary beams is from about 1 to 0.3 times the over all depth of the bracket. The over all depth of bracket is about a half of over all depth of beam. The equation appears to be in conformity with the results in this range.

(d) the inclined steel should be provided in the reentrant corner of the bracket and the beam end should be conventionally reinforced.

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NOTATION

$l_1, l_2, \cdots, l_{10}$: lengths of model members 1, 2, 10, respectively

$A_1, A_2, \cdots, A_{10}$: cross-sectional area of model members $1, 2, \cdots, 10$, respectively

$I_1, I_2, \cdots, I_{10}$: moment of inertia of model members $1, 2, \cdots, 10$, respectively

$\theta_1, \theta_2, \cdots, \theta_{10}$: inclination angle of model members $1, 2, \cdots, 10$, respectively

hm : height of model

λ : l/hm

H : $\cot\theta_1+\cot\theta_2$

H' : $\cot\theta_4+\cot\theta_5$

a : shear span, distance between load point and nearest support

l : span of beam

l_s : length of bracket base to the support

l_H : horizontal distance from the point of intersection of the corner crack with the horizontal steel to the load

l_V, l_D: lever arm, the perpendicular distance from load point to the force, which represents the resultant of all forces carried by the vertical and inclined steel, respectively

b : width of compression face of beam

h : over all depth of beam

hs : over all depth of bracket

d : distance from extreme compression fiber to centroid of tension reinforcement of beam

ds : distance from extreme compression fiber to centroid of tension reinforcement of bracket

A_{SH}, A_{SV}, A_{SD}: total area of horizontal, vertical and inclined steel all of which cross the corner crack, respectively

$\sigma_{SH}, \sigma_{SV}, \sigma_{SD}$: stress of horizontal, vertical and inclined steel all of which cross the corner crack, respectively

σ_{SW} : $=\sigma_{SV}=\sigma_{SD}$

$\sigma_{SYH}, \sigma_{SYV}, \sigma_{SYD}$: yield strength of horizontal, vertical and inclined steel all of which

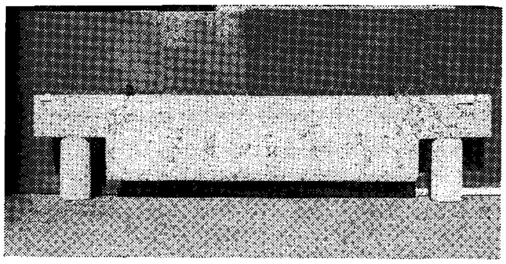

Photo. 1 Crack pattern of the specimen.

cross the corner crack, respectively

s_V, s_D: the spacing of vertical and inclined steel along the direction of the member, respectively

$\alpha\sigma_{ck}$: average compressive stress of concrete at failure

θ_D : inclination angle of inclined steel

θ_c : inclination angle of corner crack at failure

k : ratio of depth of neutral axis to effective depth at ultimate load

c : total internal compressive force in concrete

V_c : resisting shear supplied by compression concrete

V : shear acted on a given cross-section of a beam

V_u : shear at ultimate load

ϕ : capacity reduction factor

Ψ_a, Ψ_b, Ψ_c: factor [see Eq. (12b)]

q_H, q_D : factor [see Eq. (12b)]

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APPENDIX

1. Derivation of Eq. (3) for $l_9<a<(l_9+l_3)$.

Referring to Fig. 6. Let

$$D\sin\theta_1=S$$

Then

$$W=\frac{l_1S^2}{2A_1E_s\sin^2\theta_1}+\frac{l_2S^2}{2A_2E_c\sin^2\theta_2}+\frac{l_3S^2}{2A_3E_c\tan^2\theta_1}+\frac{l_4S^2}{2A_4E_s\sin^2\theta_4}\ \frac{H^2}{H'^2}$$

$$+\frac{l_5S^2}{2A_5E_c\sin^2\theta_5}\frac{H^2}{H'^2}$$
$$+\frac{l_6S^2}{2A_6E_c\tan^2\theta_4}\frac{H^2}{H'^2}+\frac{l_7H^2S^2}{2A_7E_c}+\frac{l_8H^2S^2}{2A_8E_s}$$
$$+\int_0^{l_9}\frac{[Vx]^2}{2E_cI_9}dx+\int_0^{a-l_9}\frac{[V(l_9+x)-Sx]^2}{2E_cI_3}dx$$
$$+\int_0^{l_3+l_9-a}\frac{[Va-S(a-l_9+x)]^2}{2E_cI_3}dx$$
$$+\int_0^{l_{10}}\frac{[Vx]^2}{2E_cI_{10}}dx$$
$$+\int_0^{a-l_{10}}\frac{\left[V(l_{10}+x)-Sx\dfrac{H}{H'}\right]^2}{2E_cI_6}dx$$
$$+\int_0^{l_6+l_{10}-a}\frac{\left[Va-(a-l_{10}+x)S\dfrac{H}{H'}\right]^2}{2E_cI_6}dx$$
$$+\int_0^{l_7}\frac{\left[Va-l_6S\dfrac{H}{H'}\right]^2}{2E_cI_7}dx$$

Therefore

$$\frac{\partial W}{\partial S}=\frac{l_1S}{E_sA_1\sin^2\theta_1}+\frac{l_2S}{E_cA_2\sin^2\theta_2}$$
$$+\frac{l_3S}{E_cA_3\tan^2\theta_1}+\frac{l_4S}{E_sA_4\sin^2\theta_4}\frac{H^2}{H'^2}$$
$$+\frac{l_5S}{E_cA_5\sin^2\theta_5}\frac{H^2}{H'^2}+\frac{l_6S}{E_cA_6\tan^2\theta_4}\frac{H^2}{H'^2}$$
$$+\frac{l_7H^2S}{E_cA_7}+\frac{l_8H^2S}{E_sA_8}-\frac{(a-l_9)^2(2a+l_9)}{6E_cI_3}V$$
$$+\frac{(a-l_9)^3}{3E_cI_3}S-\frac{a(l_3-l_9+a)(l_3+l_9-a)}{2E_cI_3}V$$
$$+\frac{[3l_3(a-l_9)+(l_3+l_9-a)^2](l_3+l_9-a)}{3E_cI_3}S$$
$$-\frac{(a-l_{10})^2(2a+l_{10})}{6E_cI_6}\frac{H}{H'}V+\frac{(a-l_{10})^3}{3E_cI_6}\frac{H^2}{H'^2}S$$
$$-\frac{a(l_6-l_{10}+a)(l_6+l_{10}-a)}{2E_cI_6}\frac{H}{H'}V$$
$$-\frac{[3l_6(a-l_{10})+(l_6+l_{10}-a)^2](l_6+l_{10}-a)}{3E_cI_6}\frac{H^2}{H'^2}S$$
$$-\frac{al_6l_7}{E_cI_7}\frac{H}{H'}V+\frac{l_6^2l_7}{E_cI_7}\frac{H^2}{H'^2}S$$

Hence, with a rearrangement, we have Eq. (3).

2. Derivation of Eq. (3) for $(l_9+l_3)<a<l/2$.

$$W=\frac{l_1S^2}{2E_sA_1\sin^2\theta_1}+\frac{l_2S^2}{2E_cA_2\sin^2\theta_2}$$
$$+\frac{l_3S^2}{2E_cA_3\tan^2\theta_1}+\frac{l_4S^2}{2E_sA_4\sin^2\theta_4}\frac{H^2}{H'^2}$$
$$+\frac{l_5S^2}{2E_cA_5\sin^2\theta_5}\frac{H^2}{H'^2}+\frac{l_6S^2}{2E_cA_6\tan^2\theta_4}\frac{H^2}{H'^2}$$
$$+\frac{l_7H^2S^2}{2E_cA_7}+\frac{l_8H^2S^2}{2E_sA_8}+\int_0^{l_9}\frac{[Vx]^2}{2E_cI_9}dx$$
$$+\int_0^{l_3}\frac{[V(l_9+x)-Sx]^2}{2E_cI_3}dx$$
$$+\int_0^{a-l_3-l_9}\frac{[V(l_3+l_9+x)-l_3S]^2}{2E_cI_7}dx$$
$$+\int_0^{l-2a}\frac{\left[Va-l_6S\dfrac{H}{H'}\right]^2}{2E_cI_7}dx+\int_0^{l_{10}}\frac{[Vx]^2}{2E_cI_{10}}dx$$
$$+\int_0^{l_6}\frac{\left[V(l_{10}+x)-S\dfrac{H}{H'}x\right]^2}{2E_6I_6}dx$$
$$+\int_0^{a-l_6-l_{10}}\frac{\left[V(l_6+l_{10}+x)-l_6S\dfrac{H}{H'}\right]^2}{2E_cI_7}dx$$

Therefore

$$\frac{\partial W}{\partial S}=\frac{l_1S}{E_sA_1\sin^2\theta_1}+\frac{l_2S}{E_cA_2\sin^2\theta_2}$$
$$+\frac{l_3S}{E_cA_3\tan^2\theta_1}+\frac{l_4S}{E_sA_4\sin^2\theta_4}\frac{H^2}{H'^2}$$
$$+\frac{l_5S}{E_cA_5\sin^2\theta_5}\frac{H^2}{H'^2}+\frac{l_6S}{E_cA_6\tan^2\theta_4}\frac{H^2}{H'^2}$$
$$+\frac{l_7H^2S}{E_cA_7}+\frac{l_8H^2S}{E_sA_8}-\frac{(2l_3+3l_9)l_3^2}{6E_cI_3}V$$
$$+\frac{l_3^3}{3E_cI_3}S-\frac{l_3(a+l_3+l_9)(a-l_3-l_9)}{2E_cI_7}V$$
$$+\frac{l_3^2(a-l_9-l_3)}{E_cI_7}S-\frac{al_6(l-2a)}{E_cI_7}\frac{H}{H'}V$$
$$+\frac{l_6^2(l-2a)}{E_cI_7}\frac{H^2}{H'^2}S-\frac{(2l_6+3l_{10})l_6^2}{6E_cI_6}\frac{H}{H'}V$$
$$+\frac{l_6^3}{3E_cI_6}\frac{H^2}{H'^2}S$$
$$-\frac{l_6(a+l_6+l_{10})(a-l_6-l_{10})}{2E_cI_7}\frac{H}{H'}V$$
$$+\frac{l_6^2(a-l_6-l_{10})}{E_cI_7}\frac{H^2}{H'^2}S$$

Hence we have Eq. (3).

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