PROC. OF JSCE,
NO. 205, SEPT. 1972

MONTE CARLO EVALUATION OF SAFETY OF INDETERMINATE STRUCTURE

By *Masaru* HOSHIYA*

1. INTRODUCTION

Great interest has been paid to the study of the rational evaluation of structural safety since we were confronted with the fact that structural resistance and external load are of random nature. Since then, extensive studies in the safety analysis of structure have been made, taking into account the statistical nature of the problem. The state of art in the structural reliability uptil recent years was described in references (1), (2) and (3). As pointed out in (3), one of the most difficult problems is the exact calculation of the reliability for practical redundant structures. The reason is that for redundant structural systems, particular component failure does not necessarily mean the overall collapse of structure. Failure of some components gives rise to the redistribution of the member forces and the remaining components may be still functional against the redistributed forces. In other words, failure path varies dependent upon the realization of external loads and resistances. Thus, the evaluation of the probabilities of various levels of survival and failure for redundant systems are very complicated. Alternatively, approximate methods such as upper and lower bounds, substitution of simpler structures, analysis of simulation techniques can be considered.

Yao and Yea presented a formulation of reliability of redundant structures (4). However, it seems that the counting of all possible failure paths in order to find the probability of the survival of $(m-k)$ components where the structure is originally composed of m components remains a complicated procedure except for simpler structures. Moses and Kinser (5) and (6) proposed an ordering method for redundant systems. However, the assumptions that failure of any single member is equivalent to the failure of a structure was obviously erroneous unless justified. Bounds of reliability were also analyzed as the approximation (7) and (8). In this case, the sharpness of the bounds must be examined.

* Ph.D., Assistant Professor, Musashi Institute of Technology, Civil Engineering Department, Tokyo.

Taking the above arguments into consideration, this paper clarifies the following problems;

(a) Failure path analysis can be extremely simplified by a practical Monte Carlo experiment for redundant structures.
(b) Chain failure occurrence typical for redundant structures is examined.
(c) Effect of material property of redundant structures to the reliability is discussed.

For instance, structural component of brittle material becomes completely inactive once it fails while for ductile material, the component can be still functional since it carries at least the maximum resistance capacity. Therefore, this effect must be included in the failure path analysis.

Finally, reliability of a tower truss is presented by assuming the probability of the system damage (at least one component failure) as the overall reliability. This is due to the consequence of the studies (a) to (c).

2. REDUNDANT CABLE SYSTEM

Consider a structural system of initial m components (Fig. 1)—a parallel cable system with

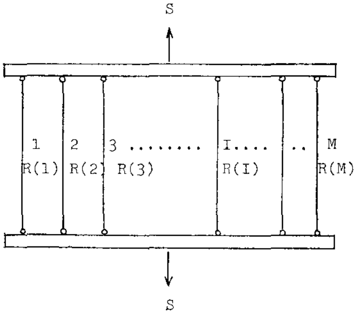

Fig. 1 Redundant Cable System of M Components.

single load S. This problem was already studied by this author for the system of brittle material (9) and (10). The Monte Carlo experiment is herein extended to the cases of both brittle and ductile material since the material property may play a role to the failure paths of the system. As in (9), consider the case where the component resistances are mutually independent but identically distributed random variables with lognormal distribution function and share equally the external load S with lognormal distribution. The experimental scheme is illustrated in Fig. 2. For

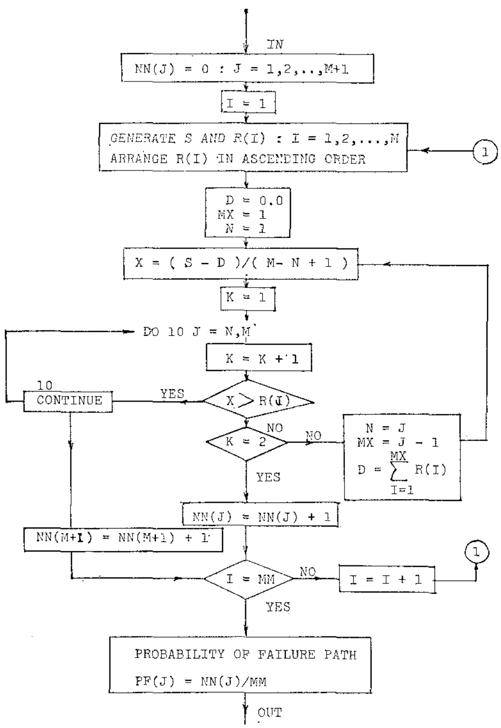

Fig. 2 Flow Chart of Cable System (Ductile & Brittle).

a ductile system, the Monte Carlo experiment starts with the generation of S and the resistances R_i, $i=1, 2, \cdots, m$ of each component from the given probability distribution functions of S and R_i. The technique of generation process is well known and described elsewhere (11), (12) and (13). Arrange the generated R_i in the ascending order. Then examine whether the member force S/m exceeds the resistances R_i for $i=1, 2, \cdots, m$. If k components fail, the load, $S-\sum_{i=1}^{k} R_i$ is to be equally redistributed over the remaining components. Since, the material is ductile, the failed k components still carry at least the maximum yielding capacities R_i, $i=1, 2, \cdots, k$ respectively. Thus, each remaining component is now subject to the member force $\left(S-\sum_{i=1}^{k} R_i\right)\Big/(m-k)$. Comparison between the remaining resistances and the redistributed member forces is then made repeatedly until the end of failure paths. Repeat the above experiment many times, keeping the tally of each failure path. Then, the probability of failure of i members can be obtained by n_i/N where n_i is the total numbers of failure of i members out of N experiments. For a brittle system, failure occurs through the fracture and the failed components become inactive. This case can be obtained simply by dropping out MX and D in the flow chart.

As the numerical example, lognormal distribution of the original member force with the mean 20 kips and with the standard deviation 5 kips, and the resistance of each cable lognormally distributed with the mean 40 kips and with the standard deviation 10 kips are considered. Thus the both variables have the coefficients of variation 0.25. Systems with 2 components to 14 components are examined for both ductile and brittle material. The results are shown in Tables 1 and 2. This experiment is based on 2 000 trials of a random sampling technique designated RANDU1 developed by Mitsubishi Electric Co. (14). The accuracy of the frequency test of the RANDU1 to produce uniform variables in the range (0, 1) was reported to be satisfactory (14). In Table 1, the numerical values shown are the probabilities vs. no. of member failure. For instance, the column 3 shows the results of a system of original 4 members in which the probabilities of non failure, and of 1 to 4 member failure are respectively 0.8905, 0.0740, 0.0205, 0.0010 and 0.0050. Reliabilities based on non component failure or at least one component success are also shown for the discussion of the results. For the further examination, the analytical upper and lower bounds of probability of non component failure are also given in the table. If component resistances are statistically independent, the probability of non component failure is theoretically given by (8)

$$P_{SS}=\int_0^\infty \prod_{i=1}^{m} \{1-F_{R_i}(c_i s)\} f_S(s)\, ds$$

where $F_{R_i}(\ )$ is a probability distribution function of R_i. $f_S(\ )$ is a probability density function of load S. $c_i s$ is the member force of ith component. If perfectly correlated component

Table 1 Failure Paths of Cable System (Ductile) 2 000 Trials.

	Original member force, $X=S/m=LN$ (20, 5) Resistance of each cable, LN (40, 10) in Kips[a]						
	Original system, m						
	2	4	6	8	10	12	14
	External Load $LN($, $)$ in Kips[a]						
Probability of member failure	(40,10)	(80,20)	(120,30)	(160,40)	(200,50)	(240,60)	(280,70)
0	0.9385	0.8905	0.8545	0.8145	0.7810	0.7575	0.7730
1	0.0470	0.0740	0.0800	0.1135	0.1265	0.1305	0.1145
2	0.0145	0.0205	0.0335	0.0325	0.0450	0.0470	0.0405
3		0.0010[b]	0.0225	0.0155	0.0160	0.0265	0.0200
4		0.0050[b]	0.0065	0.0110	0.0070	0.0105	0.0155
5			0.0[b]	0.0065	0.0075	0.0055	0.0080
6			0.0030[b]	0.0015[b]	0.0060	0.0050	0.0050[b]
7				0.0010[b]	0.0050	0.0060	0.0060[b]
8				0.0040[b]	0.0020[b]	0.0035[b]	0.0035[b]
9					0.0005[b]	0.0020[b]	0.0045[b]
10					0.0035[b]	0.0005[b]	0.0[b]
11						0.0[b]	0.0020[b]
12						0.0055[b]	0.0[b]
13							0.0010[b]
14							0.0065[b]
Reliability based on non component failure	0.9385	0.8905	0.8545	0.8145	0.7810	0.7575	0.7730
Reliability based on at least one component success	0.9855	0.9950	0.9970	0.9960	0.9965	0.9945	0.9935
Upper bound				0.9693			
Lower bound	0.9386	0.8772	0.8158	0.7544	0.6930	0.6316	0.5702

a; coefficient of variation=0.25
b; chain failure occurrence

resistances are assumed, P_{ss} may be written by

$$P'_{ss}=\text{minimum of } (P_{s1}, P_{s2}, \cdots, P_{sm})$$

where P_{si} indicates the probability of ith component success. If the individual component failure is statistically independent event, P_{ss} may be given by

$$P^{**}_{ss}=\prod_{i=1}^{m} P_{si}$$

It is shown (8) that P'_{ss} and P^{**}_{ss} give upper and lower bounds of P_{ss} respectively.

Fig. 4 shows probabilities of failure paths of 14 member systems with ductile and brittle material. For the purpose of examining the accuracy of the Monte Carlo experiment, the curve of probability of non component failure against the number of experiments is shown in Fig. 3.

The following observations and conclusions can be made from the experiment.

(1) The Monte Carlo experiment proposed herein—Flow Chart of Fig. 2, can be distinctively simple and useful for the analysis of redundant systems. The convergence of the probabilities by this experiment is satisfactory with sufficient number of experiments (see Fig. 3).

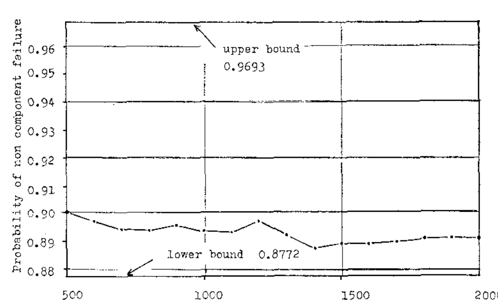

Fig. 3 Probability of Non Component Failure vs. No. of Experiments (m=4).

(2) The chain failure occurrence denoted by the suffix b can be observed both for ductile and brittle material (Tables 1 and 2). For instance, note that the probabilities of fail-

Table 2 Failure Paths of Cable System (Brittle) 2 000 Trials.

Probability of member failure	Original system, m						
	2	4	6	8	10	12	14
0	0.9385	0.8905	0.8545	0.8145	0.7810	0.7575	0.7730
1	0.0075^b	0.0305^b	0.0385^b	0.0645^b	0.0780^b	0.0795^b	0.0720^b
2	0.0540^b	0.0020^b	0.0	0.0115	0.0170	0.0245	0.0225
3		0.0^b	0.0	0.0045	0.0060	0.0075	0.0165
4		0.0770^b	0.0	0.0	0.0005	0.0015	0.0015
5			0.0	0.0	0.0	0.0005	0.0010
6			0.1070^b	0.0	0.0	0.0	0.0
7				0.0	0.0	0.0	0.0
8				0.1050^b	0.0	0.0	0.0
9					0.0	0.0	0.0
10					0.1175^b	0.0	0.0
11						0.0	0.0
12						0.1290^b	0.0
13							0.0
14							0.1135^b
Reliability based on non component failure	0.9385	0.8905	0.8545	0.8145	0.7810	0.7575	0.7730
Reliability based on at least one component success	0.9460	0.9230	0.8930	0.8950	0.8825	0.8710	0.8865

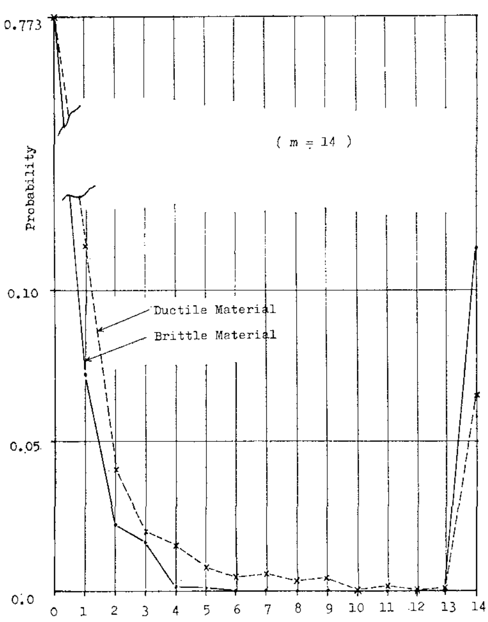

Fig. 4 Probability of Failure Paths.

ure of one to 13 components are less than the probability of all component failure for the original 14 component system of brittle material. This indicates that failure of one component produces at least partial failure sequence of other components. Furthermore, the fact that probabilities of 6 to 13 component failure are zero indicates a complete sequence of failure paths all the way until the complete collapse of the system.

(3) The comparison of probabilities of failure paths for ductile and brittle material indicates that the chain failure occurrence of brittle material is more critical than that of ductile material (Fig. 4).

(4) Reliabilities based on at least one component success of ductile material are higher than those of brittle material (Tables 1 and 2).

(5) The theoretical bounds of probability are of no practical use especially for systems with high redundancy since it is not sharp for systems with many components.

In practice, structures, determinate or indeterminate are designed in such a way that equal marginal safety of each member can be attained. Therefore for indeterminate systems, once any member fails, the remaining members must carry the redistributed larger forces and the probability of failure of the remaining members increases. With (1) and till (5), reliability of indeterminate system may be estimated as probability of non component failure where the failure mode may be assumed to be of fracture type as the conservative estimation. To justify the arguments, we need consider other systems too.

3. INDETERMINATE TRUSS

Consider an indeterminate truss in Fig. 5, where

X_i, $i=1, 2, 3$ and 4 are independent but identically distributed random loads. Two modes of failure, yielding and buckling can be considered. Assume that the yielding and buckling resistances of each member are independent random variables. The scheme of the Monte Carlo experiment of failure path analysis is illustrated in Fig. 6.

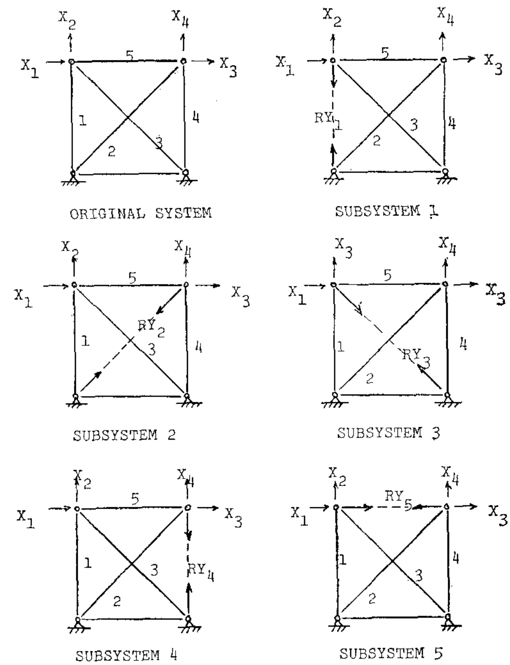

Fig. 5 Indeterminate Truss.

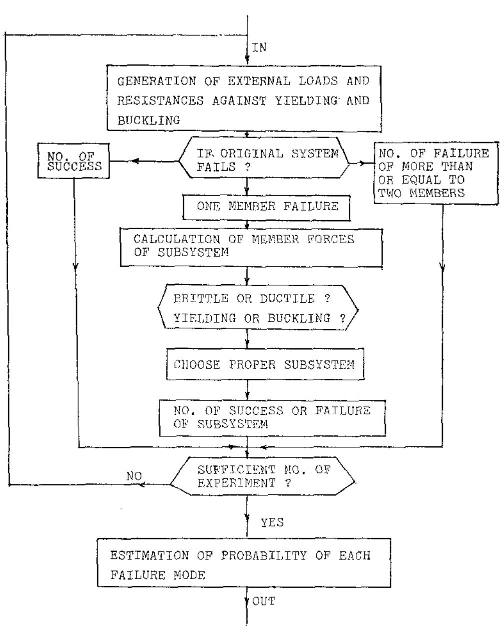

Fig. 6 Block Diagram of Reliability Analysis of Indeterminate Truss (Ductile & Brittle).

Generate the external loads X_i and the resistances against yielding and buckling. The member forces are then obtained as functions of X_i. Examine if the original system fails. If none of the members fail, keep the tally of the success of the system. If more than or equal to two member failure occur, the system is assumed to fail whatever the failure mode may be. In the case of one member failure, the member forces are redistributed over the subsystems, —5 subsystems are possible depending upon the particular member failure. Here, the material property, brittle or ductile must be considered since the way of the redistribution depends upon the material property. After the choice of proper subsystem, again examine the success or failure and keep the tally of each case. This procedure is repeated many times uptil sufficient number of the experiments is retained. The Monte Carlo experiment is made such that six cases of probability of failure paths can be obtained;

PS =probability of overall success

PYS=probability of success of the subsystem after yielding of one member of the original system.

PBS=probability of success of the subsystem after buckling of one member of the original system.

PF =probability of overall failure.

PYF=probability of failure of the subsystem after yielding of one member of the original system.

and

PBF=probability of failure of the subsystem after buckling of one member of the original system.

The member forces S_i and the external loads X_i are linearly related. The stress analysis gives the following equations, which are used for the evaluation of member forces in the above experiment.

For the original system,

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{pmatrix} = \begin{pmatrix} 0.5578, & 0.8845, & 0.4422, & -0.1155 \\ 0.6255, & 0.1634, & 0.7888, & 0.1634 \\ -0.7888, & 0.1634, & -0.6255, & 0.1634 \\ -0.4422, & -0.1155, & -0.5578, & 0.8845 \\ -0.4422, & -0.1155, & 0.4422, & -0.1155 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$

For the subsystems, $i=1, 2, \cdots, 5$ respectively

$$\begin{Bmatrix} S_2 \\ S_3 \\ S_4 \\ S_5 \end{Bmatrix} = \begin{bmatrix} 1.414, & 1.414, & 1.414, & 0 \\ 0, & 1.414, & 0, & 0 \\ -1, & -1, & -1, & 1 \\ -1, & -1, & 0, & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 - RY_1 \\ X_3 \\ X_4 \end{Bmatrix}$$

$$\begin{Bmatrix} S_1 \\ S_3 \\ S_4 \\ S_5 \end{Bmatrix} = \begin{bmatrix} 1, & 1, & 1, & 0 \\ -1.414, & 0, & -1.414, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 - 0.707RY_2 \\ X_4 - 0.707RY_2 \end{Bmatrix}$$

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_4 \\ S_5 \end{Bmatrix} = \begin{bmatrix} 0, & 1, & 0, & 0 \\ 1.414, & 0, & 1.414, & 0 \\ -1, & 0, & -1, & 1 \\ -1, & 0, & 0, & 0 \end{bmatrix} \begin{Bmatrix} X_1 + 0.707RY_3 \\ X_2 - 0.707RY_3 \\ X_3 \\ X_4 \end{Bmatrix}$$

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_5 \end{Bmatrix} = \begin{bmatrix} 1, & 1, & 1, & -1 \\ 0, & 0, & 0, & 1.414 \\ -1.414, & 0, & -1.414, & 1.414 \\ 0, & 0, & 1, & -1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 - RY_4 \end{Bmatrix}$$

and

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{Bmatrix} = \begin{bmatrix} 1, & 1, & 0, & 0 \\ 0, & 0, & 1.414, & 0 \\ -1.414, & 0, & 0, & 0 \\ 0, & 0, & -1, & 1 \end{bmatrix} \begin{Bmatrix} X_1 + RY_5 \\ X_2 \\ X_3 - RY_5 \\ X_4 \end{Bmatrix}$$

where for the brittle material, put $RY_i=0$. RY_i is the resistance capacity of member i against yielding.

As the numerical example, lognormal distribution of X_i with mean 10 kips and with a coefficient of variation of 0.25 is assumed. The members used are tabulated in Table 3. A mean yielding stress of steel 40 ksi with a coefficient of variation of 0.10 and the mean buckling capacity by Euler formula $\pi^2 EI/l^2$ with a coefficient of variation of 0.10 are assumed to evaluate the probabilistic information of the resistances. Table 4 shows the member resistances.

Table 3 Dimensions of Members.

Member	Area (in²)	I (in⁴)	Length (ft)
1 ∠2 1/2 * 2 1/2 * 3/16	0.90	0.55	10
2 ∠2 1/2 * 2 1/2 * 3/16	0.90	0.55	2 * 10
3 ∠3 1/2 * 3 1/2 * 5/16	2.09	2.5	2 * 10
4 ∠2 * 2 * 1/4	0.94	0.35	10
5 ∠2 * 2 * 3/16	0.71	0.27	10

Table 4 Resistance Capacity.

Member	Resistance against yielding*		Resistance against buckling*	
	Mean	St. dev.	Mean	St. dev.
1	36.0 Kips	3.6 Kips	10.9 Kips	1.09 Kips
2	36.0	3.6	5.5	0.55
3	83.6	8.36	25.0	2.5
4	37.6	3.76	7.0	0.70
5	28.4	2.84	5.4	0.54

* Lognormal distribution

Table 5 Probability of Failure Path

No. of Experiments	PS	PF	PYS	PYF	PBS	PBF
500	0.884	0.01			0.072	0.034
1 000	0.890	0.009			0.060	0.041
1 500	0.884	0.0087			0.0613	0.046
2 000	0.889	0.0065	zero	zero	0.0595	0.045
2 500	0.887	0.0068			0.0604	0.0456
3 000	0.885	0.0070			0.0620	0.046

The results are shown in Table 5. For this particular example, probability of one member failure through yielding was turned out to be zero for the assumption of ductility of material. (see the columns of PYS and PYF) Therefore no difference can be recognized whatever the material may be. However, it should be noted that here again there is non zero probability for the collapse of the structure after one member buckling (chain failure).—see the column of PBF.

4. TOWER TRUSS

Consider a tower truss subjected to lateral joint loads which are lognormally distributed random variables (Fig. 7). Since the structure is indeterminate with three degree redundancy, the fail-

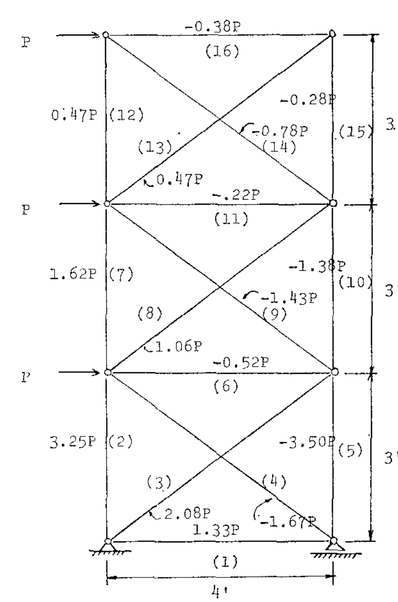

Fig. 7 Tower Truss.

ure path analysis may be very complicated even though the Monte Carlo scheme can be applied. Here as a conclusion of the foregoing two analyses, the reliability of the structure can be assumed to be equivalent to the probability of the system damage. In other words, any component failure results a chain failure so that the probability of failure of the surviving part becomes very high. This approximation makes the analysis simple and the experiment is illustrated in Fig. 8.

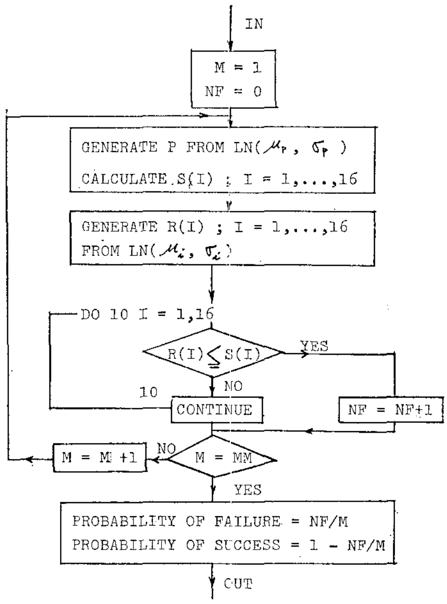

Fig. 8 Flow Chart of Tower Truss.

For the numerical example, the loads are assumed to be identical variables with mean 10 kips and with a coefficient of variation of 0.25. The member forces as a function of the load P are shown in the figure. The resistances against yielding or buckling are assumed to have means of twice the corresponding means of member forces and a coefficient of variation of 0.25 common to the all members. The probability of non component failure was evaluated to be 0.7265, where the probability bounds are 0.9750 and 0.4000 respectively. The bounds are not sharp again.

5. CONCLUSIONS

The following conclusions can be obtained;

(a) A Monte Carlo method can be used in a simple manner for the reliability analysis of indeterminate structures. The scheme of the experiment is however not unique for each problem. Thus, it must be effectively made depending upon the objectives and frequently with substitution of simpler structures. It is noted that at the present, the failure path analysis can be attained only by the Monte Carlo experiment. This is simply because the theoretical approach is too complicated.

(b) The chain failure occurrence of the examples may suggest that for practical purposes, the reliability of redundant systems can be measured by the probability of non component failure. This gives a conservative estimation of the reliability.

(c) The chain failure occurs both for brittle and ductile material. Since the probability of the chain failure of systems with brittle characteristics is higher, the reliability analysis of a system with intermediate property between brittle and ductile can be performed assuming the fracture failure (brittle) as a conservative estimation.

(d) The bounds of reliability by the analytical methods presently available can be used as a rough estimation of the reliability of simpler structures. However, for a system of many components, determinate or indeterminate, the bounds are not sharp and are of no practical use.

The numerical calculations were performed by the computer machine FACOM 270-30 at Musashi Institute of Technology.

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(*Received. Feb. 17 1972*)