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LARGE STRAIN, VISCO-ELASTIC NUMERICAL ANALYSIS BY MEANS OF FINITE ELEMENT METHOD

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1. INTRODUCTION

This paper is concerned with the extended theory of the author's previous papers[1),2)]. The author has shown that a numerical analysis of visco-elastic bodies can be easily conducted by the finite element method using generalized coordinates under small deformation assumption[1)]. The author also presented the paper dealing with numerical methods of elastic-plastic bodies based on large strain theories[2)]. This paper describes a finite element analysis of visco-elastic bodies based on large strain and generalized coordinates assuming the relation between Kirchhoff's stress tensor and Green's strain tensor.

In general, visco-elastic bodies may be defined either by using generalized coordinates, i.e. observed coordinates and hidden coordinates, or by using the assumption that Helmhaltz free energy is given by the certain functional equations. This paper discusses the visco-elastic body defined by generalized coordinates and associated generalized forces. Namely, a certain state of the body can be expressed by the generalized coordinates (state variables) q_i $(i=1, 2, \cdots, p)$ where p is the maximum number of the variables. Corresponding to q_i, generalized forces Q_i are introduced assuming that they consist of their reversible parts $Q_i^{(I)}$ and their irreversible parts $Q_i^{(R)}$. $Q_i^{(I)}$ and $Q_i^{(R)}$ may be given as the linear functions of q_i and $\dot{q}_i$, respectively, where superposed dot denotes the material time differentiation. The generalized coordinates can be divided into two parts: measurable quantities, that are called observed coordinates, and unmeasurable quantities that are hidden coordinates. In this paper, it is assumed that Green's strain tensor can be regarded as the observed coordinate, and Kirchhoff's stress tensor is introduced as its associated generalized force.

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Application of finite element method to visco-elastic body has been presented by many authors under the small deformation assumption[3)~9)]. On the basis of large strain theory, Oden and Ramiretz[10)] have derived the original method taking account of the constitutive equations as the functional form. On the contrary, this paper presents the method considering the constitutive equations as the expressions of the generalized coordinates. The final equations of the finite element method can be transformed into total form of the relation between displacements and external forces.

2. KINEMATICAL EQUATIONS

Throughout this paper, material description in the convected coordinates is used, and rectangular Cartesian coordinates X_i and x_i $(i=1, 2, 3)$, are adopted. The conventional summation convention with repeated indices is introduced. Subscripted comma $,_i$ denotes the partial differentiation $\dfrac{\partial}{\partial X_i}$.

The position vectors in the undeformed and deformed states are denoted by $\boldsymbol{R}$ and $\boldsymbol{r}$, respectively. Both are related each other as:

$$\boldsymbol{r}=\boldsymbol{R}+\boldsymbol{u}-\boldsymbol{C} \qquad \cdots\cdots(2.1)$$

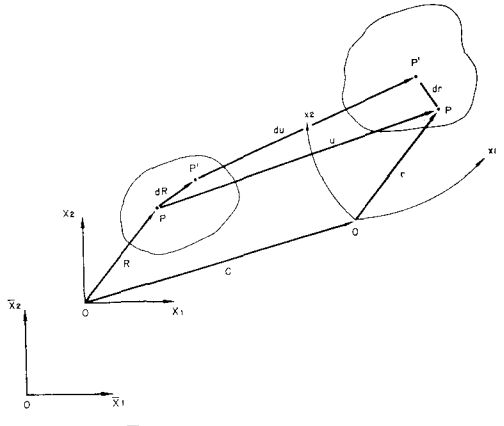

Fig. 1 Reference frame.

in which $\boldsymbol{u}$ is the displacement vector, and $\boldsymbol{C}$ is the quantities as shown Fig. 1. $\boldsymbol{R}$ is the reference position vector and its components are denoted by X_i. $\boldsymbol{r}$ is the current position vector which has components x_i. It is convenient to let the reference coordinate system X_i and the gloval coordinate system be coincident.

Let u_i be components of displacement vector $\boldsymbol{u}$ measured along X_i coordinate direction, the relation:

$$x_i=X_i+u_i \qquad \cdots\cdots(2.2)$$

can be introduced, assuming that in the reference state the current coordinate system x_i and the reference coordinate system X_i are coincident.

Denoting the base vector in x_i coordinate system by $\boldsymbol{g}_i$ and using equation (2.1), it follows that

$$\boldsymbol{g}_i=\boldsymbol{r}_{,i}=\boldsymbol{R}_{,i}+\boldsymbol{u}_{,i}=\boldsymbol{G}_i+\boldsymbol{u}_{,i} \qquad \cdots\cdots(2.3)$$

in which $\boldsymbol{G}_i$ is the base vector in the reference state. $\boldsymbol{G}_i$ is the unit vector along X_i coordinate, because X_i coordinate system is chosen to be rectangular Cartesian coordinate system.

The components of the displacement vector u_i can be expressed by:

$$\boldsymbol{u}=u_i\boldsymbol{G}_i \qquad \cdots\cdots(2.4)$$

Thus, equation (2.3) can be rewritten in the following form.

$$\boldsymbol{g}_i=(\delta_{ij}+u_{j,i})\cdot\boldsymbol{G}_j=F_{ji}\cdot\boldsymbol{G}_j \qquad \cdots\cdots(2.5)$$

where F_{ij} is the deformation gradient.

The metric tensor g_{ij} in the x_i coordinate system can be expressed from equation (2.3) as:

$$g_{ij}=\boldsymbol{g}_i\cdot\boldsymbol{g}_j=G_{ij}+\boldsymbol{u}_{,j}\boldsymbol{G}_i+\boldsymbol{u}_{,i}\boldsymbol{G}_i+\boldsymbol{u}_{,i}\boldsymbol{u}_{,j} \qquad \cdots\cdots(2.6)$$

in which

$$G_{ij}=\boldsymbol{G}_i\boldsymbol{G}_j=\delta_{ij} \qquad \cdots\cdots(2.7)$$

where δ_{ij} is the Kronecker's delta function. Green's strain tensor γ_{ij} is defined by the following equations.

$$\gamma_{ij}=\frac{1}{2}(g_{ij}-G_{ij}) \qquad \cdots\cdots(2.8)$$

Using equation (2.4) and equation (2.8), Green's strain tensor can be expressed by components of displacement vector u_i as:

$$\gamma_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{m,i}u_{m,j}) \qquad \cdots\cdots(2.9)$$

The strain rate tensor $\dot{\gamma}_{ij}$ can be derived by differentiating both sides of equation (2.9) with respect to time.

$$\dot{\gamma}_{ij}=\frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}+\dot{u}_{m,i}u_{m,j}+u_{m,i}\dot{u}_{m,j}) \qquad \cdots\cdots(2.10)$$

3. EQUILIBRIUM EQUATIONS

Consider an arbitrary volume V of the continuum bounded by the surface A in the current state, and suppose that V_0 is the corresponding volume in the reference state bounded by the surface A_0. The energy balance equation is postulated in the form:

$$\int_{V_0}(\rho_0\dot{U})\,dV_0=\int_{V_0}\rho_0(r+f_i\dot{u}_i)\,dV_0+\int_{A_0}(p_i\dot{u}_i)\,dA_0 \qquad \cdots\cdots(3.1)$$

where ρ_0 is the mass density in the reference state, U is the internal energy per unit mass, f_i is the body force per unit mass, p_i is the surface force per unit area of A_0 referred to $\boldsymbol{G}_i$, and $\boldsymbol{r}$ is the heat supply function per unit mass. In equation (3.1) and throughout this paper, the kinematic energy and the heat energy supplied across the surface A_0 are neglected.

Let it be assumed that U, r, f_i and p_i are unaltered by an arbitrary uniform rigid body translational velocity $\dot{a}_i$. Then, since $\dot{a}_i$ is arbitrary, substitution $\dot{u}_i+\dot{a}_i$ into equation (3.1), gives the equilibrium equations for body force and surface force in the following form.

$$\int_V(\rho_0 f_i)\,dV+\int_{A_0}p_i\,dA_0=0 \qquad \cdots\cdots(3.2)$$

The surface force p_i acts across the surface A whose original position in the reference state was A_0. The Kirchhoff's stress tensor S_{ij} is introduced as follows.

$$p_i=F_{ij}S_{jk}N_k=S_{jk}(\delta_{ij}+u_{i,j})N_k \qquad \cdots\cdots(3.3)$$

Using equation (3.3) and equation (3.2), the well-known equilibrium equations (3.4) can be derived:

$$(F_{ij}S_{jk})_{,k}+\rho_0 f_i=0 \qquad \cdots\cdots(3.4)$$

where N_k is the outward unit normal to A_0.

Introducing equation (3.3) into equation (3.1) and changing a surface integral into a volume integral, the local form of the energy balance equation can be derived as follows.

$$\rho_0 r-\rho_0\dot{U}+F_{ik}S_{jk}\dot{u}_{i,k}=0 \qquad \cdots\cdots(3.5)$$

The last term of the left hand side of equation (3.5) can be transformed into

$$F_{ij}S_{jk}\dot{u}_{i,k}=F_{ij}\dot{u}_{i,k}S_{jk}=S_{jk}\dot{\gamma}_{jk} \qquad \cdots\cdots(3.6)$$

Thus, the energy balance equation is rewritten as:

$$\rho_0\dot{U}=S_{jk}\dot{\gamma}_{jk}+\rho_0 r \qquad \cdots\cdots(3.7)$$

The entropy production inequality is postulated in the following form.

$$\int_{V_0}\left(\rho_0\frac{\Theta}{T}\right)dV_0=\int_{V_0}(\rho_0\dot{S})\,dV_0 -\int_{V_0}\left(\rho_0\frac{r}{T}\right)dV_0\geqq 0 \quad \cdots\cdots(3.8)$$

where Θ denotes the entropy production per unit mass, $\dot{S}$ is the entropy per unit mass and T is the absolute temperature. Application of equation (3.8) to an arbitrary volume yields

$$\rho_0\Theta=\rho_0 T\dot{S}-\rho_0 r\geqq 0 \quad \cdots\cdots(3.9)$$

For the purpose of the application of finite element method, the virtual work equation is introduced here. Multiplying both sides of equation (3.4) by virtual displacement u_i^* and integrating over the whole volume, it follows that

$$\int_{V_0}[(F_{ij}S_{jk}),_k u_i^*]\,dV_0+\int_{V_0}(\rho_0 f_i u_i^*)\,dV_0=0 \quad \cdots\cdots(3.10)$$

Application of Green-Gauss theorem to the first term of the left hand side of equation (3.10) leads to the virtual work equation in the following form.

$$\int_{V_0}(S_{ij}\gamma_{ij}^*)\,dV_0=\Omega \quad \cdots\cdots(3.11)$$

where

$$\Omega=\int_{A_0}[S_{ij}N_j(\delta_{kj}+u_{k,j})u_k^*]\,dA_0 +\int_{V_0}(\rho_0 f_i u_i^*)\,dV_0 \quad \cdots\cdots(3.12)$$

and the virtual strain γ_{ij}^* is denoted as follows.

$$\gamma_{ij}^*=\frac{1}{2}(F_{k,j}u_{k,i}^*+F_{k,i}u_{k,j}^*) \quad \cdots\cdots(3.13)$$

4. CONSTITUTIVE EQUATIONS

The constitutive equation of the visco-elastic body described in this paper is defined by the relation between the generalized coordinates q_i $(i=1, 2, \cdots, p)$, where p is the total number of the coordinates. They are considered to consist of the observed coordinates and the hidden coordinates. The Green's strain tensor is employed as the observed coordinates. The hidden coordinates are denoted by $h_{ij}{}^{\alpha}$. The generalized force Q_i is introduced in the manner that product $Q_i\dot{q}_i$ is equivalent to $S_{ij}\dot{\gamma}_{ij}$. Namely, the part of Q_i corresponding to the observed coordinates is the Kirchhoff stress tensor S_{ij} and the parts corresponding to the hidden coordinate are all zero.

Energy balance equation (3.7) can be rewritten in the form:

$$\rho\dot{U}=Q_i\dot{q}_i+\rho_0 r \quad \cdots\cdots(4.1)$$

The generalized force is assumed to be devided into the reversible parts $Q_i^{(R)}$ and the irreversible parts $Q_i^{(I)}$.

$$Q_i=Q_i^{(R)}+Q_i^{(I)} \quad \cdots\cdots(4.2)$$

Postulating the internal energy can be expressed as:

$$\rho\dot{U}=\rho_0 T\dot{S}+Q_i^{(R)}\dot{q}_i \quad \cdots\cdots(4.3)$$

the Gibbs entropy equation becomes as follows.

$$\rho_0 T\dot{S}=\rho_0\dot{U}-\dot{W}^{(R)} \quad \cdots\cdots(4.4)$$

where

$$Q_i\dot{q}_i=\dot{W}=\dot{W}^{(I)}+\dot{W}^{(R)} \quad \cdots\cdots(4.5)$$

and

$$\dot{W}^{(R)}=Q_i^{(R)}\dot{q}_i\,,\quad \dot{W}^{(I)}=Q_i^{(I)}\dot{q}_i \quad \cdots\cdots(4.6)$$

Eliminating $\rho_0\gamma$ from equation (4.1) using equation (3.7), it follows that

$$\rho_0\dot{U}-\rho_0 T\dot{S}=Q_i\dot{q}_i-\rho_0\Theta \quad \cdots\cdots(4.7)$$

From equations (4.5), (4.6) and (4.7), $\rho_0\Theta$ can be explicitly transformed into equation (4.8).

$$\rho_0\Theta=W^{(I)}=Q_i^{(I)}\dot{q}_i \quad \cdots\cdots(4.8)$$

The restriction in equation (3.9) requires that the last term in equation (4.8) has to be positive or equal to zero. Assuming linear relations between $Q_i^{(I)}$ and $\dot{q}_i$, i.e.

$$Q_i^{(I)}=b_{ij}\dot{q}_j \quad \cdots\cdots(4.9)$$

and equation (4.9) is substituted into equation (4.8), since $\rho_0\Theta\geqq 0$, it is clear that the constant array b_{ij} have to be positive semidefinite. From Onswager's principle, b_{ij} must be symmetric.

The Helmhaltz free energy can be defined by the following equation (4.10).

$$F=U-TS \quad \cdots\cdots(4.10)$$

Using equation (4.8) and equation (4.9), rearrangement of equation (4.10) leads to equation (4.11).

$$\rho_0\dot{F}=Q_i^{(R)}\dot{q}_i-\rho_0\dot{T}S \quad \cdots\cdots(4.11)$$

From equation (4.11), the reversible part of the generalized force and the entropy S can be derived as follows.

$$Q_i^{(R)}=\rho_0\frac{\partial F}{\partial q_i}\,,\quad S=-\frac{\partial F}{\partial T} \quad \cdots\cdots(4.12)$$

Let it be assumed that the free energy can be expressed in the form:

$$\rho_0 F=\frac{1}{2}a_{ij}q_i q_j-\beta_i\theta q_i-\frac{C_\varepsilon}{2T_0}\theta^2 \quad \cdots\cdots(4.13)$$

where T_0 is the temperature in the natural state, C_ε is the specific heat, and a_{ij} and β_j are constants. Because the free energy has to become minimum value in the natural state, a_{ij} is positive semidefinite. Symmetricity of stress and strain results in symmetricity of a_{ij}. θ denotes relative temperature $(T=T_0+\theta)$. Using equation

(4.13), equation (4.12) takes the following form for the reversible part of the generalized force and the entropy, respectively.

$$Q_i^{(R)}=a_{ij}q_j-\beta_i\theta \qquad \cdots\cdots(4.14)$$

$$\rho_0 S=\beta_i q_i+\frac{C_\varepsilon}{T_0}\theta \qquad \cdots\cdots(4.15)$$

Substitution of equation (4.9) and equation (4.14) into equation (4.2) yields the constitutive equation in the following form.

$$Q_i=a_{ij}q_j+b_{ij}\dot{q}_j-\beta_i\theta \qquad \cdots\cdots(4.16)$$

Neglecting nonlinear coupling equation (3.9) and equation (4.15) lead to the following equations.

$$\dot{\theta}=\frac{r}{C_\varepsilon}-\frac{T_0\beta_i}{C_\varepsilon}\dot{q}_i \qquad \cdots\cdots(4.17)$$

Using equation (4.17) and equation (4.16), the constitutive equations in the stationary adiabatic state can be obtained in the following form.

$$Q_i=\left[a_{ij}+\frac{T_0}{C_\varepsilon}\beta_i\beta_j\right]\cdot q_j+b_{ij}\dot{q}_j-\beta_i\left(\frac{H}{C_\varepsilon}\right) \qquad \cdots\cdots(4.18)$$

where

$$H=\int_0^t r\,dt \qquad \cdots\cdots(4.19)$$

5. FINITE ELEMENT ANALYSIS

To begin with, the constitutive equation need to be transformed into the more convenient form for the application of the finite element method. Neglecting thermal terms, the constitutive equa- (4.18) can be expressed as in equation (5.1) and (5.2).

$$Q_{ij}^{\alpha}=A_{ijkl}^{\alpha\beta}\cdot h_{kl}^{\beta}+B_{ijkl}^{\alpha\beta}\cdot\dot{h}_{kl}^{\beta} \qquad \cdots\cdots(5.1)$$

where

$$Q_{ij}^{1}=S_{ij}\,,\quad h_{ij}^{1}=\gamma_{ij}\,,\quad Q_{ij}^{\beta}=0\quad(\beta\neq 1) \qquad \cdots\cdots(5.2)$$

Consider a small increment in time, Δt, and suppose that $\dot{h}_{ij}^{\beta}$ can be regarded to be constant during this increment, i.e.

$$\dot{h}_{ij}^{\beta}=(h_{ij}^{\beta}-h_{ij}^{\beta}(0))/\Delta t \qquad \cdots\cdots(5.3)$$

where h_{ij}^{β} and $h_{ij}^{\beta}(0)$ are the initial and the final value for the time increment Δt, respectively.

Substituting equation (5.3) into equation (5.1), using equation (5.2) and rearranging them, equation (5.4) can be obtained.

$$Q_{ij}^{\alpha}=K_{ijkl}^{\alpha\beta}\cdot h_{kl}^{\beta}+H_{ijkl}^{\alpha\beta}\cdot h_{ij}^{\beta}(0) \qquad \cdots\cdots(5.4)$$

where

$$K_{ijkl}^{\alpha\beta}=A_{ijkl}^{\alpha\beta}+\frac{1}{\Delta t}B_{ijkl}^{\alpha\beta}\,,\quad H_{ijkl}^{\alpha\beta}=-\frac{1}{\Delta t}B_{ijkl}^{\alpha\beta}$$

Taking equation (5.2) into consideration, equation (5.4) can be devided into two equations as follows.

$$S_{ij}=K_{ijkl}^{(1)}\cdot\gamma_{kl}+K_{ijkl\alpha}^{(2)}\cdot h_{kl}^{\alpha}+H_{ijkl}^{(1)}\cdot\gamma_{kl}(0)+H_{ijkl\alpha}^{(2)}\cdot h_{kl}^{\alpha}(0) \qquad \cdots\cdots(5.5)$$

$$K_{ijkl\alpha}^{(2)}\cdot\gamma_{ij}+K_{ijkl\alpha\beta}^{(3)}\cdot h_{ij}^{\beta}+H_{ijkl\alpha}^{(2)}\cdot\gamma_{ij}(0)+H_{ijkl\alpha\beta}^{(3)}\cdot h_{ij}^{\beta}(0)=0 \qquad \cdots\cdots(5.6)$$

Solving equation (5.6) for h_{ij}^{β} and substituting it into equation (5.5), the following equations can be formulated.

$$S_{ij}=D_{ijkl}\cdot\gamma_{kl}+\bar{S}_{ij} \qquad \cdots\cdots(5.7)$$

$$h_{ij}^{\alpha}=-S_{\alpha}^{(1)}\cdot\gamma_{ij}-S_{\alpha}^{(2)}\cdot\gamma_{ij}(0)-S_{\alpha\beta}^{(3)}\cdot h_{ij}^{\beta}(0) \qquad \cdots\cdots(5.8)$$

where

$$S_{\alpha}^{(1)}=L_{ijkl\alpha\gamma}^{(3)}\cdot K_{ijkl\gamma}^{(2)}$$

$$S_{\alpha}^{(2)}=L_{ijkl\alpha\gamma}^{(3)}\cdot H_{ijkl\gamma}^{(2)}$$

$$S_{\alpha\beta}^{(3)}=L_{ijkl\alpha\gamma}^{(3)}\cdot H_{ijkl\gamma\beta}^{(3)}$$

$$K_{ijkl\alpha\beta}^{(3)}\cdot L_{mnpq\gamma\delta}^{(3)}=\delta_{im}\delta_{jn}\delta_{kp}\delta_{lq}\delta_{\alpha\gamma}\delta_{\beta\delta}$$

$$D_{ijkl}=K_{ijkl}^{(1)}-K_{ijkl\gamma}^{(2)}\cdot L_{ijkl\gamma\delta}^{(3)}\cdot K_{ijkl\delta}^{(2)}$$

$$\bar{S}_{ij}=(H_{ijkl}^{(1)}-K_{ijkl\beta}^{(2)}\cdot S_{\beta}^{(2)})\cdot\gamma_{kl}(0)+(H_{ijkl\gamma}^{(2)}-K_{ijkl\beta}^{(2)}\cdot S_{\beta\gamma}^{(3)})\cdot h_{kl}^{\gamma}(0)$$

Equation (5.7) states that the stress strain equation of the visco-elastic body has the same form in the Green's strain and Kirchhoff's stress tensor as for the Hookean elastic body.

According to the conventional finite element displacement procedure, suppose that the continuous medium is divided into several media called finite elements. The displacement of αth node in the ith direction is denoted by $u_{\alpha i}$ on each finite element. With the aid of shape function Φ_α, the displacement inside the finite element u_i can be expressed approximately as follows.

$$u_i=\Phi_\alpha u_{\alpha i} \qquad \cdots\cdots(5.9)$$

Substituting this into equation (2.9), γ_{ij} can be expressed in the following form.

$$\gamma_{ij}=\Phi_{\alpha,i}u_{\alpha j}+\Phi_{\alpha,j}u_{\alpha i}+\Phi_{\alpha,i}\Phi_{\beta,j}u_{\alpha k}u_{\beta k} \qquad \cdots\cdots(5.10)$$

Making use of equation (5.9) and equation (3.13), the virtual strain γ_{ij}^{*} can be described as in equation (5.11).

$$2\gamma_{ij}^{*}=\Phi_{\alpha,i}u_{\alpha j}^{*}+\Phi_{\alpha,j}u_{i\alpha}^{*}+\Phi_{\alpha,i}\Phi_{\beta,j}u_{\alpha m}^{*}u_{\beta m}+\Phi_{\alpha,i}\Phi_{\beta,j}u_{\alpha m}u_{\beta m}^{*} \qquad \cdots\cdots(5.11)$$

Introducing equation (5.7) into equation (3.11), it follows that

$$\int_V (D_{ij}\gamma_{ij}\gamma_{ij}^*)\,dV_0+\int_{V_0}(\bar{S}_{ij}\gamma_{ij}^*)\,dV_0=\Omega \qquad \cdots\cdots(5.12)$$

where

$$\Omega=\int_A(\rho_i u_i^*)\,dA_0+\int_{V_0}(\rho_0 F_i u_i^*)\,dV_0 \cdots(5.13)$$

Substitution of equations (5.9), (5.10) and (5.11) into equations (5.12) and (5.13) and rearrangement of the resulted equation lead to the equations of the finite element formulation. Since u_{ai}^* at each node of finite elements is arbitrary, it follows that

$$K_{\beta i\alpha j}\cdot u_{\alpha j}=\Omega_{\beta i}-\bar{\Omega}_{\beta i} \qquad \cdots\cdots(5.14)$$

where

$$K_{\beta i\alpha j}=\int_{V_0}\left[\Phi_{\beta,k}(\delta_{li}+\Phi_{\delta,l}u_{\delta i})D_{klmn}\times\left(\delta_{mn}+\frac{1}{2}\Phi_{\gamma,m}u_{\gamma j}\right)\Phi_{\alpha,n}\right]dV_0 \qquad \cdots\cdots(5.15)$$

$$\Omega_{\beta i}=\int_{A_0}(p_i\Phi_\beta)\,dA_0+\int_{V_0}(\rho_0 f_i\Phi_\beta)\,dV_0 \qquad \cdots\cdots(5.16)$$

$$\bar{\Omega}_{\beta i}=\int_{V_0}\{\bar{S}_{kl}(\delta_{li}+\Phi_{\gamma,l}u_{\gamma i})\}\Phi_{\beta,k}\}\,dV_0 \qquad \cdots\cdots(5.17)$$

The term $K_{\beta i\alpha j}$ in the left hand side of equation (5.14) is related to the modulus of the visco-elastic body and the small time increment. $K_{\beta i\alpha j}$ corresponds to the stiffness matrix in the conventional finite element method. The term $\bar{\Omega}_{\beta i}$ expresses the influence of the history of deformation.

In the same manner as is used for the conventional finite element method, the simultaneous equations on the whole continuum are easily constructed by using the equilibrium equations of the nodal force and the continuity equations of the nodal displacements. The displacements at each node of finite elements can be obtained from these equations for the given external force. The strain can be obtained from equation (5.10). Using the γ_{ij}, obtained above, and $\gamma_{ij}(0)$ and $h_{ij}^\alpha(0)$ previously calculated, the new hidden coordinate h_{ij}^α can be obtained from equation (5.8). The computation can be proceeded by setting these calculated values as the initial values for the next cycle.

6. FORMS OF CONSTITUTIVE EQUATION OF ORDINARY FUNCTIONAL TYPE AND GENERALIZED COORDINATE TYPE

The relation between the constitutive equation (4.16) and the ordinary constitutive equation given in the functional form is considered here. Firstly, the thermomechanically simple theory is obtained in the following discussions. The array b_{ij} is assumed to be given by the function of temperature θ i.e.

$$b_{ij}=a_T[\theta(\tau)]\cdot b'_{ij} \qquad \cdots\cdots(6.1)$$

where τ is the time parameters. The reduced time is introduced in the following form.

$$\xi=\int_0^t\frac{d\tau}{a_T[\theta(\tau)]} \qquad \cdots\cdots(6.2)$$

Making use of equation (6.2), the constitutive equation (4.16) can be rewritten as follows.

$$Q_i=a_{ij}\cdot q_j+b'_{ij}\frac{\partial q_i}{\partial\xi}-\beta_i\cdot\theta \qquad \cdots\cdots(6.3)$$

Equation (6.3) is the revised form of the constitutive equation of the thermomechanically simple body. However, in the method presented in this paper, it is convenient to use the different b_{ij} for every time increment Δt assuming that b_{ij} is constant during each Δt.

In the following part of this section, $\dot{}$ denotes the differentiation with respect to ξ. Equation (6.3) can be rewritten in the following form.

$$Q_i=A_{ij}^{(1)}\cdot q_j+A_{ik}^{(2)}\cdot h_k+B_{ij}^{(1)}\cdot\dot{q}_j+B_{ik}^{(2)}\cdot\dot{h}_k-\beta^{(1)}\cdot\theta \qquad \cdots\cdots(6.4)$$

$$O=A_{hj}^{(2)}\cdot q_j+A_{hk}^{(3)}\cdot h_k+B_{hj}^{(2)}\cdot\dot{q}_j+B_{hk}^{(3)}\cdot\dot{h}_k-\beta_h^{(2)}\cdot\theta \qquad \cdots\cdots(6.5)$$

in which h_k is the hidden coordinate. Evaluating the eigenvalue from

$$\det(B_{hk}^{(3)}\cdot\sigma+A_{hk}^{(3)})=0 \qquad \cdots\cdots(6.6)$$

and substituting h_k obtained from equations (6.5) and (6.6) into equation (6.4), the generalized force Q_i can be described by the following equation.

$$Q_i=D_{ij}q_j+D'_{ij}\cdot\dot{q}_j+\int_0^t D_{ij}^{(S)}(\xi-\xi')\frac{\partial q_j}{\partial t'}\cdot dt'-B_i\theta-\int_0^t B_i^{(S)}(\xi-\xi')\frac{\partial\theta}{\partial t'}\,dt' \qquad \cdots\cdots(6.7)$$

where

$$D_{ij}=A_{ij}^{(1)}-\frac{A_{iS}^{(2)\prime}\cdot A_{Sj}^{(2)\prime}}{\sigma_S},$$

$$D'_{ij}=B_{ij}^{(1)}-B_{iS}^{(2)\prime}\cdot B_{Sj}^{(2)\prime},$$

$$D_{ij}^{(S)}(\xi-\xi')=\sum^S e^{-\sigma_S(\xi-\xi')}\times\left(\frac{A_{iS}^{(2)\prime}\cdot A_{Sj}^{(2)\prime}}{\sigma_S}-A_{iS}^{(2)\prime}\cdot B_{Sj}^{(2)\prime}-B_{Si}^{(2)\prime}\cdot A_{Sj}^{(2)\prime}-\sigma_S B_{iS}^{(2)\prime}\cdot B_{Sj}^{(2)\prime}\right),$$

$$B_i=\beta_i^{(1)}\frac{A_{iS}^{(2)\prime}\cdot\beta_S^{(2)\prime}}{\sigma_S},$$

$$B_i^{(S)}=\frac{A_{iS}^{(2)\prime}\cdot\beta_S^{(2)\prime}}{\sigma_S}-B_{iS}^{(2)\prime}\cdot\beta_S^{(2)\prime},$$

$$A_{Sj}^{(2)\prime}=\phi_{hS}A_{hj}^{(2)},\quad B_{Sj}^{(2)\prime}=\phi_{hS}B_{hj}^{(2)},$$

$$\beta_S^{(2)\prime}=\phi_{hS}\beta_h^{(2)},$$

in which σ_S and ϕ_{hS} are the eigenvalue and eigenvectar obtained from equation (6.6). It is clear that in equation (6.7), $1/\sigma_S$ corresponds to the relaxation time and $D_{ij}^{(S)}$ is the relaxation function.

As the constitutive equations in previous works, Tayler et. al.[3] used the equation neglecting first, second and fourth terms of equation (6.7), i.e.

$$\sigma_{ij}=\int_0^t C_{ij}^{kl}[\xi(t)-\xi'(t);\ T_0]\times\frac{\partial}{\partial\tau}[\varepsilon_{kl}(\tau)-\theta_{kl}(\tau)]\,d\tau \qquad \cdots\cdots(6.8)$$

White[4] used the relation:

$$\sigma_{ij}=2G(0)\cdot\varepsilon_{ij}-2\int_0^t\frac{\partial G(\xi-\xi')}{\partial t}\cdot\varepsilon_{ij}\,dt+\delta_{ij}\left[K-\frac{2}{3}G(0)\right]\cdot\varepsilon_m+\delta_{ij}\frac{2}{3}\int_0^t\frac{\partial G(\xi-\xi')}{\partial t}\varepsilon_m\,dt-\delta_{ij}3\alpha K\theta \qquad \cdots\cdots(6.9)$$

Becker and Nickel[5] neglected the first, second and fourth terms in equation (6.7),

$$\sigma_{ij}=\int_0^t G_{ijkl}(\xi-\xi')\frac{\partial\varepsilon_{kl}}{\partial t'}dt'-\int_0^t B_{ijkl}(\xi-\xi')\frac{\partial\theta}{\partial t'}\,dt' \qquad \cdots\cdots(6.10)$$

Linch[6], Malone and Conner[7] and others employed the elation

$$\sigma_{ij}=\int_0^t G_{ijkl}(t-\tau)\frac{\partial\varepsilon_{kl}}{\partial\tau}\,d\tau \qquad \cdots\cdots(6.11)$$

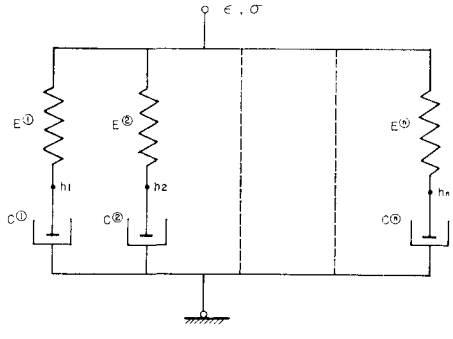

Fig. 2 Generalized maxwell model.

which was obtained from equation (6.7) neglecting first, second and thermal terms.

Referring to reference[1], the constitutive equations of the generalized Maxwell model shown in Fig. 2 can be obtained for the uniaxial case as follows.

$$\sigma=\int_0^t G(t-t')\frac{\partial\varepsilon}{\partial t'}\,dt' \qquad \cdots\cdots(6.12)$$

where

$$G(\tau)=\sum^{S}E^{(S)}e^{-\tau/\tau^{(S)}},\qquad \tau^{(S)}=\frac{C^{(S)}}{E^{(S)}}$$

σ denotes stress and ε is strain. Equation (6.12) corresponds to the basic equations presented by Boltzmann.

In this case, equation (4.16) can be written in the following form.

$$\begin{pmatrix}\sigma\\ \cdots\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}=\left(\begin{array}{c|ccc} E①+E②+\cdots+E⑪ & -E① & -E②\cdots & -E⑪\\ \hline -E① & E①+C①\frac{\partial}{\partial t} & & \\ -E② & & E②+C②\frac{\partial}{\partial t} & \\ \vdots & & & \ddots\\ -E⑪ & & & E⑪+C⑪\frac{\partial}{\partial t}\end{array}\right)\times\begin{pmatrix}\varepsilon\\ \cdots\\ h_1\\ h_2\\ \\ h_n\end{pmatrix} \qquad \cdots\cdots(6.13)$$

where $E①, E②, \cdots E⑪$, $C①, C②, \cdots,$ and $C⑪$ are the visco-elastic constants, and $h_1, h_2, \cdots, h_n$ denote hidden coordinates. Comparing equation (6.12) with equation (6.13), it is evident that $\tau^{(S)}$ corresponds to the relaxation time and $E^{(S)}$ is the relaxation spectrum in the generalized Maxwell model.

7. NUMERICAL EXAMPLES

In order to illustrate the method presented in this paper, simple numerical examples are considered. Fig. 3 shows the results of calculation obtained for the creep problem under the constant load P. Vertical components of displacement at node point No. 2 is illustrated against non-dimensional time. The constitutive equation is considered to be given by the three-element model shown in the figure. In this case, equation (5.1) can be written as:

$$S'_{ij}=2G^{①}\cdot\gamma'_{ij}-2G^{①}h_{ij}$$
$$S_{kk}=3K\cdot\varepsilon_{kk} \qquad \cdots\cdots(7.1)$$
$$O=-2G^{①}\cdot\gamma'_{ij}+2(G^{①}+G^{②})h_{ij}+2C^{③}\dot{h}_{ij}$$

where

$$\gamma'_{ij}=\gamma_{ij}-\frac{\gamma_{kk}}{3}$$
$$\cdots\cdots(7.2)$$
$$S'_{ij}=S_{ij}-\frac{S_{kk}}{3}$$

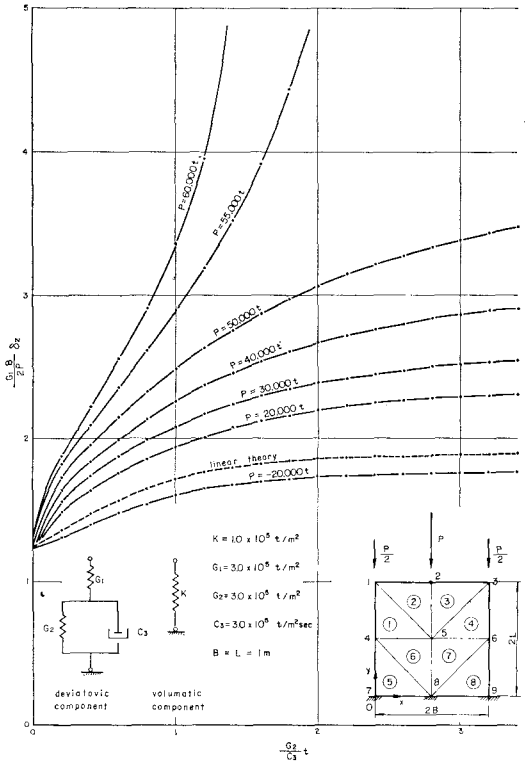

Fig. 3 Time displacement diagram (y direction component of displacement at node point No. 2, denoting δ_2).

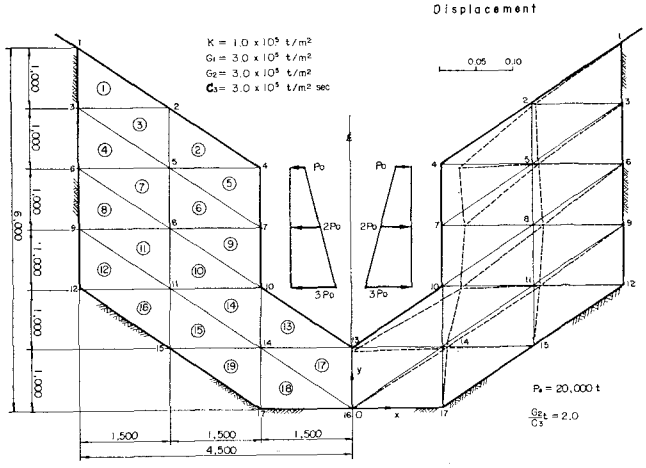

Fig. 4 Finite element idealization.

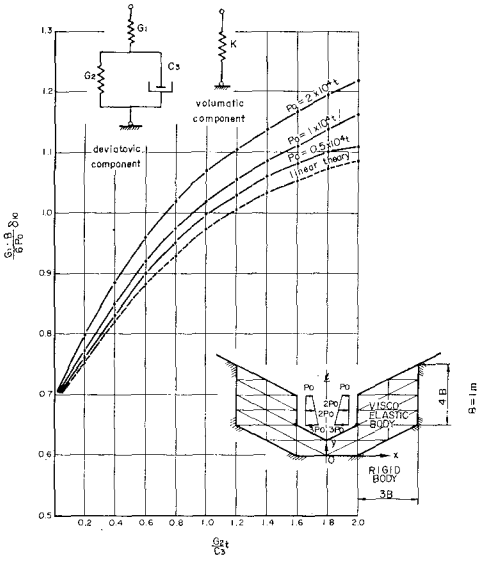

Fig. 5 Time displacement diagram (x direction component of displacement at node point No. 10, denoting δ_{10}).

in which $G^{①}$, $G^{②}$, K and $C^{③}$ are the visco-elastic constants and h_{ij} is the hidden coordinates. According to the value of external load P, the tertiary creep as well as secondary creep can be calculated.

More complicated boundary value problems are shown in Fig. 4 through 7. In these examples, the same constitutive equations described by equation (7.1) are used. The mesh idealizations and deformation diagram are shown in Fig. 4 and Fig. 6. Time-

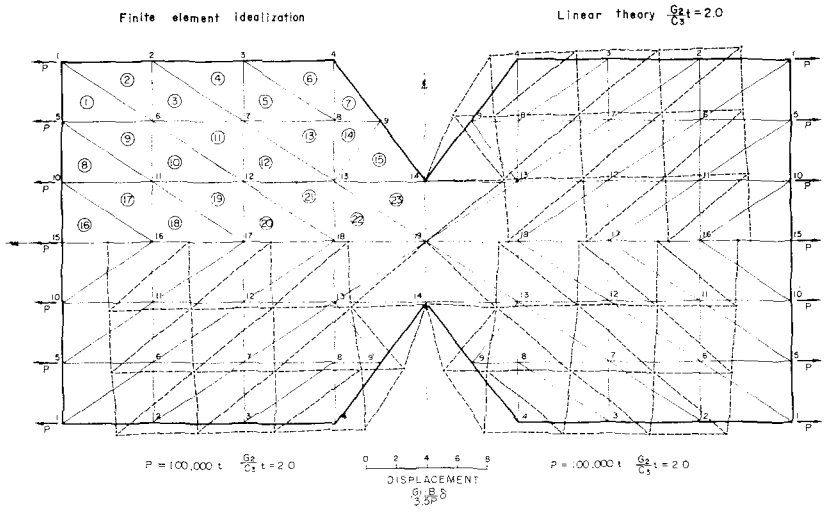

Fig. 6 Displacement.

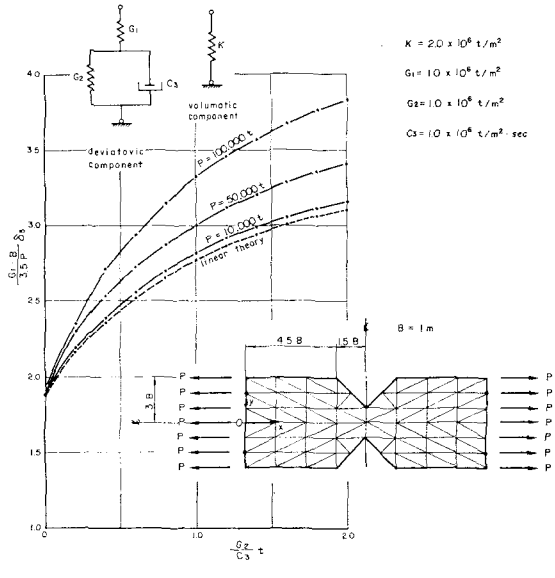

Fig. 7 Time displacement diagram (x direction component of displacement at node point No. 5, denoting δ_5).

displacement diagrams for different values of external load are illustrated in Fig. 5 and Fig. 7.

In the above three examples, triangular finite elements and constant strain type shape function using area coordinates are employed. The shape function in equation (5.9) is expressed as follows.

$$\Phi_\alpha = \frac{A_\alpha}{A} \qquad \cdots\cdots(7.3)$$

where A is the area of a triangular finite element,

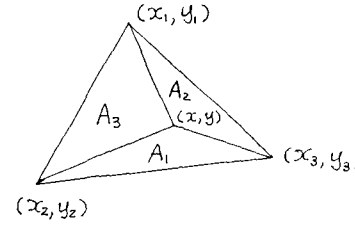

Fig. 8 Area coordinate.

and A_α denotes the area of the triangle corresponding to node α. (Fig. 8).

8. CONCLUDING REMARKS

A finite element method applied to the large strain visco-elastic body is presented using the concept of the generalized coordinates. The present paper shows that the constitutive equation can be defined by the relation between the Kirchhoff's stress tensor, Green's strain tensor and hidden coordinates. The final equations of the finite element method can be reduced to similar equations as that of Hookean body based on the large strain theory.

The final equation are the third order simultaneous equations with respect to the nodal displacements. In order to solve these equations it is convenient to use the Newton-Raphson method described in reference[2]. Secondary and tertiary creep analysis were successfully carried out for illustrative examples. The method is applied Relaxation analysis as well as creep analysis by using numerical precedures described in reference[1].

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