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# A BASIC STUDY ON REGIONAL INCOME DISPARITY ARISING FROM REGIONAL ALLOCATION OF PUBLIC INVESTMENTS

*By Etsuo* YAMAMURA*

## 1. INTRODUCTION

The existence and persistence of regional disparities at every level of development and in the history of all developed countries are familiar phenomena to governmental bodies.

This is becoming one of the most important problems for the civil planners constructing public facilities. The allocation of public facilities among regions is closely related to the promotion of social capital formation in each region.

Table 1 summarizes the public utility of Japan government expenditures. We can take several expenditure categories for civil such as Forestry conservancy & river improvement, Road, Harbor, Fishingport, Airport facilities, Living & enviroment and Disaster restration. The percentages of total expenditure for civil categories are 79% (in 1966), 78% (in 1967), 77.5% (in 1968) and 77% (in 1969).

It is clear that the civil body of public utility indicates high level and plays an important role. However, only a relatively small amount of research has been made on regional inequalities in relation to the regional allocation of public investments. In the literature, regional development models recently were developed by Rahman-Sakashita.[1),2),3)]

One of the most important problems in regional income disparities is the determination of restrictions of the minimum proportion of public investment and saving ratio among regions. These restrictions and the differential regional rates of saving are closely related to the maximization of national income. However, a detailed comparison of the sensitivity between these restrictions and the national income has not been made.

In this paper, we shall consider detailed simulations of the regional income disparities concentrating on the minimum proportion of public investment and the differential regional rates of saving. An analysis of these models consists of an application of the Discrete Maximum Principle.

**Table 1** Public Utility of Government Expenditures for Real Program Outputs

(one hundred million yen)

| Expenditure category | 1966 | 1967 | 1968 | 1969 |
|---|---|---|---|---|
| Forestry conservancy and river improvement | 1,446 (16.4%) | 1,681 (16.5%) | 1,766 (16.6%) | 2,035 (16.9%) |
| Road facility | 3,600 (40.7%) | 4,169 (40.9%) | 4,340 (40.7%) | 4,975 (41.4%) |
| Harbour facility | 464 (5.3%) | 536 (5.3%) | 568 (5.3%) | 655 (5.4%) |
| Fishing port facility | 119 (1.3%) | 143 (1.4%) | 155 (1.5%) | 184 (1.5%) |
| Airport facility | 73 (0.8%) | 97 (1.0%) | 114 (1.0%) | 142 (1.2%) |
| Housing | 487 (5.5%) | 648 (6.4%) | 697 (6.5%) | 794 (6.6%) |
| Living and enviroment | 262 (3.0%) | 343 (3.4%) | 391 (3.7%) | 480 (4.0%) |
| Agriculture base equipment | 1,098 (12.4%) | 1,306 (12.8%) | 1,394 (13.5%) | 1,625 (13.5%) |
| Forest road and Industrial Water | 217 (2.5%) | 228 (2.2%) | 245 (2.3%) | 281 (2.3%) |
| Disaster restoration | 1,017 (11.5%) | 969 (9.5%) | 923 (8.7%) | 787 (6.6%) |
| Other | 52 (0.6%) | 59 (0.6%) | 62 (0.6%) | 69 (0.6%) |
| Total | 8,835 (100%) | 10,179 (100%) | 10,655 (100%) | 12,027 (100%) |

## 2. MATHEMATICAL FORMATION OF MODELS

This chapter presents the mathematical formation of regional development models. The analysis of these models consists of an application

* Assistant, Dept. of Civil Engineering, Hokkaido University.

of the Discrete Maximum Principle, then this principle is briefly reviewed.

In the process with the continuous state variables in $s$-dimensional space, the state variables equations can be written in the form of a system of differential equations.

$$\dot{x}_i = f_i(x, m, t) \quad (i=1, \cdots, s) \qquad \cdots\cdots(1)$$

The initial condition is as follows:

$$x_i(t_0) = x_i^0 \qquad (i=1, \cdots, s) \qquad \cdots\cdots(2)$$

The $t$-dimensional control vector $m$ is satisfied the following restrictions:

$$g_j(m) \leqq 0 \qquad (j=1, \cdots) \qquad \cdots\cdots(3)$$

We shall find the optimal control vector to take $J$ on the largest possible value:

$$J = \sum_i^s b_i \cdot x_i(T_f) \rightarrow \text{Max} \qquad \cdots\cdots(4)$$

$T_f$: the terminal time

$b_i$: known values

At the terminal time $T_f$, the state variable is unknown (with right hand side). Then, the procedure for solving such the optimization problem by the Pontryagin Maximum Principle is to introduce the Hamiltonian function $H$ and auxiliary variable $\psi$.

$$H(x, m, \psi, t) = \sum_i^s \psi_i(t) \cdot f_i(x, m, t) \cdots\cdots(5)$$

Where the auxiliary variables $\psi_i(t)$ are satisfied the following equations:

$$\dot{\psi}_i = -\sum_j^s \psi_j(t) \frac{\partial f_j(x, m, t)}{\partial x_i} \qquad (j=1, \cdots, s) \cdots\cdots(6)$$

and we take the system of equations:

$$\left.\begin{array}{l} \dot{x}_i = \partial H / \partial \psi_i \\ \dot{\psi}_i = -\partial H / \partial x_i \\ x_i(t_0) = x_i^0 \\ \psi_i(T_f) = -b_i \end{array}\right\} \qquad (i=1, \cdots, s) \cdots\cdots(7)$$

The optimal control variables are the admissible control variables satisfied the following condition at each time:

$$H(x, m, \psi, t) = \sum_i^s \psi_i(t) \cdot f_i(x, m, t) \rightarrow \text{Min} \cdots\cdots(8)$$

Next, we shall consider the Discrete Maximum Principle. Let take the process with the discrete state variables in $s$-dimensional space. The meaning of the discrete state variables at each cycle $T$ during total process. Then, this process takes the following form:

$$x_i(k \cdot T) - x_i((k-1) \cdot T) = T \cdot f_i\{x((k-1) \cdot T), m(k \cdot T), k \cdot T\} \qquad (i=1, \cdots, s) \quad (k=1, \cdots, N) \cdots\cdots(9)$$

The initial condition is as follows:

$$x_i(0) = x_i^0 \quad (i=1, \cdots, s) \qquad \cdots\cdots(10)$$

The $t$-dimensional control vector $m$ is satisfied the following restrictions:

$$g_i(m(k \cdot T)) \leqq 0 \quad (k=1, \cdots, N) \qquad \cdots\cdots(11)$$

We shall find the optimal control vector to take $J$ on largest possible value at each time.

$$J = \sum_i^s c_i \cdot x_i(N \cdot T) \rightarrow \text{Max} \qquad \cdots\cdots(12)$$

where

$N \cdot T = T_f$

$c_i$: known values

The procedure for solving such problem is as follows. First, we determine the variables such as:

1) variable $x(k \cdot T)$ is determined by the both variables $x((k-1) \cdot T)$ and $m(k \cdot T)$.
2) auxiliary variable $\psi((k-1) \cdot T)$ is determined by the auxiliary variable $\psi(k \cdot T)$, control variable $m(k \cdot T)$ and state variable $x((k-1) \cdot T)$.

We introduce the Hamiltonian function.

$$H\{x((k-1) \cdot T), m(k \cdot T), \psi(k \cdot T), k \cdot T\} = T \cdot \sum_j^s \psi_j(k \cdot T) \cdot f_j\{x((k-1) \cdot T), m(k \cdot T), k \cdot T\} \qquad \cdots\cdots(13)$$

where the auxiliary variables $\psi_i(k \cdot T)$ are as follows:

$$\psi_i(k \cdot T) - \psi_i((k-1) \cdot T) = -T \cdot \sum_j^s \psi_j(k \cdot T) \times \frac{\partial f_j\{x((k-1) \cdot T), m(k \cdot T), k \cdot T\}}{\partial x_i((k-1) \cdot T)} \qquad (i=1, \cdots, s) \cdots\cdots(14)$$

and we take the following equations of the system:

$$x_i(k \cdot T) - x_i((k-1) \cdot T) = \partial H / \partial \psi_i(k \cdot T) \qquad \cdots\cdots(15)$$

$$\psi_i(k \cdot T) - \psi_i((k-1) \cdot T) = -\partial H / \partial x_i((k-1) \cdot T) \qquad \cdots\cdots(16)$$

$$x_i(0) = x_i^0 \qquad \psi_i(N \cdot T) = -c_i$$
$$(i=1, \cdots, s) \quad (k=1, \cdots, N)$$

The optimal control variables are the admissible control variables satisfied the following conditions at each time.

$$H\{x((k-1) \cdot T), m(k \cdot T), \psi(k \cdot T), k \cdot T\} \rightarrow \text{Min} \cdots\cdots(17)$$

We shall translate the discrete processes into the multistage decision processes according to the following procedure in which the stage at each time $t = k \cdot T$ ($k=1, \cdots\cdots, N$) is the $k$th step one.

The equations (15), (16) are rewritten as follows.

$$x_i{}^k - x_i{}^{k-1} = \partial H/\partial \phi_i{}^k \quad \cdots\cdots(18)$$
$$\phi_i{}^k - \phi_i{}^{k-1} = -\partial H/\partial x_i{}^{k-1}$$
$$(i=1,\ \cdots,\ s)\quad (k=1,\ \cdots,\ N) \quad \cdots\cdots(19)$$

The Hamiltonian function is rewritten as follows:

$$H = T\cdot \sum_j^s \phi_j{}^k \cdot f_j(x^{k-1},\ m^k) \quad \cdots\cdots(20)$$

where

$$x(k\cdot T) = x^k$$

then,

$$x_i{}^k = x_i{}^{k-1} + \partial H/\partial \phi_i{}^k$$
$$= x_i{}^{k-1} + T\cdot f_i(x^{k-1},\ m^k) \quad \cdots\cdots(21)$$
$$\phi_i{}^{k-1} = \phi_i{}^k + \partial H/\partial x_i{}^{k-1}$$
$$= \phi_i{}^k + T\cdot \sum_j^s \phi_j{}^k \frac{\partial f_j(x^{k-1},\ m^k)}{\partial x_i{}^{k-1}}$$
$$(i=1,\ \cdots,\ s)\quad (k=1,\ \cdots,\ N) \cdots\cdots(22)$$

We can rewrite the conditions (21), (22) as follows:

$$x_i{}^k = \partial H^k/\partial \phi_i{}^k \quad \cdots\cdots(23)$$
$$\phi_i{}^{k-1} = \partial H^k/\partial x_i{}^{k-1} \quad \cdots\cdots(24)$$
$$H^k = T\cdot \sum_j^s \phi_j \cdot f_j(x^{k-1},\ m^k)$$
$$\equiv \sum_j^s \phi_j{}^k (x_j{}^{k-1} + T\cdot f_j(x^{k-1},\ m^k)) \cdots\cdots(25)$$

With respect to the detail conception of the Pontryagin Maximum Principle, the reader may refer to the books.[4),5),6)]

Next, we shall consider the mathematical formation of regional development models where the following conditions hold.

1) the allocation of public investment is aimed at maximizing the national income at the end of the planning period;
2) the process of economic growth should not bring about any wide disparity in regional living standards at the end of the planning period;
3) the supply of public funds for investment will be limited to the sum of private savings in each region; and
4) the productivity of investment and the saving ratio differ at each time process in each region.

The analysis is an explicit planning model for a closed economy, and it is assumed that planned saving equals planned investment through central direction.

We define the notations as follows:

$S_j{}^i$ = the saving ratio of region $j$ at $i$ time.

$P_j{}^i$ = the productivity of investment of region $j$ at $i$ time.

$U_j{}^i$ = the proportion of investment shared to the region $j$ at $i$ time.

$$\left(\sum_j^M U_j{}^i = 1\right) \quad (i=1,\ \cdots,\ N)$$

$D^i$ = the minimum proportion of investment at $i$ time.

$(0 \leqq D^i \leqq 1/M)$

The minimum proportion of investment cannot exceed a certain proportion $(1/M)$.

$X_j{}^i$ = the regional income of region $j$ at $i$ time.

$$(X_j{}^i - X_j{}^{i-1} \geqq 0) \quad (i=1,\ \cdots,\ N)$$

There cannot be any net consumption of capital or disinvestment in any region.

$Z^i$ = the national income at $i$ time.

The national income of the country in any time $i$ equals the sum of the $M$ regional incomes, and may be written as:

$$\left(z^i = \sum_j^M X_j{}^i\right)$$

The performance equations from the condition (3) are as follows:

$$\sum_j^M (X_j{}^i - X_j{}^{i-1})/P_j{}^i = \sum_j^M S_j{}^i \cdot X_j{}^i \quad \cdots\cdots(26)$$

where,

$$X_j{}^i - X_j{}^{i-1} = P_j{}^i \cdot U_j{}^i \left(\sum_j^M S_j{}^i \cdot X_j{}^i\right)$$
$$(i=1,\ \cdots,\ N) \cdots\cdots(27)$$

The left-hand side represents total investment and the right-hand side represents total saving in the whole country at $i$ time.

The boundary conditions are as follows:

$$X_j{}^0 = C_j \quad D^i \leqq U_j{}^i \leqq 1 - D^i \quad \cdots\cdots(28)$$

The objective function is aimed at maximizing the national income at the end of planning period.

$$J = Z^N \to \text{Max} \quad \left(Z^N = \sum_j^M X_j{}^N\right) \quad \cdots\cdots(29)$$

The Hamiltonian function and auxiliary variable are as follows:

$$H^i = \sum_j^M \phi_j{}^i \left\{P_j{}^i \cdot U_j{}^i \cdot \left(\sum_j^M S_j{}^i \cdot X_j{}^i\right) + X_j{}^{i-1}\right\}$$
$$+ \phi^i_{M+1} \left\{\sum_j^M \left(P_j{}^i \cdot U_j{}^i \cdot \left(\sum_j^M S_j{}^i \cdot X_j{}^i\right) + X_j{}^{i-1}\right)\right\}$$
$$\cdots\cdots(30)$$
$$\phi_j{}^{i-1} = \partial H^i/\partial X_j{}^{i-1} \quad \cdots\cdots(31)$$

where,

$$\left.\begin{array}{ll} \phi^i_{M+1} = 1 & (i=1,\ \cdots,\ N) \\ \phi_j{}^N = 0 & (j=1,\ \cdots,\ M) \end{array}\right\} \quad \cdots\cdots(32)$$

Next, we shall consider the determination of the optimal proportions of investment $U = \{U_j{}^i\}$ encountered on the computation of this problem. The algorithm for the problem can be carried out as follows:

Step 1. Assume the proportions of investment $U=\{U_j^i\}$. Here, $X=\{X_j^0\}$ is shown from (28) and $\phi=\{\phi_j^i\}$ from (32).

Step 2. Calculate the regional Incomes $X=\{X_j^i\}$ from the equation (27).

Step 3. Calculate the auxiliary variables $\phi=\{\phi_j^{i-1}\}$ from the equation (31).

Step 4. Hamiltonian function can be calculated from the equation (30). Find out the optimal proportion of investment $\bar{U}=\{\bar{U}_j^i\}$ so as to maximize the Hamiltonian function $H=\{H^i\}$.

Step 5. If the proportions of investment assumed at step 1 ($U$) are equal to the optimal proportions of investment calculated at step 4 ($\bar{U}$), the computation is completed. If $U \neq \bar{U}$, repeat from step 1 to step 4 until $U=\bar{U}$.

## 3. SIMULATIONS OF THE MODELS

In this chapter, we shall introduce several typical simulations of the model because we do not have enough space to describe all simulations.

### (1) Model 1

This model is a simple one in which the productivity of investment and saving ratio are assumed to be a constant over time in a two-region economy. The data used in the computation is shown as follows.

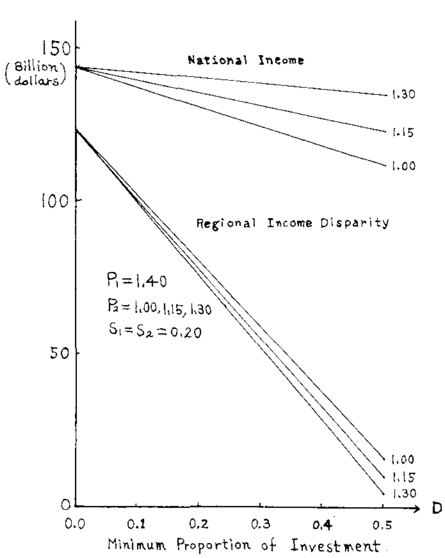


**Fig. 1** The National Income and Regional Income Disparity by the Minimum Proportion of Investment.

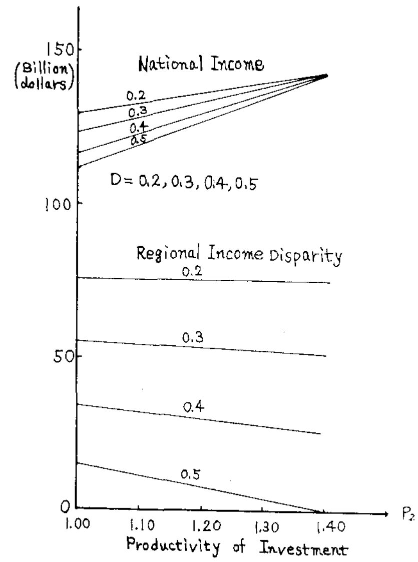


**Fig. 2** The National Income and Regional Income Disparity by the Productivity of Investment.

$P_1=1.400$ $P_2=1.000$, 1.150, 1.300

$S_1=S_2=0.200$ $X_1^0=X_2^0=10$ (Billion dollars)

Where the minimum proportions of investment are changed in the order of magnitude from 0.000 to 0.500. The results of the model 1 are shown in Fig. 1 and 2.

It is clear from the graphs described above that the decreasing rate of national income is small and the regional income disparity between two regions shows a rapidly decreasing rate as the minimum proportion of investment increases.

A detailed investigation of this situation reveals that if the difference of the productivity of investment between region 1 and region 2 become smaller, the decreasing rate of the regional income disparity is affected remarkably by the increase of the minimum proportion of investment.

### (2) Model 2

In this model, the productivity of investment is assumed to be linear increasing functions.

$P_1^t=0.050 \cdot t+1.050$

$P_2^t=0.025 \cdot t+1.150$ ($t=1$, ……, 8)

Two simulations are considered here. One is a simulation in which the regional income disparity and the maximum regional income disparity are analyzed by the changes of the saving ratio. The other is a simulation in which these disparities are analyzed by the changes of the distribution function of the saving ratio.

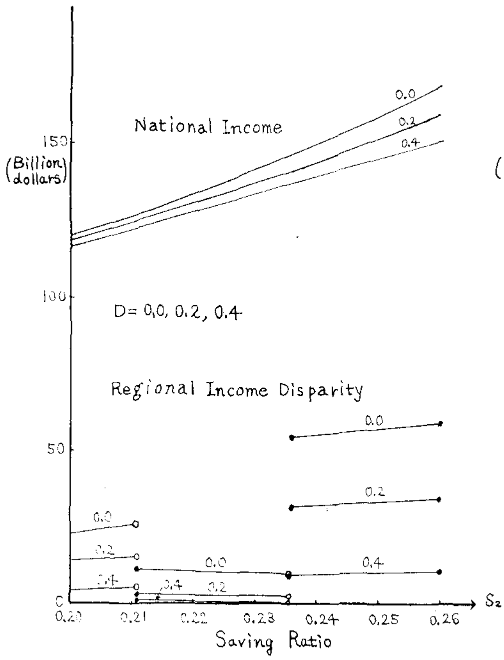


**Fig. 3** The National Income and Regional Income Disparity by the Saving Ratio.

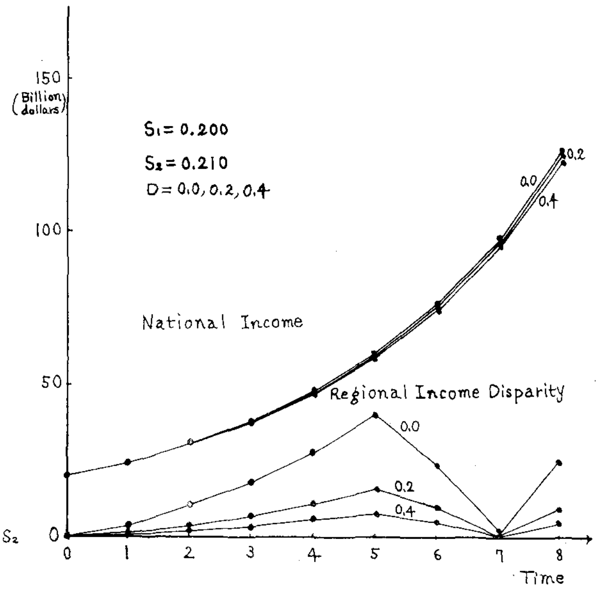


**Fig. 5** The National Income and Regional Income Disparity by the Minimum Proportion of Investment (1).

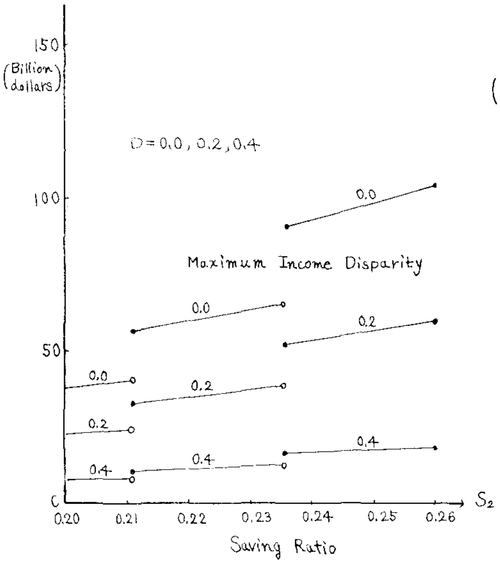


**Fig. 4** The Maximum Income Ddisparity by the Minimum Proportion of Investment.

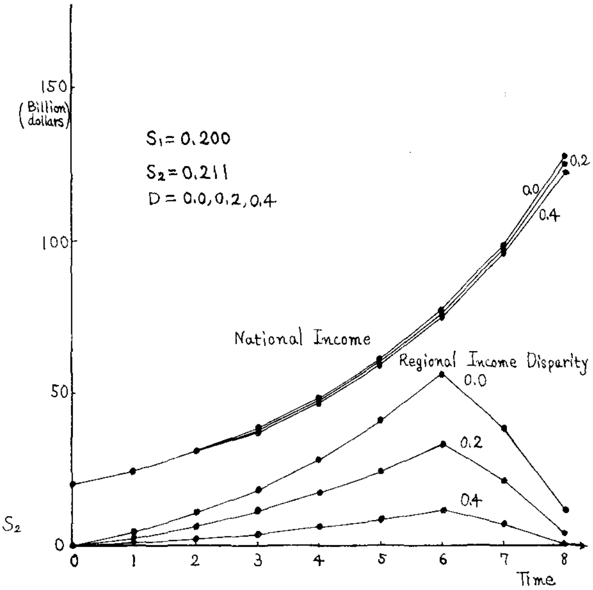


**Fig. 6** The National Income and Regional Income Disparity by the Minimum Proportion of Investment (2).

**a)** The first simulation

The results of the simulations are shown from Fig. 3 to Fig. 8. In Fig. 3 the minimum values of disparity are affected remarkably by the changes of the phases of the saving ratio ($S_2$).

In this case, three phases are observed as follows:

Phase 1. $S_2=0.200\sim0.210$
Phase 2. $S_2=0.211\sim0.235$
Phase 3. $S_2=0.236\sim0.260$

It is clear that the values of the regional income disparity have the minimal values in phase 2 and even when the minimum proportion of investment increases, the disparity becomes remarkably small. However, although the saving ratio is increased from more than $S_2=0.235$, the minimum values of the disparity are not always found in phase 3 mentioned above.

In Fig. 4, the maximum income disparity between two regions increases in each phase as the $S_2$ increases, but in the case of the large portion of minimum proportion of investment, this disparity becomes remarkably small.

Next, to clarify the cause for the growth of the gaps of disparity as the $S_2$ increases, the detailed time processes of national income and regional income disparity will be analyzed. Detailed graphs are shown in Fig. 5 to Fig. 8. From the facts presented in the graphs, the cause of the gaps between $S_2=0.210$ and $S_2=0.211$ is the change of times processes from 6th time stage to 7th and the one between $S_2=0.235$ and $S_2=0.236$ is from 7th time stage to 8th.

**b)** The second simulation

The purpose of the second simulation is to analyze the regional income disparity by the changes of the distribution function of saving ratio. The distribution of these functions are shown in Fig. 9. In this case, four types are assumed.

The results of simulation are shown in Fig. 10. From the facts presented in the graph, the regional income disparity and the maximum regional income disparity decrease by the changes of the types from 1 to 4, and the difference of national income among four types at the end of planning period is remarkably small. From the facts obtained from these simulations, it may be concluded that the

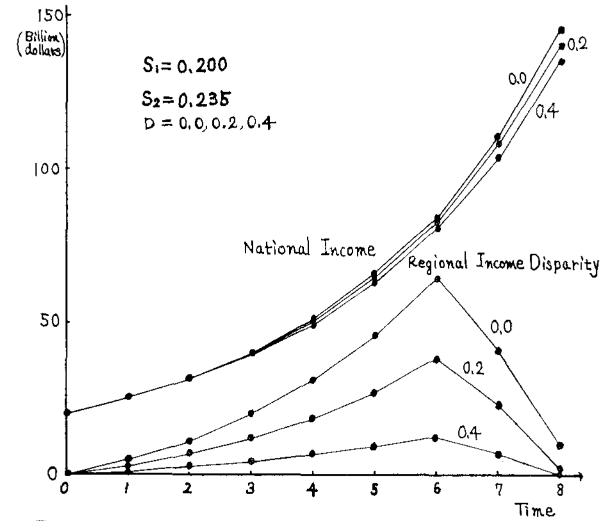


**Fig. 7** The National Income and Regional Income Disparity by the Minimum Proportion of Investment (3).

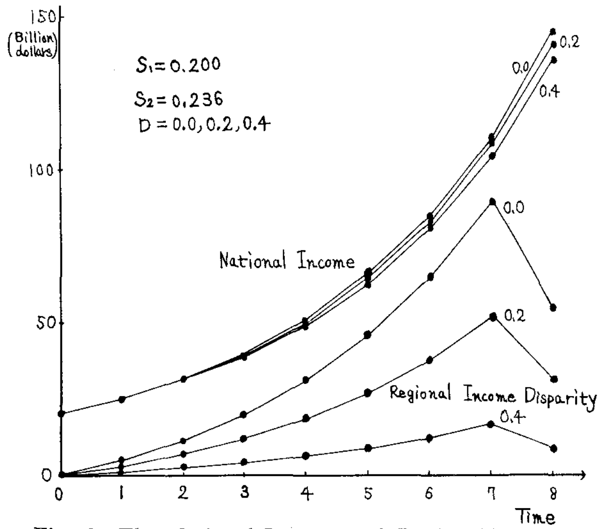


**Fig. 8** The National Income and Regional Income Disparity by the Minimum Propertion of Investment (4).

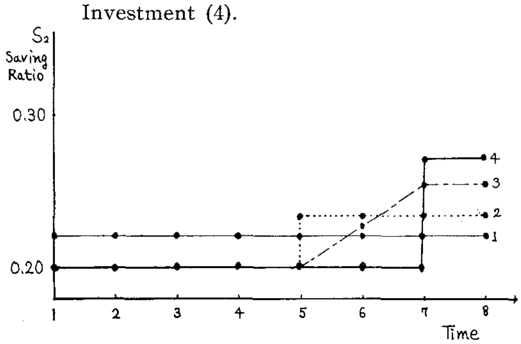


**Fig. 9** The Distribution Functions of Saving Ratio.

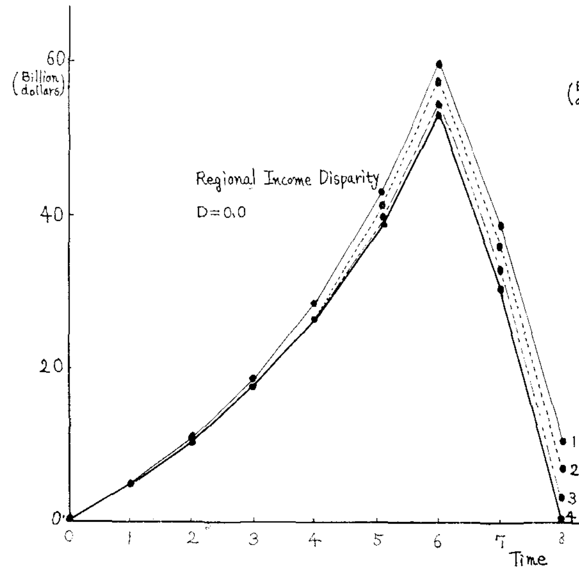


**Fig. 10** The Regional Income Disparities of Four Types by $D=0.00$.

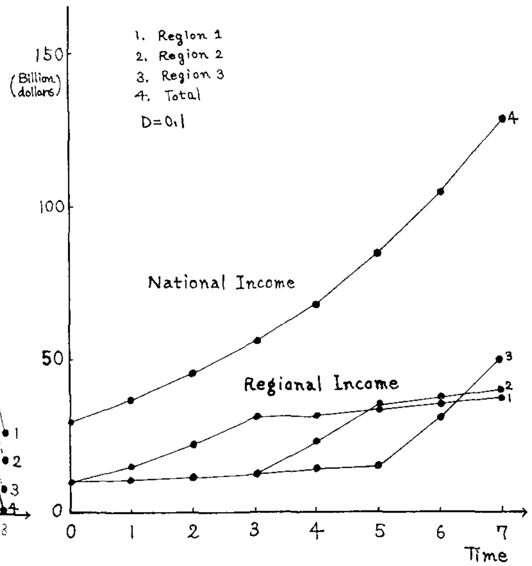


**Fig. 11** The National Income and Regional Income by $D=0.1$ in a Three-Region Economy (1).

functions of type 3 and 4 are far better than those of type 1 and 2 at the same level of national income.

### (3) Model 3

This model illustrates complex simulations in which the functions of productivity of investment are assumed to be a nonlinear over time in a three region economy:

$P_1{}^t=1.20\cdot(0.98)^t$

$P_2{}^t=1.15$

$P_3{}^t=1.10\cdot(1.01)^t \quad (t=1, \cdots\cdots, 7)$

where

$X_1{}^0=X_2{}^0=X_3{}^0=10$ (Billion dollars)

$S_1=S_2=S_3=0.200$

In addition, the minimum proportions of investment are assumed in the order of 0.100, 0.200 and 0.300. The results of the simulations are shown in Fig. 11 to Fig. 18. The time processes of national and regional income are illustrated with $D$=0.100, 0.200, and 0.300 in Fig. 11 to Fig. 13. And the difference of the national income is observed as a small decline.

With respect to the disparity in Fig. 14 to Fig. 16, the total absolute disparity of regional income at the end of the planning period decreased from 25 (in $D$=0.100) to 5 (in $D$=0.300) and also the maximum absolute disparity decreased remarkably from 40 (in $D$=0.100) to 6 (in $D$=0.300). As indicated in the case of a two-region economy, these disparities are also affected by the increase of the minimum proportion of investment.

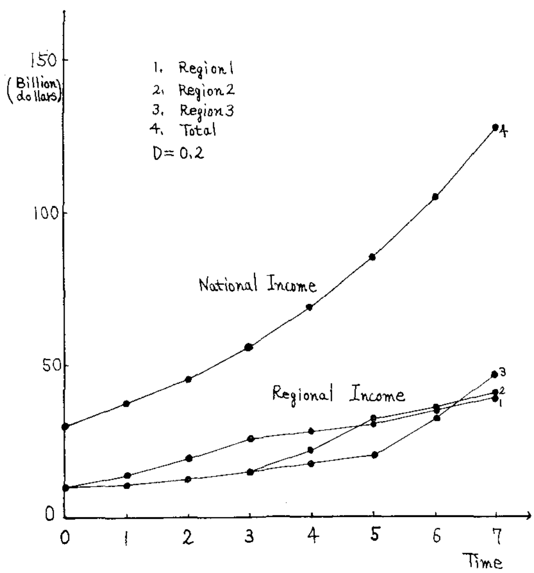


**Fig. 12** The National Income and Regional Income by $D=0.2$ in a Three-Region Economy (2).

Next, we shall compare the differences of the graphs presented in Fig. 15 and Fig. 18. The control variables are shown in Table 2. By comparing these graphs and tables, it may be observed that the differences of the national income is indicated as a negligible decline, but

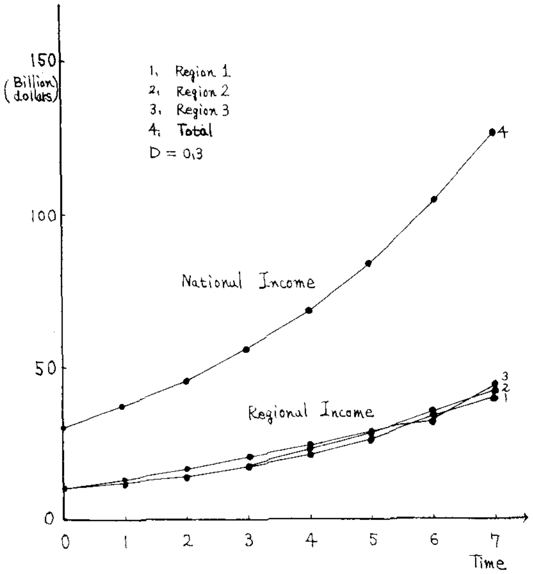


Fig. 13 The National Income and Regional Income by $D=0.3$ in a Three-Region Economy (3).

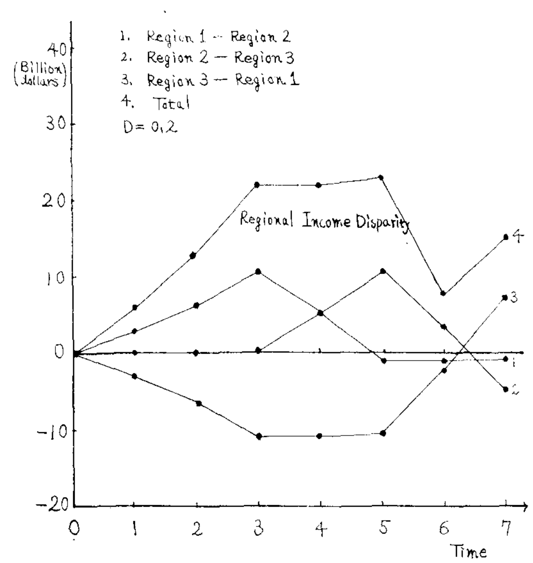


Fig. 15 The Regional Income Disparity by $D=0.2$ in a Three-Region Economy.

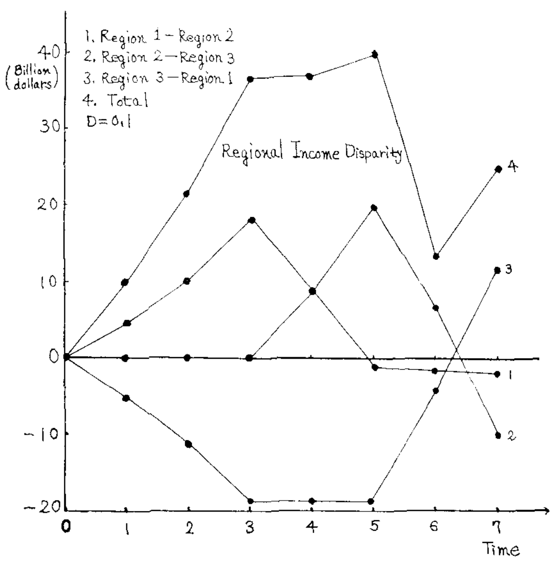


Fig. 14 The Regional Income Disparity by $D=0.1$ in a Three-Region Economy.

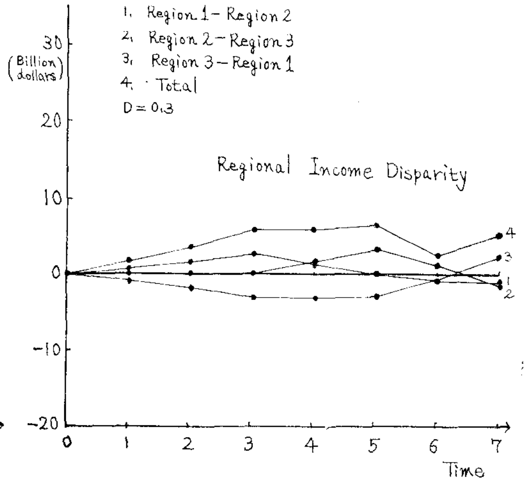


Fig. 16 The Regional Income Disparity by $D=0.3$ in a Three-Region Economy.

the total absolute disparity at the end of the planning period decreased from 15 to 10 by the flight change of the control variables. Thus, these simulations indicate that the second optimal policy of the maximization of national income becomes an important field for the study of regional disparity.

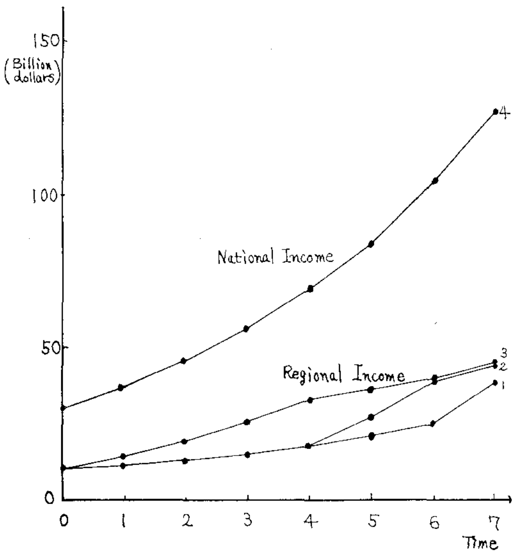


**Fig. 17** The National Income and Regional Income by the Second Optimal Policy.

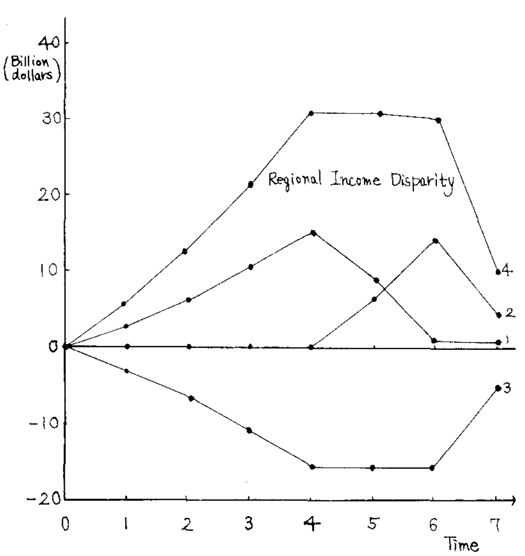


**Fig. 18** The Regional Income Disparity by the Second Optimal Policy.

**Table 2**

| Stage | Optimal Policy | | | Second Optimal Policy | | |
|---|---|---|---|---|---|---|
| | Reg. 1 | Reg. 2 | Reg. 3 | Reg. 1 | Reg. 2 | Reg. 3 |
| 1 | 0.6 | 0.2 | 0.2 | 0.6 | 0.2 | 0.2 |
| 2 | 0.6 | 0.2 | 0.2 | 0.6 | 0.2 | 0.2 |
| 3 | 0.6 | 0.2 | 0.2 | 0.6 | 0.2 | 0.2 |
| 4 | 0.2 | 0.6 | 0.2 | 0.6 | 0.2 | 0.2 |
| 5 | 0.2 | 0.6 | 0.2 | 0.2 | 0.6 | 0.2 |
| 6 | 0.2 | 0.2 | 0.6 | 0.2 | 0.6 | 0.2 |
| 7 | 0.2 | 0.2 | 0.6 | 0.2 | 0.2 | 0.6 |

## 4. CONCLUSION

In summary, we have investigated several typical simulations of regional allocation model based on the Discrete Maximum Principle. By applying this principle, we can formulate a more generalized model.

From the facts obtained in the simulations described above, the following four points may be concluded.

First, in Model 1, it seems to be clear that when the difference of productivity of investment between two regions become smaller, the decreasing rate of regional income disparity is affected remarkably by the increase of the minimum proportion of investment.

Second, in Model 2 (a), we can find an optimal phase of saving ratio by which to minimize the disparity by controlling the saving ratio.

Third, in Model 2 (b), it seems to be clear that ones of the optimal types of saving functions are observed in type 3 and 4. It is considered that the distribution function of saving ratio is one of the important factors.

Fourth, in Model 3, the function of productivity of investment is considered to be a nonlinear over time in a three-region economy. We may conclude that these simulations indicate the same results as seen in a two-region economy and the second optimal policy of national income becomes an important field for the study of regional disparity.

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