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DIFFUSION DUE TO RANDOM WAVES

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SYNOPSIS

The knowledge of the physical processes of oceanic mixing has been advanced in the last decade owing to enormous efforts by both scientists and engineers in this field. Nevertheless, it is far from completion. In this paper the effect of random waves on oceanic mixing is specified. The author described in a preceding investigation that the linearized random waves had no diffusivity due to the basic condition of the covariance stationary, second order stochastic process. The analysis is extended up to the third-order approximation including wave-wave interactions. Turbulent diffusion coefficient is given in terms of wave spectra.

Derived diffusion coefficient is classified into two regions. In one region diffusion coefficient D is linearly proportional to time elapsed after the release of dye patches. In the other region D is constant. The criterion between these two regions is the interaction time of random waves. However, precise description of the interaction time is not sufficient because of relatively poor knowledge of wave-wave interactions.

The magnitude of the theoretically derived diffusion coefficient is compared with that observed in ocean and in a wave tank. It is a subject of further study to perform experiments under the same conditions as those assumed in the theory.

INTRODUCTION

The objective of this investigation is to specify the effect of random waves on oceanic mixing. Analysis is concentrated on the mixing of dynamically passive contaminants the presence of which does not affect the dynamics of the ambient fluid and the effect of surface tension is excluded because the main concern is placed on wind waves. It is intended to express the coefficient of eddy diffusion in terms of wave spectra, since we have relatively firm basis for the form of wave spectra among many unknown factors concerning with ocean. A brief review of previous works in the related fields is given below.

A comprehensive survey of theoretical models describing the horizontal distribution of concentration due to turbulent diffusion in the sea was made by Okubo[1]. According to theories, there are two classes in solution of radially symmetric horizontal diffusion in ocean. The horizontal variance of released patch increases as t^2 or t^3.

Although vertical mixing is a weak process compared with horizontal mixing in ocean, the difference in the mean flow gives rise to an effective mixing in the longitudinal direction when combined with transverse diffusion due to small-scale random motions. Okubo[2] explained the shear effect model in two- and three-dimensional schemes with linear distribution of velocity. He demonstrated dye release experiments made in the Cape Kennedy area solely on the basis of shear-diffusion models with considerable success.

Hereafter, wave theories related to the present study are discussed briefly. Non-linear interactions between pairs of intersecting gravity wave trains of arbitrary wavelength and direction on the surface were studied by Phillips[3]. It is shown that the third-order terms can give rise to tertiary components whose phase velocity is equal to the phase velocity of a free infinitesimal wave of the same wave number, and when this condition is satisfied the amplitude of the tertiary component grows linearly with time in a resonant manner, and there is a continuing flux of potential energy from one wave-number to another. The characteristic development time for the amplitude of the tertiary interactions to become comparable with that of the primary is obtained to be of order of the (-2)-power of the geometric mean of the primary wave slopes times the period of the tertiary wave.

Pierson[4] solved the Lagrangian equations of motion by a perturbation method up to the second order for a long-crested wave train. According to

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this model, he obtained the probability that some particle is involved in a breaking wave in terms of wave spectra.

Hino[5] investigated the one-dimensional turbulent diffusion induced by wave motions in the Eulerian representation. He obtained the Eulerian expression of the velocity of a particle in a wave by the sum of the mean orbital velocity and a fluctuation. The specific form of the fluctuating velocity and its gradient, however, is left to be decided in measurements.

Tamai[6] considered the diffusion due to linearized random waves according to the Lagrangian representation of wave motion. He expressed diffusion coefficient in terms of wave spectra following Taylor's[7] method of approach to diffusion of continuous movement. Derived turbulent diffusion coefficient comes to be zero for linearized random waves.

This paper presents the analysis of the diffusion due to non-linear wave-wave interactions up to the third-order approximation. Comparison is made between the theoretical result and the previous observation in ocean and in an experimental wave tank.

THEORETICAL CONSIDERATIONS

The Lagrangian equations are stated in the following paragraphs. Let α, β and δ be the x, y and z coordinates of a particle of fluid. In the Lagrangian system of equations a solution to the equations consists of finding the positions x, y and z of all the particles in the fluid as a function of time and the initial positions of the particles, α, β and δ. The Lagrangian equations are given by Eq. (1) where subscripts denote partial differentiation.

$$\left.\begin{aligned} &x_{tt}x_\alpha+y_{tt}y_\alpha+(z_{tt}+g)z_\alpha+p_\alpha/\rho=0\\ &x_{tt}x_\beta+y_{tt}y_\beta+(z_{tt}+g)z_\beta+p_\beta/\rho=0\\ &x_{tt}x_\delta+y_{tt}y_\delta+(z_{tt}+g)z_\delta+p_\delta/\rho=0 \end{aligned}\right\}\cdots\cdots(1)$$

The equation of continuity is expressed by

$$\frac{\partial(x,\ y,\ z)}{\partial(\alpha,\ \beta,\ \delta)}=1 \qquad\cdots\cdots(2)$$

Such solutions need not be irrotational, but, if a function, $F(\alpha,\ \beta,\ \delta,\ t)$, can be found such that

$$\begin{aligned} dF=&(x_tx_\alpha+y_ty_\alpha+z_tz_\alpha)\,d\alpha+(x_tx_\beta+y_ty_\beta\\ &+z_tz_\beta)\,d\beta+(x_tx_\delta+y_ty_\delta+z_tz_\delta)\,d\delta\cdots\cdots(3) \end{aligned}$$

is a perfect differential, there is no vorticity.

A zero order solution to these equations is given by

$$x=\alpha,\quad y=\beta,\quad z=\delta,\quad p=p_0-g\rho\delta\ \cdots\cdots(4)$$

in which all fluid particles are at rest in hydrostatic equilibrium under the force of gravity. Here δ is set to be positive in the upward direction.

We expand about the zero order solution in terms of a small parameter, ε, as in Eq. (5) in which F_0 is a constant. Here we think of ε as equal to ak, but the first order terms must have the dimensions of a length, and so $a=\varepsilon/k$ is used in the various solutions that are obtained. The parameter ε never appears explicitly.

$$\left.\begin{aligned} &x=\alpha+\varepsilon x_{\mathrm{I}}+\varepsilon^2x_{\mathrm{II}}+\varepsilon^3x_{\mathrm{III}}\\ &y=\beta+\varepsilon y_{\mathrm{I}}+\varepsilon^2y_{\mathrm{II}}+\varepsilon^3y_{\mathrm{III}}\\ &z=\delta+\varepsilon z_{\mathrm{I}}+\varepsilon^2z_{\mathrm{II}}+\varepsilon^3z_{\mathrm{III}}\\ &p=p_0-g\rho\delta+\varepsilon p_{\mathrm{I}}+\varepsilon^2p_{\mathrm{II}}+\varepsilon^3p_{\mathrm{III}}\\ &F=F_0+\varepsilon F_{\mathrm{I}}+\varepsilon^2F_{\mathrm{II}}+\varepsilon^3F_{\mathrm{III}} \end{aligned}\right\}\cdots\cdots(5)$$

where subscripts of Roman numerals explain the order of approximation in the solution.

The equations in ε become

$$\left.\begin{aligned} &x_{\mathrm{I}_{tt}}+gz_{\mathrm{I}_\alpha}+p_{\mathrm{I}_\alpha}/\rho=0\\ &y_{\mathrm{I}_{tt}}+gz_{\mathrm{I}_\beta}+p_{\mathrm{I}_\beta}/\rho=0\\ &z_{\mathrm{I}_{tt}}+gz_{\mathrm{I}_\delta}+p_{\mathrm{I}_\delta}/\rho=0\\ &x_{\mathrm{I}_\alpha}+y_{\mathrm{I}_\beta}+z_{\mathrm{I}_\delta}=0\\ &dF_{\mathrm{I}}=x_{\mathrm{I}_t}d\alpha+y_{\mathrm{I}_t}d\beta+z_{\mathrm{I}_t}d\delta \end{aligned}\right\}\cdots\cdots(6)$$

The equations in ε^2 become

$$\left.\begin{aligned} &x_{\mathrm{II}_{tt}}+gz_{\mathrm{II}_\alpha}+p_{\mathrm{II}_\alpha}/\rho=-x_{\mathrm{I}_{tt}}x_{\mathrm{I}_\alpha}-y_{\mathrm{I}_{tt}}y_{\mathrm{I}_\alpha}-z_{\mathrm{I}_{tt}}z_{\mathrm{I}_\alpha}\\ &y_{\mathrm{II}_{tt}}+gz_{\mathrm{II}_\beta}+p_{\mathrm{II}_\beta}/\rho=-x_{\mathrm{I}_{tt}}x_{\mathrm{I}_\beta}-y_{\mathrm{I}_{tt}}y_{\mathrm{I}_\beta}-z_{\mathrm{I}_{tt}}z_{\mathrm{I}_\beta}\\ &z_{\mathrm{II}_{tt}}+gz_{\mathrm{II}_\delta}+p_{\mathrm{II}_\delta}/\rho=-x_{\mathrm{I}_{tt}}x_{\mathrm{I}_\delta}-y_{\mathrm{I}_{tt}}y_{\mathrm{I}_\delta}-z_{\mathrm{I}_{tt}}z_{\mathrm{I}_\delta}\\ &x_{\mathrm{II}_\alpha}+y_{\mathrm{II}_\beta}+z_{\mathrm{II}_\delta}=z_{\mathrm{I}_\alpha}x_{\mathrm{I}_\delta}+z_{\mathrm{I}_\beta}y_{\mathrm{I}_\delta}+x_{\mathrm{I}_\beta}y_{\mathrm{I}_\alpha}\\ &\qquad\qquad-y_{\mathrm{I}_\beta}z_{\mathrm{I}_\delta}-z_{\mathrm{I}_\delta}x_{\mathrm{I}_\alpha}-x_{\mathrm{I}_\alpha}y_{\mathrm{I}_\beta}\\ &dF_{\mathrm{II}}=(x_{\mathrm{II}_t}+x_{\mathrm{I}_t}x_{\mathrm{I}_\alpha}+y_{\mathrm{I}_t}y_{\mathrm{I}_\alpha}+z_{\mathrm{I}_t}z_{\mathrm{I}_\alpha})\,d\alpha\\ &\qquad+(y_{\mathrm{II}_t}+x_{\mathrm{I}_t}x_{\mathrm{I}_\beta}+y_{\mathrm{I}_t}y_{\mathrm{I}_\beta}+z_{\mathrm{I}_t}z_{\mathrm{I}_\beta})\,d\beta\\ &\qquad+(z_{\mathrm{II}_t}+x_{\mathrm{I}_t}x_{\mathrm{I}_\delta}+y_{\mathrm{I}_t}y_{\mathrm{I}_\delta}+z_{\mathrm{I}_t}z_{\mathrm{I}_\delta})\,d\delta \end{aligned}\right\}\cdots\cdots(7)$$

The equations in ε^3 are written as

$$\left.\begin{aligned} &x_{\mathrm{III}_{tt}}+gz_{\mathrm{III}_\alpha}+p_{\mathrm{III}_\alpha}/\rho=-x_{\mathrm{II}_{tt}}x_{\mathrm{I}_\alpha}-x_{\mathrm{I}_{tt}}x_{\mathrm{II}_\alpha}\\ &\qquad-y_{\mathrm{II}_{tt}}y_{\mathrm{I}_\alpha}-y_{\mathrm{I}_{tt}}y_{\mathrm{II}_\alpha}-z_{\mathrm{II}_{tt}}z_{\mathrm{I}_\alpha}-z_{\mathrm{I}_{tt}}z_{\mathrm{II}_\alpha}\\ &y_{\mathrm{III}_{tt}}+gz_{\mathrm{III}_\beta}+p_{\mathrm{III}_\beta}/\rho=-x_{\mathrm{II}_{tt}}x_{\mathrm{I}_\beta}-x_{\mathrm{I}_{tt}}x_{\mathrm{II}_\beta}\\ &\qquad-y_{\mathrm{II}_{tt}}y_{\mathrm{I}_\beta}-y_{\mathrm{I}_{tt}}y_{\mathrm{II}_\beta}-z_{\mathrm{II}_{tt}}z_{\mathrm{I}_\beta}-z_{\mathrm{I}_{tt}}z_{\mathrm{II}_\beta}\\ &z_{\mathrm{III}_{tt}}+gz_{\mathrm{III}_\delta}+p_{\mathrm{III}_\delta}/\rho=-x_{\mathrm{II}_{tt}}x_{\mathrm{I}_\delta}-x_{\mathrm{I}_{tt}}x_{\mathrm{II}_\delta}\\ &\qquad-y_{\mathrm{II}_{tt}}y_{\mathrm{I}_\delta}-y_{\mathrm{I}_{tt}}y_{\mathrm{II}_\delta}-z_{\mathrm{II}_{tt}}z_{\mathrm{II}_\delta}-z_{\mathrm{I}_{tt}}z_{\mathrm{II}_\delta}\\ &x_{\mathrm{III}_\alpha}+y_{\mathrm{III}_\beta}+z_{\mathrm{III}_\delta}=-x_{\mathrm{II}_\alpha}(y_{\mathrm{I}_\beta}+z_{\mathrm{I}_\delta})\\ &\qquad-y_{\mathrm{II}_\beta}(z_{\mathrm{I}_\delta}+x_{\mathrm{I}_\alpha})-z_{\mathrm{II}_\delta}(x_{\mathrm{I}_\alpha}+y_{\mathrm{I}_\beta})\\ &\qquad+(x_{\mathrm{II}_\beta}y_{\mathrm{I}_\alpha}+x_{\mathrm{II}_\delta}z_{\mathrm{I}_\alpha}+y_{\mathrm{II}_\delta}z_{\mathrm{I}_\beta}+y_{\mathrm{II}_\alpha}z_{\mathrm{I}_\beta}+z_{\mathrm{II}_\alpha}x_{\mathrm{I}_\delta}\\ &\qquad+z_{\mathrm{II}_\beta}y_{\mathrm{I}_\delta})+x_{\mathrm{I}_\alpha}y_{\mathrm{I}_\delta}z_{\mathrm{I}_\beta}+y_{\mathrm{I}_\beta}x_{\mathrm{I}_\delta}z_{\mathrm{I}_\alpha}+z_{\mathrm{I}_\delta}y_{\mathrm{I}_\alpha}x_{\mathrm{I}_\beta}\\ &\qquad-x_{\mathrm{I}_\alpha}y_{\mathrm{I}_\beta}z_{\mathrm{I}_\delta}-y_{\mathrm{I}_\alpha}z_{\mathrm{I}_\beta}x_{\mathrm{I}_\delta}-z_{\mathrm{I}_\alpha}x_{\mathrm{I}_\beta}y_{\mathrm{I}_\delta}\\ &dF_{\mathrm{III}}=[(x_{\mathrm{III}_t}+x_{\mathrm{II}_t}x_{\mathrm{I}_\alpha}+x_{\mathrm{II}_\alpha}x_{\mathrm{I}_t})\\ &\qquad+(y_{\mathrm{II}_t}y_{\mathrm{I}_\alpha}+y_{\mathrm{II}_\alpha}y_{\mathrm{I}_t})+(z_{\mathrm{II}_t}z_{\mathrm{I}_\alpha}+z_{\mathrm{II}_\alpha}z_{\mathrm{I}_t})]\,d\alpha\\ &\qquad+[(x_{\mathrm{II}_t}x_{\mathrm{I}_\beta}+x_{\mathrm{II}_\beta}x_{\mathrm{I}_t})+(y_{\mathrm{III}_t}+y_{\mathrm{II}_t}y_{\mathrm{I}_\beta}\\ &\qquad+y_{\mathrm{II}_\beta}y_{\mathrm{I}_t})+(z_{\mathrm{II}_t}z_{\mathrm{I}_\beta}+z_{\mathrm{II}_\beta}z_{\mathrm{I}_t})]\,d\beta\\ &\qquad+[(x_{\mathrm{II}_t}x_{\mathrm{I}_\delta}+x_{\mathrm{II}_\delta}x_{\mathrm{I}_t})+(y_{\mathrm{II}_t}y_{\mathrm{I}_\delta}+y_{\mathrm{II}_\delta}y_{\mathrm{I}_t})\\ &\qquad+(z_{\mathrm{III}_t}+z_{\mathrm{II}_t}z_{\mathrm{I}_\delta}+z_{\mathrm{II}_\delta}z_{\mathrm{I}_t})]\,d\delta \end{aligned}\right\}\cdots\cdots(8)$$

Equations in (6) are linear. A solution of them in turn determines the right hand side of Eq. (7). Since the left hand side of Eq. (7) is linear, in principle at least, a solution can be found. Also in principle it is possible to proceed to as high an order as desired by this procedure.

Because the linearized model was described in the author's previous paper[6], it is omitted in this paper to make space for remaining part. The long-crested wave model is more amenable to an investigation of second-order effects because it is simpler. We consider the problem of two waves, and then generalize to the randomized process. For two wave trains 1 and 2 the linear solution is given by

$$\left.\begin{aligned}
X_{\mathrm{I}} &= -a_1 e^{k_1\delta}\sin(k_1\alpha-\omega_1 t+\epsilon_1)\\
&\quad -a_2 e^{k_2\delta}\sin(k_2\alpha-\omega_2 t+\epsilon_2)\\
Z_{\mathrm{I}} &= a_1 e^{k_1\delta}\cos(k_1\alpha-\omega_1 t+\epsilon_1)\\
&\quad +a_2 e^{k_2\delta}\cos(k_2\alpha-\omega_2 t+\epsilon_2)\\
P_{\mathrm{I}} &= 0\\
F_{\mathrm{I}} &= \frac{a_1\omega_1}{k_1}e^{k_1\delta}\sin(k_1\alpha-\omega_1 t+\epsilon_1)\\
&\quad +\frac{a_2\omega_2}{k_2}e^{k_2\delta}\sin(k_2\alpha-\omega_2 t+\epsilon_2)
\end{aligned}\right\}\cdots\cdots(9)$$

where ϵ_1 and ϵ_2 are random phase lag expressed by Gaussian function for each wave train. Capital letters in X, Z, P and F explain the product of ε and each physical quantity, for example, $X_{\mathrm{I}}=\varepsilon x_{\mathrm{I}}$, $X_{\mathrm{II}}=\varepsilon^2 x_{\mathrm{II}}$, and so on, because in the analysis of wave-wave interactions the resulted value of ε in a higher order solution is not the same as that in a lower order solution. Equation (9) is irrotational to first order. For simplicity, we assume that $\omega_2>\omega_1$.

The second-order terms become

$$\left.\begin{aligned}
X_{\mathrm{II}} &= -\frac{a_1a_2}{g}\left(\frac{\omega_1^3+\omega_2^3}{\omega_2-\omega_1}\right)e^{(k_2+k_1)\delta}\\
&\quad\times\sin((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
&\quad+\frac{a_1a_2}{g}(\omega_2+\omega_1)\,\omega_2\,e^{(k_2-k_1)\delta}\\
&\quad\times\sin((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
&\quad+a_1^2\omega_1k_1e^{2k_1\delta}t+a_2^2\omega_2k_2e^{2k_2\delta}t\\
Z_{\mathrm{II}} &= \frac{a_1a_2}{g}(\omega_1^2+\omega_1\omega_2+\omega_2^2)\,e^{(k_1+k_2)\delta}\\
&\quad\times\cos((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
&\quad-\frac{a_1a_2}{g}\omega_2(\omega_2+\omega_1)\,e^{(k_2-k_1)\delta}\\
&\quad\times\cos((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
P_{\mathrm{II}} &= g\rho\frac{a_1^2k_1}{2}(e^{2k_1\delta}-1)+g\rho\frac{a_2^2k_2}{2}(e^{2k_2\delta}-1)\\
&\quad-2\rho a_1a_2\omega_2\omega_1\,e^{(k_1+k_2)\delta}\\
&\quad\times\cos((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
&\quad+2\rho a_1a_2\omega_2\omega_1\,e^{(k_2-k_1)\delta}\\
&\quad\times\cos((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
F_{\mathrm{II}} &= a_1a_2\omega_2e^{(k_2-k_1)\delta}\\
&\quad\times\sin((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
&\quad+a_1a_2(\omega_2-\omega_1)\,e^{(k_2+k_1)\delta}\\
&\quad\times\sin((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)
\end{aligned}\right\}\cdots\cdots(10)$$

The full solution is obtained by combining Eqs. (4), (9) and (10). This solution satisfies the equations to second order. If more terms added to the linear solution subject to the condition that $\omega_1<\omega_2<\omega_3<\cdots<\omega_n$, the terms in the linear solution interact in a predictable way to generate appropriate second-order terms. The randomized second-order solution for $x(\alpha,\delta,t)$ and $z(\alpha,\delta,t)$ are given by Eq. (11).

$$\begin{aligned}
x(\alpha,\delta,t) &= \alpha-\sum_i a_i e^{k_i\delta}\sin(k_i\alpha-\omega_i t+\epsilon_i)\\
&\quad-\sum_j a_j e^{k_j\delta}\sin(k_j\alpha-\omega_j t+\epsilon_j)\\
&\quad-\sum_{j>i}\sum_i\frac{a_ia_j}{g}\left(\frac{\omega_i^3+\omega_j^3}{\omega_j-\omega_i}\right)e^{(k_j+k_i)\delta}\\
&\quad\times\sin((k_j-k_i)\alpha-(\omega_j-\omega_i)t+\epsilon_j-\epsilon_i)\\
&\quad+\sum_{j>i}\sum_i\frac{a_ia_j}{g}(\omega_j+\omega_i)\,\omega_j\,e^{(k_j-k_i)\delta}\\
&\quad\times\sin((k_j-k_i)\alpha-(\omega_j-\omega_i)t+\epsilon_j-\epsilon_i)\\
&\quad+\sum_i a_i^2\omega_ik_ie^{2k_i\delta}t+\sum_j a_j^2\omega_jk_je^{2k_j\delta}t\\
z(\alpha,\delta,t) &= \delta+\sum_i a_i e^{k_i\delta}\cos(k_i\alpha-\omega_i t+\epsilon_i)\\
&\quad+\sum_j a_j e^{k_j\delta}\cos(k_j\alpha-\omega_j t+\epsilon_j)\\
&\quad+\sum_{j>i}\sum_i\frac{a_ia_j}{g}(\omega_i^2+\omega_i\omega_j+\omega_j^2)\,e^{(k_i+k_j)\delta}\\
&\quad\times\cos((k_j-k_i)\alpha-(\omega_j-\omega_i)t+\epsilon_j-\epsilon_i)\\
&\quad-\sum_{j>i}\sum_i\frac{a_ia_j}{g}\omega_j(\omega_j+\omega_i)\,e^{(k_j-k_i)\delta}\\
&\quad\times\cos((k_j-k_i)\alpha-(\omega_j-\omega_i)t+\epsilon_j-\epsilon_i)
\end{aligned}\qquad\cdots\cdots(11)$$

Although there are many cases of combination of primary waves in tertiary interactions as described by Phillips[3], in this paper the interaction between two groups of a primary wave train and a generated secondary wave is discussed. The third-order terms are written as

$$\begin{aligned}
X_{\mathrm{III}} &= -\frac{a_1^2a_2}{2g^2}\frac{\omega_2^3(\omega_2+\omega_1)(\omega_2^2-2\omega_2\omega_1+2\omega_1^2)}{(\omega_2-\omega_1)(\omega_2-2\omega_1)}\\
&\quad\times e^{(2k_1+k_2)\delta}\sin((k_2-2k_1)\alpha\\
&\quad-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)\\
&\quad+\frac{a_1^2a_2}{g^2}\frac{\omega_2(\omega_2+\omega_1)(\omega_2^3-\omega_2^2\omega_1-\omega_2\omega_1^2+2\omega_1^3)}{\omega_2-2\omega_1}\\
&\quad\times e^{k_2\delta}\sin((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)\\
&\quad-\frac{a_1^2a_2}{2g^2}\frac{\omega_2(\omega_2^2-2\omega_1^2)^2}{\omega_2-2\omega_1}e^{(k_2-2k_1)\delta}\\
&\quad\times\sin((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)\\
&\quad+\frac{a_1a_2^2}{2g^2}\frac{\omega_1^3(\omega_2+\omega_1)(2\omega_2^2-2\omega_2\omega_1+\omega_1^2)}{(\omega_2-\omega_1)(2\omega_2-\omega_1)}\\
&\quad\times e^{(k_1+2k_2)\delta}\sin((2k_2-k_1)\alpha\\
&\quad-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)\\
&\quad-\frac{a_1a_2^2}{2g^2}\frac{\omega_1^3\omega_2(2\omega_2^2-\omega_1^2)(3\omega_2-2\omega_1)}{(\omega_2-\omega_1)^2(2\omega_2-\omega_1)}e^{(2k_2-k_1)\delta}\\
&\quad\times\sin((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)\\
&\quad-\frac{a_1^3}{g^2}\omega_1^5e^{3k_1\delta}t\cos(k_1\alpha-\omega_1t+\epsilon_1)
\end{aligned}$$

$$-\frac{a_1a_2^2}{g^2}\omega_1^2\omega_2^3 e^{(k_1+2k_2)\delta}\, t\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{a_2^3}{g^2}\omega_2^5 e^{3k_2\delta}\, t\cos(k_2\alpha-\omega_2t+\epsilon_2)$$
$$-\frac{a_1^2a_2}{g_2}\omega_1^3\omega_2^2 e^{(2k_1+k_2)\delta}\, t\cos(k_2\alpha-\omega_2t+\epsilon_2)$$
$$-\frac{3}{2}\frac{a_1^3}{g^2}\omega_1^4 e^{3k_1\delta}\sin(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{3}{2}\frac{a_2^3}{g^2}\omega_2^4 e^{3k_2\delta}\sin(k_2\alpha-\omega_2t+\epsilon_2)$$
$$-\frac{a_1a_2^2}{g^2}\frac{2\omega_2^6-\omega_2^5\omega_1+\omega_2^4\omega_1^2-\omega_2^3\omega_1^3}{\omega_1(\omega_2-\omega_1)}$$
$$\times e^{(k_1+2k_2)\delta}\sin(k_1\alpha-\omega_1t+\epsilon_1)$$
$$+\frac{a_1a_2^2}{g^2}\frac{\omega_2(\omega_2+\omega_1)}{\omega_1}(2\omega_2^3-\omega_2^2\omega_1-\omega_2\omega_1^2+\omega_1^3)$$
$$\times e^{(2k_2-k_1)\delta}\sin(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{a_1^2a_2}{g^2}\frac{\omega_2^3\omega_1^3-\omega_2^2\omega_1^4+\omega_2\omega_1^5-2\omega_1^6}{\omega_2(\omega_2-\omega_1)}$$
$$\times e^{(2k_1+k_2)\delta}\sin(k_2\alpha-\omega_2t+\epsilon_2) \qquad \cdots\cdots(12)$$

$$Z_{\mathrm{III}}=\frac{a_1^2a_2}{2g^2}\frac{\omega_2^3(\omega_2^3-\omega_2^2\omega_1-2\omega_1^3)}{(\omega_2-\omega_1)(\omega_2-2\omega_1)}e^{(2k_1+k_2)\delta}$$
$$\times\cos((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$-\frac{a_1^2a_2}{g_2}\frac{\omega_2^2(\omega_2+\omega_1)(\omega_2^2-\omega_2\omega_1-\omega_1^2)}{\omega_2-2\omega_1}e^{k_2\delta}$$
$$\times\cos((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$+\frac{a_1^2a_2}{2g^2}\frac{\omega^2(\omega_2^2-2\omega_1^2)^2}{\omega_2-2\omega_1}e^{(k_2-2k_1)\delta}$$
$$\times\cos((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$-\frac{a_1a_2^2}{2g^2}\frac{\omega_1^3(2\omega_2^3+\omega_2\omega_1^2-\omega_1^3)}{(\omega_2-\omega_1)(2\omega_2-\omega_1)}e^{(k_1+2k_2)\delta}$$
$$\times\cos((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)$$
$$+\frac{a_1a_2^2}{2g^2}\omega_1^3\omega_2\frac{(2\omega_2^2-\omega_1^2)(3\omega_2-2\omega_1)}{(\omega_2-\omega_1)^2(2\omega_2-\omega_1)}e^{(2k_2-k_1)\delta}$$
$$\times\cos((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)$$
$$-\frac{a_1^3}{g_2}\omega_1^5 e^{3k_1\delta}\, t\sin(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{a_1a_2^2}{g^2}\omega_1^2\omega_2^3 e^{(k_1+2k_2)\delta}\, t\sin(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{a_2^3}{g^2}\omega_2^5 e^{3k_2\delta}\, t\sin(k_2\alpha-\omega_2t+\epsilon_2)$$
$$-\frac{a_1^2a_2}{g^2}\omega_1^3\omega_2^2 e^{(2k_1+k_2)\delta}\, t\sin(k_2\alpha-\omega_2t+\epsilon_2)$$
$$+\frac{1}{2}\frac{a_1^3}{g^2}\omega_1^4 e^{3k_1\delta}\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$+\frac{1}{2}\frac{a_2^3}{g^2}\omega_2^4 e^{3k_2\delta}\cos(k_2\alpha-\omega_2t+\epsilon_2)$$
$$+\frac{a_1a_2^2}{g^2}\frac{\omega_2^3(\omega_2^2+\omega_2\omega_1-\omega_1^2)}{\omega_2-\omega_1}e^{(k_1+2k_2)\delta}$$
$$\times\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{a_1a_2^2}{g^2}\omega_2(\omega_2+\omega_1)(\omega_2^2+\omega_2\omega_1-\omega_1^2)$$
$$\times e^{(2k_2-k_1)\delta}\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$+\frac{a_1^2a_2}{g^2}\frac{\omega_1^3(\omega_2^2-\omega_2\omega_1-\omega_1^2)}{\omega_2-\omega_1}$$
$$\times e^{(2k_1+k_2)\delta}\cos(k_2\alpha-\omega_2t+\epsilon_2) \qquad \cdots\cdots(13)$$

$$\frac{P_{\mathrm{III}}}{\rho}=-\frac{a_1^2a_2}{g}\omega_1\omega_2^3\frac{2\omega_2-3\omega_1}{\omega_2-2\omega_1}e^{(2k_1+k_2)\delta}$$
$$\times\cos((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$+\frac{2a_1^2a_2}{g}\frac{\omega_1\omega_2(\omega_2^2-\omega_1^2)(2\omega_2-3\omega_1)}{\omega_2-2\omega_1}e^{k_2\delta}$$
$$\times\cos((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$-\frac{a_1^2a_2}{g}\frac{\omega_1\omega_2(\omega_2^2-2\omega_1^2)(2\omega_2-3\omega_1)}{\omega_2-2\omega_1}e^{(k_2-2k_1)\delta}$$
$$\times\cos((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$-\frac{a_1a_2^2}{g}\frac{\omega_1^3\omega_2(3\omega_2-2\omega_1)}{2\omega_2-\omega_1}e^{(k_1+2k_2)\delta}$$
$$\times\cos((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)$$
$$+\frac{a_1a_2^2}{g}\frac{\omega_1^3\omega_2(3\omega_2-2\omega_1)}{2\omega_2-\omega_1}e^{(2k_2-k_1)\delta}$$
$$\times\cos((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)$$
$$-\frac{a_1^3}{g}\omega_1^4 e^{3k_1\delta}\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{a_2^3}{g}\omega_2^4 e^{3k_2\delta}\cos(k_2\alpha-\omega_2t+\epsilon_2)$$
$$-\frac{2a_1a_2^2}{g}\omega_1\omega_2^3 e^{(k_1+2k_2)\delta}\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$+\frac{2a_1a_2^2}{g}\omega_2\omega_1(\omega_2^2-\omega_1^2)e^{(2k_2-k_1)\delta}$$
$$\times\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{2a_1^2a_2}{g}\omega_1^3\omega_2 e^{(2k_1+k_2)\delta}\cos(k_2\alpha-\omega_2t+\epsilon_2)$$
$$\cdots\cdots(14)$$

$$F_{\mathrm{III}}=\frac{a_1^2a_2}{2g}\frac{\omega_2^3(\omega_2-3\omega_1)}{\omega_2-\omega_1}e^{(k_2+2k_1)\delta}$$
$$\times\sin((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$-\frac{a_1^2a_2}{g}\omega_2(\omega_2+\omega_1)(\omega_2-2\omega_1)e^{k_2\delta}$$
$$\times\sin((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$+\frac{a_1^2a_2}{2g}\omega_2(\omega_2^2-2\omega_1^2)e^{(k_2-2k_1)\delta}$$
$$\times\sin((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)$$
$$-\frac{a_1a_2^2}{2g}\frac{\omega_1^3(3\omega_2-\omega_1)}{\omega_2-\omega_1}e^{(2k_2+k_1)\delta}$$
$$\times\sin((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)$$
$$+\frac{a_1a_2^2}{2g}\frac{\omega_1^3\omega_2(3\omega_2-2\omega_1)}{(\omega_2-\omega_1)^2}e^{(2k_2-k_1)\delta}$$
$$\times\sin((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)$$
$$+\frac{a_1^3}{g}\omega_1^4 e^{3k_1\delta}\, t\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$+\frac{a_2^3}{g}\omega_2^4 e^{3k_2\delta}\, t\cos(k_2\alpha-\omega_2t+\epsilon_2)$$
$$+\frac{a_1a_2^2}{g}\omega_1\omega_2^3 e^{(2k_2+k_1)\delta}\, t\cos(k_1\alpha-\omega_1t+\epsilon_1)$$
$$+\frac{a_1^2a_2}{g}\omega_2\omega_1^3 e^{(k_2+2k_1)\delta}\, t\cos(k_2\alpha-\omega_2t+\epsilon_2)$$
$$-\frac{a_1^3}{2g}\omega_1^3 e^{3k_1\delta}\sin(k_1\alpha-\omega_1t+\epsilon_1)$$
$$-\frac{a_2^3}{2g}\omega_2^3 e^{3k_2\delta}\sin(k_2\alpha-\omega_2t+\epsilon_2)$$

$$+\frac{a_1 a_2^2}{g}\frac{\omega_2^3\omega_1}{\omega_2-\omega_1}e^{(2k_2+k_1)\delta}\sin(k_1\alpha-\omega_1 t+\epsilon_1)$$
$$-\frac{a_1^2 a_2}{g}\frac{\omega_2\omega_1^3}{\omega_2-\omega_1}e^{(k_2+2k_1)\delta}\sin(k_2\alpha-\omega_2 t+\epsilon_2)$$
$$-\frac{a_1 a_2^2}{g}\omega_2\omega_1(\omega_2+\omega_1)e^{(2k_2-k_1)\delta}$$
$$\times\sin(k_1\alpha-\omega_1 t+\epsilon_1) \qquad \cdots\cdots(15)$$

The randomized version of tertiary waves is obtained as in the case of the second-order theory. In Eqs. (12) and (13), there are terms whose amplitude grows lineary with time t and the phase of these terms is in advance of the primary wave by $\pi/2$. As it stands, the solution of this type is only valid for $\omega t \ll (1/2a^2k^2)^{-1}$, after which the developing tertiary wave has an amplitude comparable with that of the primary wave. Notice that the interaction time, or the time scale of development of the resonant tertiary component is of the order of (−2)-power of the maximum primary wave slope times the wave period. As for the condition at the free surface, P_{III} does not vanish when δ is set to be zero. Although tertiary components with differences of wave-numbers and frequencies of the primary components cancel out among themselves, tertiary components with the same wave-numbers and frequencies as the primary components does not reduce to zero. However, the amplitude of remaining terms is bounded in time and remains in third order for any combination of frequencies of two wave groups. Therefore, expressions of Eqs. (12) and (13) are used hereafter neglecting a bounded third-order deviation in the surface condition.

The condition of the existence of a growing tertiary wave with differences of wave-numbers and frequencies of the primary components is checked as follows. If the solution of such a type satisfies the homogeneous differential equation and boundary conditions, the solution obtained can be added to Eqs. (12) through (14) because of the linearity of the basic perturbation equation. Suppose we have a solution for third-order waves as

$$Z_{\mathrm{III}}=B\,e^{(k_2-2k_1)\delta}\,t\sin((k_2-2k_1)\alpha-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1) \qquad \cdots\cdots(16)$$

This is a homogeneous solution to the differential equation for Z_{III}. In order to satisfy the surface condition that $P_{\mathrm{III}}=0$ for $\delta=0$, the following condition must hold.

$$(\omega_2-2\omega_1)^2=g(k_2-2k_1)$$

Considering the relation that $\omega^2=gk$ for the first approximation, we obtain $\omega_2=3\omega_1/2$. Another condition that $\omega_2=2\omega_1$ is required to fullfil the surface condition. These two conditions mentioned above are unable to coexist. Even if the bounded third order terms are neglected for the surface condition, the value of coefficient B remains undetermined. Therefore, it is concluded that the growing tertiary component with differences of wave-numbers and frequencies of the primary wave does not exist in the Lagrangian representation.

CALCULATION OF TURBULENT DIFFUSION COEFFICIENT

Turbulent diffusion coefficient is calculated for secondary waves generated by interactions of two wave trains in the same method as described for a linear model. Because the expectation of the products of different random phase lag function comes to be zero due to their independence, the contribution of secondary waves to diffusion if it exists will be originated by their square products. Therefore, we can discuss the effect of secondary waves on diffusion separately.

In this section diffusion about the mean position in the direction of wave propagation is discussed for a particle at the water surface. The x-component of the velocity of particle is obtained by partial differentiation of the first equation of Eq. (11) with respect to time, t. Excluding the uniform secondary mean drift current, the expression for the fluctuating velocity is easily extended to a spectral representation of two wave groups.

$$u=\int_{\omega_2}\int_{\omega_1}\frac{\omega_1^2(\omega_1+\omega_2)}{g}\cos((k_2-k_1)\alpha-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\sqrt{\Phi(\omega_1)\,d\omega_1}\sqrt{\Phi(\omega_2)\,d\omega_2} \qquad \cdots\cdots(17)$$

Here $\Phi(\omega)$ is the frequency power spectrum of surface displacement, where the frequency is regarded only as positive.

Covariance of the velocity due to the secondary component is denoted by $c_2(\tau)$ and is given as

$$c_2(\tau)=\int_{\omega_1}^{\infty}\int_0^{\infty}\frac{\omega_1^4(\omega_1+\omega_2)^2}{2g^2}\Phi(\omega_1)\Phi(\omega_2)\times\cos((\omega_2-\omega_1)\tau)\,d\omega_1 d\omega_2 \qquad \cdots\cdots(18)$$

Exchange an integral variable from ω_2 to r through the relationship of $\omega_2-\omega_1=r$. Integration is first made along the line $\omega_2-\omega_1=r$ for a fixed value of r and then r is varied from zero to infinity to cover the domain where $\omega_2>\omega_1$. $c_2(\tau)$ is given by

$$c_2(\tau)=\int_0^{\infty}\left[\int_0^{\infty}\frac{\omega_1^4(2\omega_1+r)^2}{2g^2}\Phi(\omega_1)\Phi(\omega_1+r)\,d\omega_1\right]\times\cos r\tau\,dr \qquad \cdots\cdots(19)$$

The Fourier transform pair of Eq. (19) is given by

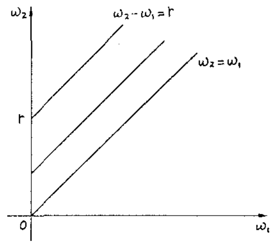

Fig. 1 Exchange of an integral variable

$$\frac{2}{\pi}\int_0^{\infty} c_2(\tau)\cos r\tau\,d\tau=\int_0^{\infty}\frac{\omega_1^4(2\omega_1+r)^2}{2g^2}\times\Phi(\omega_1)\Phi(\omega_1+r)\,d\omega_1 \qquad \cdots\cdots(20)$$

Following Taylor's approach, turbulent diffusion coefficient is equal to $\int_0^{\infty} c_2(\tau)\,d\tau$, which is obtained putting $r=0$ in Eq. (20) to yield Eq. (21).

$$D=\int_0^{\infty} c_2(\tau)\,d\tau=\frac{\pi}{2}\int_0^{\infty}\frac{2\omega_1^6}{g^2}\Phi(\omega_1)\Phi(\omega_1)\,d\omega_1 \qquad \cdots\cdots(21)$$

However, the condition of $r=0$ which is equivalent to that $\omega_2=\omega_1$ cannot be accomplished in the interactions of two different wave trains because the consideration mentioned above is applied to secondary interactions between different two wave groups. Therefore, it is manifested that the secondary wave has no constant energy flux among different frequency components and that the cloud of materials floating on the water surface does not expand continuously with time.

The diffusivity of the tertiary waves is first considered about the terms with differences of wave-numbers and frequencies of the primary components for very large values of time because the growing tertiary waves are considered to exist only for finite value of time within which the amplitude is smaller than that of primary waves. For small values of time the diffusivity of the growing tertiary waves is discussed later.

The x-component of the velocity of a particle at the water surface is calculated through partial differentiation of first five terms of the right-hand side of Eq. (12) regarding with time t, and is written in a spectral representation.

$$\begin{aligned}u=&\int_{\omega_2}\int_{\omega_1}\frac{1}{g^2}\frac{\omega_2^2\omega_1^4}{\omega_2-\omega_1}\cos((k_2-2k_1)\alpha\\&-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)\sqrt{\Phi_{\rm II}(\omega_1)\,d\omega_1}\sqrt{\Phi(\omega_2)\,d\omega_2}\\&-\int_{\omega_2}\int_{\omega_1}\frac{1}{2g^2}\frac{(-4\omega_2^4+2\omega_2^3\omega_1+2\omega_2^2\omega_1^2-\omega_1^4)}{(\omega_2-\omega_1)^2}\\&\times\cos((2k_2-k_1)\alpha-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)\\&\times\sqrt{\Phi(\omega_1)\,d\omega_1}\sqrt{\Phi_{\rm II}(\omega_2)\,d\omega_2} \qquad \cdots\cdots(22)\end{aligned}$$

where $\Phi_{\rm II}$ explains the spectrum for square of the surface displacement.

$\Phi_{\rm II}(\omega)$ is calculated after Hino's[8] method as

$$\Phi_{\rm II}(\omega)=2\int_0^{\infty}\Phi(\omega-x)\Phi(x)\,dx \qquad \cdots\cdots(23)$$

Covariance of the velocity due to the tertiary components with differences of wave-numbers and frequencies of the primary wave is denoted by $c_3(\tau)$ and is shown as

$$\begin{aligned}c_3(\tau)=&\int_{\omega_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{1}{g^4}\frac{\omega_2^4\omega_1^8}{(\omega_2-\omega_1)^2}\Phi(\omega_1-x)\Phi(x)\Phi(\omega_2)\\&\times\cos((\omega_2-2\omega_1)\tau)\,dx\,d\omega_1\,d\omega_2\\&+\int_{\omega_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{1}{4g^4}f(\omega_1,\ \omega_2)\Phi(\omega_2-x)\Phi(x)\Phi(\omega_1)\\&\times\cos((2\omega_2-\omega_1)\tau)\,d\omega_1\,dx\,d\omega_2 \qquad \cdots\cdots(24)\end{aligned}$$

where

$$f(\omega_1,\ \omega_2)=\frac{\omega_1^6(-4\omega_z^4+2\omega_2^3\omega_1+2\omega_2^2\omega_1^2-\omega_1^4)^2}{(\omega_2-\omega_1)^4}$$

Exchange an integral variable from ω_2 to s and q through the relation

$$\omega_2-2\omega_1=s \quad\text{and}\quad 2\omega_2-\omega_1=q \qquad \cdots\cdots(25)$$

Integration is carried out along the lines $\omega_2-2\omega_1=s$ and $2\omega_2-\omega_1=q$ for fixed values of s and q. The ranges of s and q are $-\infty<s<\infty$ and $0<q$ to cover the domain where $\omega_2>\omega_1$. The range of ω_1 is determined as

$$\left\{\begin{array}{lll}0\leq\omega_1<\infty & \text{for} & s\geq 0\\ -s<\omega_1<\infty & \text{for} & s<0\\ 0<\omega_1<q & & \end{array}\right. \qquad \cdots\cdots(26)$$

$c_3(\tau)$ reduces to the following expression.

$$\begin{aligned}c_3(\tau)=&\int_0^{\infty}\left[\int_0^{\infty}\int_0^{\infty}\frac{1}{g^4}\frac{(2\omega_1+s)^4\omega_1^8}{(\omega_1+s)^2}\right.\\&\left.\times\Phi(\omega_1-x)\Phi(x)\Phi(2\omega_1+s)\,dx\,d\omega_1\right]\cos s\tau\,ds\\&+\int_0^{\infty}\left[\int_{-s}^{\infty}\int_0^{\infty}\frac{1}{g^4}\frac{(2\omega_1-s)^4\omega_1^8}{(\omega_1-s)^2}\right.\\&\left.\times\Phi(\omega_1-x)\Phi(x)\Phi(2\omega_1-s)\,dx\,d\omega_1\right]\cos s\tau\,ds\\&+\int_0^{\infty}\left[\int_0^{q}\int_0^{\infty}\frac{1}{4g^4}f\left(\frac{q}{2}+\frac{\omega_1}{2},\ \omega_1\right)\Phi\left(\frac{\omega_1}{2}+\frac{q}{2}-x\right)\right.\\&\left.\times\Phi(x)\Phi(\omega_1)\,dx\,d\omega_1\right]\cos q\tau\,dq \qquad \cdots\cdots(27)\end{aligned}$$

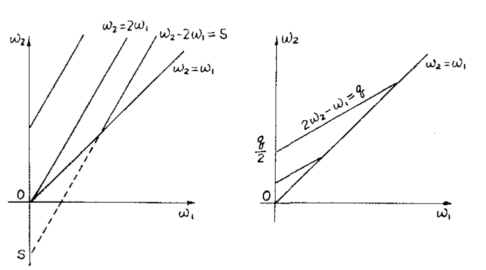

Fig. 2 Exchange of integral variables

The sum of first two terms in the right-hand side of Eq. (27) is denoted by $[c_3(\tau)]_{\rm I}$ and the last term by $[c_3(\tau)]_{\rm II}$. Therefore,

$$c_3(\tau)=[c_3(\tau)]_{\rm I}+[c_3(\tau)]_{\rm II}$$

The Fourier transform pair of Eq. (27) is written as follows:

$$\left\{\begin{aligned}&\frac{2}{\pi}\int_0^\infty[c_3(\tau)]_{\rm I}\cos s\tau\,d\tau=\int_0^\infty\int_0^\infty\frac{1}{g^4}\frac{(2\omega_1+s)^4\omega_1^8}{(\omega_1+s)^2}\\&\quad\times\Phi(\omega_1-x)\Phi(x)\Phi(2\omega_1+s)\,dx\,d\omega_1\\&\quad+\int_{-s}^\infty\int_0^\infty\frac{1}{g^4}\frac{(2\omega_1-s)^4\omega_1^8}{(\omega_1-s)^2}\\&\quad\times\Phi(\omega_1-x)\Phi(x)\Phi(2\omega_1-s)\,dx\,d\omega_1\\&\frac{2}{\pi}\int_0^\infty[c_3(\tau)]_{\rm II}\cos q\tau d\tau=\int_0^q\int_0^\infty\frac{1}{4g^4}f\left(\frac{\omega_1}{2}+\frac{q}{2},\,\omega_1\right)\\&\quad\times\Phi\left(\frac{\omega_1}{2}+\frac{q}{2}-x\right)\Phi(x)\Phi(\omega_1)\,dx\,d\omega_1\end{aligned}\right.\qquad\cdots\cdots(28)$$

We can obtain $\int_0^\infty c_3(\tau)\,d\tau$ by setting $s=q=0$ in the Fourier transform of Eq. (27).

$$\begin{aligned}D&=\int_0^\infty c_3(\tau)\,d\tau\\&=\frac{16\pi}{g^4}\int_0^\infty\int_0^\infty\omega^{10}\Phi(\omega-x)\Phi(x)\Phi(2\omega)\,dx\,d\omega\end{aligned}\qquad\cdots\cdots(29)$$

The condition that $s=0$ is compatible with the condition that $\omega_2>\omega_1$.

Phillips[9] summarized results taken under the wide range of experimental conditions and stated that, at frequencies appreciably above that of each spectral peak, the spectra of ocean surface waves are all clustered about a single line, $\Phi(\omega)=\beta g^2\omega^{-5}$, regardless of the wind speed or fetch. Its steep forward face rising to a sharp maximum is characteristic. Several sets of experiments can be used to find the value of the numerical constant β. The mean of these values is 1.17×10^{-2}. Therefore, the functional form of the wave spectra is defined as

$$\left\{\begin{aligned}\Phi(\omega)&=\beta g^2\omega^{-5}\qquad\text{for}\quad\omega\geq\omega_p\\&=0\qquad\qquad\text{otherwise}\end{aligned}\right.\qquad\cdots\cdots(30)$$

where ω_p shows the frequency of spectral peak.

Because $\Phi(\omega-x)$ is defined for $\omega_p<\omega-x$, the effective range for x is $x<\omega-\omega_p$. Furthermore x should be greater than ω_p in the function of $\Phi(x)$. Therefore, $\omega>2\omega_p$ is necessary for the integrand to have a finite value different from zero. D is rewritten as

$$D=\frac{\pi\beta^3g^2}{2}\int_{2\omega_p}^\infty\omega^5\int_{\omega_p}^{\omega-\omega_p}\frac{1}{(\omega-x)^5x^5}\,dx\,d\omega\cdots\cdots(31)$$

The upper bound of ω is infinity in theory, which makes the inner integral infinitely large. This means that energy contained in a secondary wave is infinitely large, which is unrealistic physically. Spectral density decreases proportionally to (-5)-power of frequency so that the density of energy involved in the frequency range above certain finite value of frequency becomes approximately zero from practical point of view and also higher frequency range corresponds to capillary waves. The possible maximum value for ω is denoted by ω_m and the corresponding value for W is represented by W_m.

$$D=\frac{\pi\gamma\beta^2g^2}{2}\omega_p^{-3}\qquad\cdots\cdots(32)$$

where $W=(\omega-\omega_p)/\omega_p$, and

$$\begin{aligned}\gamma=\int_1^{W_m}\frac{1}{(W+1)^4}\Big[&\frac{W^4-W^{-4}}{2}+\frac{16}{3}(W^3-W^{-3})\\&+28(W^2-W^{-2})+112(W-W^{-1})+140\ln W\Big]dW\end{aligned}$$

It is concluded that ω_m is about five times of the frequency of spectral peak, referring to several measured data shown in Phillip's books[9]. Then turbulent diffusion coefficient due to tertiary components with differences of wave-numbers and frequencies of primary components is given by Eq. (33) with the value of $\gamma=11.9$.

$$D=2.90\times10^{-3}\omega_p^{-3}\quad(\text{m}^2/\text{sec})\qquad\cdots\cdots(33)$$

Making use of the expression for ω_p proposed by Mitsuyasu and Nakayama[10], D is written down in another form.

$$D=6.50\times10^{-9}U_{10}^{1.3}Fe^{0.85}\qquad\cdots\cdots(34)$$

Here U_{10} is the wind speed at the level of 10 m above the surface and Fe is fetch.

Computed diffusion coefficient is a constant with respect to time. Therefore, the variance of the cloud of substances at water surface increases proportionally to the square root of time. The order of magnitude of the obtained diffusion coefficient is much smaller than that observed in the same scale of ocean, partly because the calculated coefficient is deduced from tertiary waves which have smaller magnitude than that of primary waves and partly because the considered case is concerned with the absolute turbulent diffusion as studied by Taylor. However, measurements are taken in an open sea where dominant scale of eddies increases as diffusion proceeds and the characteristic length scale of diffusion increases. Therefore, the concept of neighbor-distance movement or relative diffusion becomes important. In order to take account of relative diffusion, it is necessary to proceed to the analysis on diffusion due to growing tertiary waves for relatively short times of duration.

Considerations stated above excludes developing tertiary waves and tertiary waves with the same wave numbers and frequencies as those of primary waves. The characteristic feature of the developing tertiary waves is that the phase is in advance of that of the primary waves by $\pi/2$. In the following study on the diffusivity of the growing tertiary waves, another direct way of approach is applied. Statistical variance of the position of a particle is considered directly rather than covariance of fluctuating velocity.

The horizontal position of a particle after time t which is initially placed at a point $(\alpha, \delta)=(0, 0)$ is expressed in Eq. (35) including up to third-order terms.

$$
\begin{aligned}
x=&\overset{①}{-a_1}\sin(-\omega_1 t+\epsilon_1)\overset{②}{-a_2}\sin(-\omega_2 t+\epsilon_2)\\
&\overset{③}{-}\frac{a_1a_2}{g}\left(\frac{\omega_1^3+\omega_2^3}{\omega_2-\omega_1}\right)\sin(-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
&\overset{④}{+}\frac{a_1a_2}{g}(\omega_2+\omega_1)\omega_2\sin(-(\omega_2-\omega_1)t+\epsilon_2-\epsilon_1)\\
&\overset{⑤}{+a_1^2\omega_1k_1t}\overset{⑥}{+a_2^2\omega_2k_2t}\\
&\overset{⑦}{-}\frac{a_1^2a_2}{2g^2}\frac{\omega_2^3(\omega_2+\omega_1)(\omega_2^2-2\omega_2\omega_1+2\omega_1^2)}{(\omega_2-\omega_1)(\omega_2-2\omega_1)}\\
&\times\sin(-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)\\
&\overset{⑧}{+}\frac{a_1^2a_2}{g^2}\frac{\omega_2(\omega_2+\omega_1)(\omega_2^3-\omega_2^2\omega_1-\omega_2\omega_1^2+2\omega_1^3)}{\omega_2-2\omega_1}\\
&\times\sin(-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)\\
&\overset{⑨}{-}\frac{a_1^2a_2}{2g^2}\frac{\omega_2(\omega_2^2-2\omega_1^2)^2}{\omega_2-2\omega_1}\\
&\times\sin(-(\omega_2-2\omega_1)t+\epsilon_2-2\epsilon_1)\\
&\overset{⑩}{+}\frac{a_1a_2^2}{2g^2}\frac{\omega_1^3(-4\omega_2^4+2\omega_2^3\omega_1+2\omega_2^2\omega_1^2-\omega_1^4)}{(\omega_2-\omega_1)^2(2\omega_2-\omega_1)}\\
&\times\sin(-(2\omega_2-\omega_1)t+2\epsilon_2-\epsilon_1)\\
&\overset{⑪}{-}\frac{a_1^3}{g^2}\omega_1^5\cos(-\omega_1 t+\epsilon_1)\\
&\overset{⑫}{-}\frac{a_2^3}{g^2}\omega_2^5 t\cos(-\omega_2 t+\epsilon_2)\\
&\overset{⑬}{-}\frac{a_1a_2^2}{g^2}\omega_1^2\omega_2^3 t\cos(-\omega_1 t+\epsilon_1)\\
&\overset{⑭}{-}\frac{a_1^2a_2}{g^2}\omega_1^3\omega_2^2 t\cos(-\omega_2 t+\epsilon_2)\\
&\overset{⑮}{-}\frac{3}{2}\frac{a_1^3}{g^2}\omega_1^4\sin(-\omega_1 t+\epsilon_1)\\
&\overset{⑯}{-}\frac{3}{2}\frac{a_2^3}{g^2}\omega_2^4\sin(-\omega_2 t+\epsilon_2)\\
&\overset{⑰}{-}\frac{a_1a_2^2}{g^2}\frac{2\omega_2^6-\omega_2^5\omega_1+\omega_2^4\omega_1^2-\omega_2^3\omega_1^3}{\omega_1(\omega_2-\omega_1)}\\
&\times\sin(-\omega_1 t+\epsilon_1)\\
&\overset{⑱}{-}\frac{a_1^2a_2}{g^2}\frac{\omega_2^3\omega_1^3-\omega_2^2\omega_1^4+\omega_2\omega_1^5-2\omega_1^6}{\omega_2(\omega_2-\omega_1)}\\
&\times\sin(-\omega_2 t+\epsilon_2)\\
&\overset{⑲}{+}\frac{a_1a_2^2}{g^2}\frac{\omega_2(\omega_2+\omega_1)}{\omega_1}(2\omega_2^3-\omega_2^2\omega_1-\omega_2\omega_1^2+\omega_1^3)\\
&\times\sin(-\omega_1 t+\epsilon_1) \qquad \cdots\cdots(35)
\end{aligned}
$$

Each term in Eq. (35) is numbered from ① to ⑲. Expected position of the particle over a period of wave is given by

$$E[x]=⑤+⑥ \qquad \cdots\cdots(36)$$

Variance and diffusion coefficient are defined by

$$\langle r^2\rangle=E[(x-E[x])^2] \qquad \cdots\cdots(37)$$

$$D=\frac{1}{2}\frac{d\langle r^2\rangle}{dt} \qquad \cdots\cdots(38)$$

Therefore, we have to obtain the expectation of the square of x without terms of ⑤ and ⑥.

Because terms with different random phase lag are independent of each other, the expectations of products between terms of different random phase lag comes to zero. Terms in Eq. (35) are classified into three groups from the point of random phase lag.

First group is consisted of terms with differences of wave-numbers and frequencies of primary components. They are terms of ③, ④, ⑦, ⑧, ⑨ and ⑩ in the right-hand side of Eq. (35). Expectations of the square of these terms are constant. Although the diffusion coefficient obtained through Eq. (38) reduces to zero, constant value can be made infinitely large for particular combination of frequencies of two wave groups. The diffusion coefficient of this type of secondary and tertiary waves is discussed previously by means of covariance of fluctuating velocity.

Second group is composed of terms with the same wave-numbers and frequencies as those of the primary component. Coefficients of these terms do not become infinitely large for any combination of frequencies of different wave groups. It has been shown that linearized wave has no diffusivity. Therefore, the contribution to diffusion by terms of ①, ②, ⑮, ⑯, ⑰, ⑱ and ⑲ in the right-hand side of Eq. (35) is concluded to be zero.

Third group is concerned with growing tertiary waves. Wave-numbers and frequencies are the same as those of primary waves. However, phase is in advance by $\pi/2$. Therefore, the expectations of products between terms which belong to the second group and the third group become to be zero. Only the square of growing tertiary waves, which is expected to raise diffusion, is considered hereafter. The randomized version of the turbulent diffusion coefficient is given by

$$D=\frac{1}{2g^4}[\sum_i a_i^6\,\omega_i^{10}+\sum_{j>i}\sum_i a_i^4 a_j^2\omega_i^6\omega_j^3(2\omega_i+\omega_j)+\sum_{j>i}\sum_i a_i^2 a_j^4\omega_i^3\omega_j^6(\omega_i+2\omega_j)+\sum_j a_j^6\omega_j^{10}]\,t \qquad \cdots\cdots(39)$$

The first term involves the 6th power of surface displacement. Expectation of this higher order term is obtained as

$$E[\zeta^6]=\Psi_{\mathrm{III}}(0)=18\left(\int_0^\infty \Phi(\omega)\,d\omega\right)\times\left(\int_0^\infty\int_0^\infty \Phi(\omega-x)\,\Phi(x)\;dx\;d\omega\right)$$

However, we cannot know how the factor ω^{10} is related to two definite integrals with respect to ω. Therefore, the magnitude of the diffusion coefficient is discussed for one of the interaction terms generated by two wave groups. One of interaction terms is expressed in terms of the wave spectrum

$$(D)_1=\frac{2t}{g^4}\left(\int_0^\infty \omega_1^7\int_0^\infty \Phi(\omega_1-x)\,\Phi(x)\,dx\,d\omega_1\right)\times\left(\int_0^\infty \omega_2^3\,\Phi(\omega_2)\,d\omega_2\right)+\frac{t}{g^4}\left(\int_0^\infty \omega_1^6\int_0^\infty \Phi(\omega_1-x)\,\Phi(x)\,dx\,d\omega_1\right)\times\left(\int_0^\infty \omega_2^4\,\Phi(\omega_2)\,d\omega_2\right) \qquad \cdots\cdots(40)$$

Introducing the functional form of the wave spectrum and referring the physical interpretation of the maximum frequency range, Eq. (40) is rewritten in the form of

$$(D)_1=1.6\,t\beta^3g^2\omega_p^{-1}\int_{2\omega_p}^{\omega_m}\omega_1^7\int_{\omega_p}^{\omega_1-\omega_p}\frac{1}{(\omega_1-x)^5x^5}\,dx\,d\omega_1+1.6\,t\beta^3g^2\int_{2\omega_p}^{\omega_m}\omega_1^6\int_{\omega_p}^{\omega_1-\omega_p}\frac{1}{(\omega_1-x)^5x^5}\,dx\,d\omega_1 \qquad \cdots\cdots(41)$$

Eq. (41) is reduced to a following expression.

$$(D)_1=1.6\beta^3g^2\omega_p^{-2}(A+B)\,t \qquad \cdots\cdots(42)$$

where

$$A=\int_1^{W_m}\left[\frac{W^4}{2(W+1)^2}-\frac{1}{2W^4(W+1)^2}+\frac{16}{3}\frac{W^3}{(W+1)^2}-\frac{16}{3}\frac{1}{W^3(W+1)^2}+\frac{28W^2}{(W+1)^2}-\frac{28}{W^2(W+1)^2}+\frac{112W}{(W+1)^2}-\frac{112}{W(W+1)^2}+140\frac{\ln W}{(W+1)^2}\right]dW,$$

and

$$B=\int_1^{W_m}\left[\frac{W^4}{2(W+1)^3}-\frac{1}{2W^4(W+1)^3}+\frac{16}{3}\frac{W^3}{(W+1)^3}-\frac{16}{3}\frac{1}{W^3(W+1)^3}+\frac{28W^2}{(W+1)^3}-\frac{28}{W^2(W+1)^3}+\frac{112W}{(W+1)^3}-\frac{112}{W(W+1)^3}+140\frac{\ln W}{(W+1)^3}\right]dW$$

Calculation is performed to produce $A=114$ and $B=39$ for $W_m=4$ which is the consequence of $\omega_m=5\omega_p$ mentioned previously.

Diffusion coefficient due to one of interaction terms in developing tertiary waves is expressed by Eq. (43) after the calculation of a factor.

$$(D)_1=3.76\times10^{-2}\omega_p^{-2}t\ (\mathrm{m^2/sec}) \qquad \cdots\cdots(43)$$

There are three pairs of comparable term expressed by wave system in Eq. (39). Therefore, the order of magnitude of diffusion coefficient obtained by Eq. (39) is estimated as in Eq. (44).

$$D\simeq10^{-1}\omega_p^{-2}t\ (\mathrm{m^2/sec}) \qquad \cdots\cdots(44)$$

The mean value of interaction time, $\bar{t}$, for various values of angular frequency is computed to be

$$\bar{t}\ll\frac{2}{\beta}=1.71\times10^2\quad(\mathrm{sec}) \qquad \cdots\cdots(45)$$

The mean interaction time is independent of the scale of wave field, if the wave system is in the equilibrium range.

DISCUSSION OF RESULTS

The theoretical results obtained are summarized and compared with observation. It is shown that turbulent diffusivity of random waves has its origin in the tertiary components. Longitudinal turbulent diffusion coefficient is expressed by

$$\begin{cases}(\mathrm{I})\ D=10^{-1}\omega_p^{-2}t\quad(\mathrm{m^2/sec})\ \text{for}\ t\ll\bar{t}\cdots\cdots(44)\\(\mathrm{II})\ D=3\times10^{-3}\omega_p^{-3}\ (\mathrm{m^2/sec})\ \text{for}\ t\gg\bar{t}\cdots\cdots(33)\end{cases}$$

where $\bar{t}$ is the mean interaction time.

The relations derived show the same functional form as Taylor obtained for two extreme cases for correlation time. However, in Taylor's theory the criterion of two regions is decided by the characteristic time of the decay of turbulent eddies. In an unbounded environment (ocean is approximately unbounded in horizontal direction) the decay time of eddies becomes larger and larger as the scale of diffusion expands. Therefore, it is said that practically all problems lie in region (I) as far as horizontal oceanic mixing is concerned.

The characteristic feature of diffusion coefficient in region (I) can describe the general nature of relative diffusion. Diffusion coefficient given in Eq. (44) increases as the time elapses after the release of diffusive substances, which is the case encountered in ocean.

As for the non-linear interaction between dif-

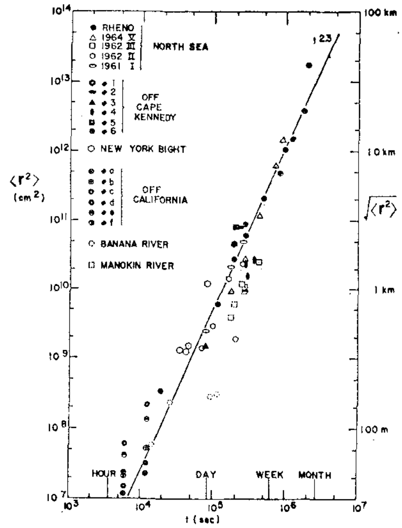

Fig. 3 Variance, $\langle r^2 \rangle$, versus diffusion time, t, from dye-diffusion experiments. (After Okubo[2])

ferent wave groups, we can consider that it occurs continuously, which means that one growing tertiary wave approaches to the size of the primary wave component and breaks, while other tertiary component is always under developing. Therefore, we can imagine the mixing power of waves holds the standard of intensity shown in region (I) in Eq. (44).

Diffusion coefficient computed by Eq. (44) is probably larger than the order of $10^{-3}t$ (m²/sec) considering the frequency at the spectral peak is less than 10 rad. sec⁻¹ in most cases encountered in ocean. Field measurements summarized by Okubo[2] are shown in Fig. 3 in terms of variance versus diffusion time. The functional form between diffusion coefficient and diffusion time is concluded as

$$D \simeq 10^{-6} t^{1.3} \quad (\mathrm{m^2/sec}) \qquad \cdots\cdots(46)$$

The diffusion process explained by Eq. (44) shows much more rapid mixing than observed, though the exponent of t is a little less than that obtained through observation. Although it is said that the interaction between two wave trains appears only when the waves have been running long enough, the diffusion generated by the growing tertiary waves is quite strong once after the interaction occurs. The estimation of effective time is very difficult for diffusion due to nonlinear interaction, because the interaction between different wave-numbers and frequencies depends upon the previous history of the wave system under consideration. Therefore, it is concluded that the discrepancy in factors of diffusion coefficient cannot be explained essentially until the process of developing tertiary waves is observed and fully understood.

For region (II) in diffusion problems the experimental result in a wind-wave flume is referred. Because the maximum eddy scale is bounded by the size of equipment, the diffusion process in an laboratory tank is expected to belong to region (II). Masch[11] made one study in a wave tank which was wide enough (1.2 m) that the wave spectrum was essentially two-dimensional, as is the case in nature. Polyethelene spheres (average diameter 2.8 mm) with a specific gravity of 0.97 were used so that they floated on the surface in order to preclude vertical mixing. Masch states that the wind did not act directly upon the spheres. Wind blowing over the water surface creates a surface current as well as waves. Both the current and the waves were measured. Masch found no evidence of the four-thirds law, except near the source. Over most of the range of the horizontal plume lateral eddy diffusivity was found to be a constant.

Assuming that the longitudinal diffusion coefficient is equal to the lateral diffusion coefficient, the computed values through Eq. (33) are compared with experimental measurements.

Estimated values shown in Table 1 are much smaller than those obtained in experiments. It seems that the contribution of random waves to the measured diffusion process is of the order of 1/400～1/1 000 of the total diffusivity and mixing is effectively caused by surface current. However, the difference of experimental condition from the theoretical postulation should be pointed out to explain cirtain portion of the discrepancy between the theory and the measurements. Masch used

Table 1 Comparison between the theoretical results and the experimental results

Run	$T_{1/3}$ sec	ω_p rad/sec	ω_p^{-3}	D cm²/sec	
				Theory	Experiment
HM 3-1	0.48	13.1	4.47×10^{-4}	1.34×10^{-2}	5.57
2	0.53	11.8	6.10 〃	1.83 〃	4.68
3	0.56	11.2	7.14 〃	2.14 〃	9.77
4	0.56	11.2	7.14 〃	2.14 〃	15.2
5	0.59	10.6	8.40 〃	2.52 〃	19.9
HM 5-1	0.33	19.0	1.46 〃	4.38×10^{-3}	3.53
2	0.31	20.2	1.22 〃	3.66 〃	5.82
3	0.36	17.4	1.90 〃	5.70 〃	8.00
4	0.38	16.5	2.23 〃	6.69 〃	11.9

floating particles of mean diameter of 2.8 mm. Althrough the specific weight of particles was almost the same as that of surrounding water, it is uncertain whether the particles followed exactly the motion of fluid particles. Because the specific weight of polyethelene spheres was 0.97, they have tendency to float and therefore they inclined to be kept in a zone with high positive velocity near the water surface. This makes diffusivity of particles much more intense than in the case theoretically considered in which the orbit of particles is nearly a circle and particles encounter the negative velocity with respect to the direction of wave propagation during the orbital motion.

The reasoning stated above is also supported from another point of view. Masch explained the obtained lateral diffusion coefficient as a function of the sum of the surface current and the half of the maximum orbital speed of water particle. This means the orbital speed is considered always positive and the arithmetic average is taken. Therefore, it is said that the floating particles were exposed to positive orbital velocity for longer time due to buoyancy of the particles and resulted diffusion was strongly affected by the orbital speed.

Another set of data[12] is referred in order to compare the theory with observation. These data were obtained in the measurements 700 m off Tokai-mura coast. Turbulent diffusion coefficient was obtained by two ways, that is, through the measurement of velocity fluctuation and through dye release experiments. Lagrangian correlation was calculated from Eulerian correlation which was actually measured. Observation time was in the range between 10^3 sec. and 10^5 sec. The velocity fluctuation was measured at 3 m beneath the sea surface.

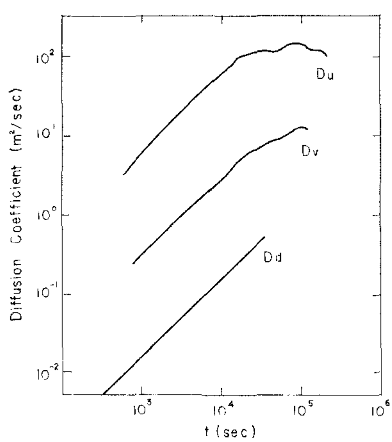

Fig. 4 Diffusion coefficients measured in the sea off Tokai-mura. (D_u and D_v express time-averaged diffusion coefficients along the coastline and in the direction perpendicular to the coastline, respectively. Both were obtained through velocity fluctuation. D_d shows geometric average of diffusion coefficients in two horizontal directions obtained by dye release experiments. After Ref. 12)

Diffusion coefficients obtained by different methods show the same functional relation with time except its coefficient for smaller range of diffusion time as explained in Fig. 4. They increase linearly with time. However, diffusion coefficient based on the correlation of velocity fluctuation tends to approach to a constant value when the observation time becomes greater than 10^4 sec. Dye release experiment does not show such a trend. Reported values for linear part with time are as follows:

(i) diffusion coefficient along the coastline

$$D_u = 5.9\times10^{-3}\,t \quad (\mathrm{m^2/sec}) \qquad \cdots\cdots(47)$$

(ii) diffusion coefficient in the direction perpendicular to the coastline

$$D_v = 3.2\times10^{-4}\,t \quad (\mathrm{m^2/sec}) \qquad \cdots\cdots(48)$$

(iii) measured value by dye release experiment

$$D_d = 1.6\times10^{-5}\,t \quad (\mathrm{m^2/sec}) \qquad \cdots\cdots(49)$$

The value of diffusion coefficient obtained by dye release experiments is very small compared with that obtained by the correlation of velocity fluctuation, though diffusion coefficient shown in Eq. (46) is smaller in magnitude than that given by Eq. (49). Equation (46) is derived as a best fit curve of dye-diffusion experiments covering the range of observation time between 7.4×10^3 sec. and 2×10^6 sec. Therefore, it may be said that the magnitude of diffusion coefficient obtained by dye release experiments may give smaller values than those obtained through velocity fluctuation, leaving some scattering of data due to the different experimental condition.

The discrepancy between diffusion coefficient along the coastline and that in the direction perpendicular to the coastline probably depends on geographical features of the coast. Theoretically derived diffusion coefficient based on the velocity fluctuation solely due to random waves shown in Eq. (44) is of the order of $10^{-3}\,t$ ($\mathrm{m^2/sec}$). Therefore, the measured diffusion coefficient along the coastline given in Eq. (47) shows agreement with the theoretical prediction in the order of a factor as well as its functional form.

Velocity fluctuation was measured at 3 m beneath the sea surface. The magnitude of velocity fluctuation at the sea surface is considered to be greater than that of velocity fluctuation at the level of 3 m beneath the sea surface. On the other hand, the theory is concerned with diffusion of a fluid particle at the sea surface. Therefore, in order to compare the theory with field observation, the measurement of velocity fluctuation should be done at the sea surface. It is reasonable to suppose that larger values of diffusion coefficient than those described in Eqs. (47) and (48) would have been obtained, if the velocity fluctuation had been observed at the surface of sea. Furthermore, it is appropriate to consider that the magnitude of diffusion coefficient measured in real sea will be greater than that derived from the theory which takes account of the sole effect of random waves even if other conditions are the same, because there are many other factors which generate diffusion of substances in real ocean, that is, wind drifts, littoral currents, tidal currents and so on.

Diffusivity of random waves comes from the interaction between primary waves and mass transfer velocity which is second order in magnitude in wave theory. In reality there exist other currents as mentioned above whose velocity might have stronger magnitude than that of mass transfer velocity. Therefore, diffusion measured in real sea is under the combined influence of total velocity field not only generated by random waves. It is difficult to estimate the exact rate of contribution of random waves to the whole diffusive process at the present stage of investigation until the precise measurement is carried out under the same condition as the theory postulates and the effect of each factor is analyzed.

It must be pointed out that there are other differences between the postulation of the theory and the condition of measurements. The theory considers the case of deep water condition, but the measurement was supposed to be performed in shallow water. Therefore, the transformation of waves in shallow water has to be considered for rigorous discussion. Also the effect of short-crested waves should be taken into consideration. Theoretically derived expression of Eq. (44) is applicable only when the amplitude of growing tertiary waves is smaller than that of primary waves. The time scale of development of tertiary waves is estimated to have the order of 10^2 sec. This is out of time scale of reported measurements in Fig. 4. Although the extrapolation of measured data implies the same relationships as those shown in Eqs. (47) and (48) for the shorter range of observation time, the detailed picture of the higher order interaction of waves is the future subject of the investigation.

In the last part the remarks on the vertical diffusion coefficient due to random waves are described briefly. There are growing tertiary wave components in the expression of surface deformation. Therefore, the same order of diffusion will occur shown in Eq. (44) for region (I). For region (II) it is derived that the tertiary waves with differences of wave-numbers and frequencies of primary components have no diffusivity by considering covariance of the fluctuating velocity. Although there exists the same intensity of mixing in the vertical direction as in the horizontal direction in this analysis, ocean is usually stratified in the vertical direction. Therefore, diffusion in the vertical direction is strongly suppressed by the exsistance of stable stratification.

CONCLUDING REMARKS

Concluding remarks of the study on the turbulent diffusion due to random waves are briefly summarized as follows.

1) It is shown that the diffusivity of random waves has its origin in non-linear interactions between different wave groups. The lowest order of wave components which show diffusive nature is found to be tertiary components.

2) There exist growing tertiary wave components with the same wave-number and frequency as those of the primary wave. The phase of these terms is in advance of the primary wave by $\pi/2$. The solution of this type is only valid for $t \ll 2/(a^2k^2\omega)$.

3) The horizontal diffusion coefficient obtained for the long-crested waves is classified into two regions and the functional form with time t is the same as described by Taylor. The criterion of these regions may be the mean interaction time in this case.

4) The diffusion coefficient derived from growing tertiary waves is linearly proportional to time, t. The mean interaction time during which the magnitude of tertiary components is small compared with that of the primary wave is of the order of 10^2 sec. The factor of t in the theoretically derived diffusion coefficient is much greater than that observed by dye-diffusion experiments in ocean. Diffusion coefficient along the coastline obtained by the measurement of velocity fluctuation, which follows the same procedure as in the theory, shows fairly good agreement with the

theoretical prediction of diffusion coefficient due to random waves. Details of conditions, however, do not coincide exactly between measurements and the theory. Although it is shown that diffusion due to random waves is able to describe essential features of the oceanic diffusion process, more extensive study is necessary to specify the exact contribution of random waves to the whole diffusion process.

5) The diffusion coefficient derived from the tertiary components with differences of wave-numbers and frequencies of the primary components is constant. Comparison of the calculated results with measurements made in a wave tank indicates that the contribution of random waves to the measured diffusion process is very small. However, quantitative estimation on the ratio of the contribution due to random waves is not satisfactory due to the discrepancy between assumptions of the theory and the experimental conditions.

6) Experimental verification of the process of tertiary interactions and of the sole effect of random waves is required for further understanding of mixing phenomena due to random waves.

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