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DAM-BREAK FLOW OVER A UNIFORMLY SLOPING BOTTOM

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A solution is derived for dam-break flow over a uniformly sloping bottom, under the assumption that each receding characteristic line of flow is parabolic. The hydraulic resistance of the bottom is not considered in the derivation. The validity and characteristics of the solution are examined through comparisons with Matsutomi's approximate solution under the same flow conditions, and with Ritter's and Peregrine et al.'s solutions. The solution is expected to be useful in examining or solving practical problems in hydraulic or coastal engineering, and any result obtained from it can be easily compared with experimental data.

Key Words: *dam-break flow, analytic solution, practical use*

1. INTRODUCTION

The knowledge of dam-break flows is sometimes useful in examining or solving practical problems in hydraulic or coastal engineering. In fact, such knowledge has been applied to discuss issues such as the behavior of a strong bore, caused by a tsunami, over a uniformly sloping beach[1], the impulsive force on a vertical wall due to the collision of a strong bore, caused by a tsunami, over a horizontal bottom[2], and the overtopping of a wave from a coastal dike with a uniform slope[3].

There are various kinds of dam-break flow problems, e.g., the problem of a flow immediately after breakage of the dam[4], the problem of a flow with or without water on the downstream side of the dam[5,6], the problem of a flow with or without the hydraulic resistance of the bottom taken into account[7-9], and the problem of a flow over a uniformly sloping bottom[1,3]. **Tables 1** and **2** summarize the past analytic studies on dam-break flow over horizontal and uniformly sloping bottoms, respectively. These studies are all based on the nonlinear shallow-water theory.

So far there have been three kinds of solutions for dam-break flow over a dry bottom without taking account

Table 1 The past studies on a dam-break flow over a horizontal bottom.

	Without water ahead	With water ahead
Without friction	Ritter (1892)[5]	Stoker (1948)[6]
With friction	Dressler (1952)[7] Whitham (1955)[8]	Matsutomi (1985)[9]

Table 2 The past studies on a dam-break flow over a uniformly sloping bottom.

	Without water ahead	With water ahead
Horizontal surface	Matsutomi (1985)[1]	Matsutomi (1985)[1]
Inclined surface	Peregrine et al. (2001)[3]	

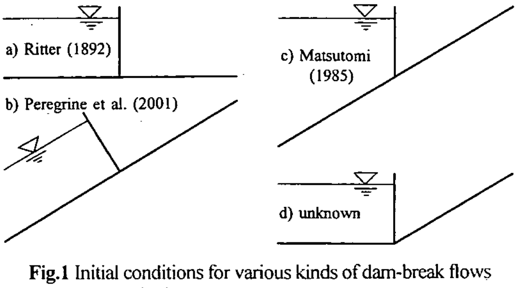

Fig.1 Initial conditions for various kinds of dam-break flows over a dry bottom.

of the hydraulic resistance (see the tables and **Fig.1**).

a) Ritter's solution for a horizontal bottom[5].
b) Peregrine et al.'s solution for a uniformly sloping bottom[3].
c) Matsutomi's approximate solution for a uniformly sloping bottom[1].

Figures 1 a) ~ c) show the initial conditions for the respective solutions. Matsutomi's approximate solution was derived assuming that (1) each receding characteristic line of flow is parabolic and (2) the relationship between the coefficients of the first- and second-power terms of the quadratic is linear. There is still no solution for the case shown in **Fig.1 d)** that is useful in many practical problems.

In light of the above circumstances, this study aims at offering a more sophisticated solution for dam-break flow over a uniformly sloping bottom (see **Fig.1 c)**) compared with Matsutomi's approximate solution, i.e., the solution is derived only under the assumption of each receding characteristic line of flow being parabolic.

2. BASIC EQUATIONS AND RELATED ISSUES

The hydraulic resistance of the bottom is ignored. The nonlinear shallow-water equations are adopted as the governing equations for dam-break flow. The characteristic forms of the equations for flow over a uniformly sloping bottom are expressed as follows:

$$u + 2c + igt = \text{const.} \quad \text{on} \quad \frac{dx}{dt} = u + c, \tag{2.1}$$

$$u - 2c + igt = \text{const.} \quad \text{on} \quad \frac{dx}{dt} = u - c, \tag{2.2}$$

where u is the water particle velocity, $c = \sqrt{gh}$ is the long wave velocity, g is the acceleration due to gravity, h is the local water depth, i is the bottom slope, t is time and x is the horizontal axis of coordinates. The symbols are defined schematically in **Fig.2**.

When h_G is the initial height from the bottom at x=0 in the upstream region of a dam, the following relation must be satisfied along a receding characteristic line:

$$u - 2c + igt = 2c_1 - 4\sqrt{gh_G}, \tag{2.3}$$

where $c_1 = \sqrt{gh_1}$ with h_1 being the initial water depth at x=0 in the upstream region of the dam. h_G can have a value between 0 and h_1, and is constant along the receding characteristic line.

As seen from Eq. (2.3), the following relation must be satisfied along the depression wavefront trajectory into

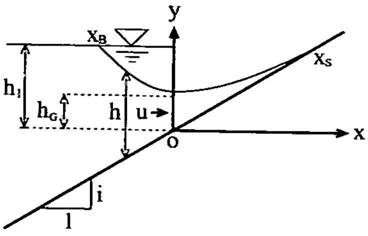

Fig.2 Definition sketch.

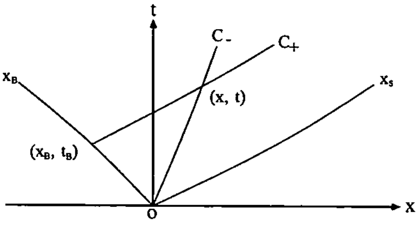

Fig.3 Advancing and receding characteristic lines.

still water behind the dam:

$$-2c_B + igt_B = -2c_1, \tag{2.4}$$

where the subscript B denotes the values concerning the depression wavefront trajectory (see **Fig.3**). Therefore, the following relation must be satisfied along an advancing characteristic line:

$$u + 2c + igt = 2c_1 + 2igt_B. \tag{2.5}$$

Solving Eqs. (2.3) and (2.5) simultaneously, the following expressions can be obtained:

$$c = \sqrt{gh_G} + \frac{1}{2}igt_B, \tag{2.6}$$

$$u = 2c_1 - 2\sqrt{gh_G} + igt_B - igt. \tag{2.7}$$

Therefore, the basic equations in this study are the two characteristic differential equations (2.1) and (2.2) plus Eqs. (2.6) and (2.7).

Incidentally, the advancing wavefront trajectory x_s and depression wavefront trajectory x_B of flow are expressed as follows[1]:

$$x_s = 2c_1 t - \frac{1}{2}igt^2, \tag{2.8}$$

$$x_B = -c_1 t - \frac{1}{4}igt^2. \tag{2.9}$$

Both the advancing and depression wavefront trajectories are parabolic receding characteristic lines. If the flow does not form any shock, receding characteristic lines between the two wavefront trajectories which start from the origin of coordinates (x, t) never intersect one another and seem also to be parabolic.

3. PEREGRINE ET AL.'S (2001) SOLUTION

Let us derive Peregrine et al.'s solution following the manner described in § 2, because the solution is obtained inductively and its characteristics, particularly the flow state in the upstream region of a dam, are not yet sufficiently clear.

The relation to be satisfied along a receding characteristic line is the same as Eq. (2.3):

$$u - 2c + igt = 2c_1 - 4\sqrt{gh_G}. \tag{3.1}$$

Therefore, the relation to be satisfied along the depression wavefront trajectory is

$$u_B - 2c_B + igt_B = -2c_1. \tag{3.2}$$

Since $c_B = c_1$, Eq. (3.2) leads to the following relation:

$$u_B = -igt_B. \tag{3.3}$$

This velocity is the same as that of a particle that freely descends over a uniformly sloping bottom with slope i under gravity.

The water particle velocity is not zero at the depression wavefront or behind it. This flow state is different from that in Ritter's solution. Therefore, the following relation must be satisfied along an advancing characteristic line:

$$u + 2c + igt = 2c_1. \tag{3.4}$$

Solving Eqs. (3.1) and (3.4) simultaneously, the following expressions can be obtained:

$$c = \sqrt{gh_G}, \tag{3.5}$$

$$u = 2c_1 - 2\sqrt{gh_G} - igt. \tag{3.6}$$

The difference between the expression of c or u in the present and in Peregrine et al.'s problems is whether the term igt_B exists or not.

The receding characteristic line can be obtained by integrating the following differential equation:

$$\begin{aligned}\frac{dx}{dt} &= u - c\\ &= 2c_1 - 3\sqrt{gh_G} - igt,\end{aligned} \tag{3.7}$$

where it is necessary to note that h_G is constant along each receding characteristic line. Solving Eq. (3.7) under the condition that $x=0$ at $t=0$, we obtain the following expression for the receding characteristic line:

$$x = \left(2c_1 - 3\sqrt{gh_G}\right)t - \frac{1}{2}igt^2. \tag{3.8}$$

Equation (3.8) indicates that if h_G is specified, the receding characteristic line can be determined and is parabolic. Substitutions of $h_G=0$ and $h_G=h_1$ into Eq. (3.8) respectively yield the following expressions:

$$x_s = 2c_1 t - \frac{1}{2}igt^2, \tag{3.9a}$$

$$x_B = -c_1 t - \frac{1}{2}igt^2. \tag{3.9b}$$

It is interesting to note that the behavior of the advancing wavefront is the same in both cases shown in **Figs.1 b)** and **c)**.

The advancing characteristic line can be obtained by integrating the following differential equation:

$$\begin{aligned}\frac{dx}{dt} &= u + c\\ &= 2c_1 - \sqrt{gh_G} - igt.\end{aligned} \tag{3.10}$$

Substituting $\sqrt{gh_G}$ derived from Eq. (3.8) into Eq. (3.10), the differential equation can be rewritten as

$$\frac{dx}{dt} = \frac{4}{3}c_1 + \frac{1}{3}\frac{x}{t} - \frac{5}{6}igt. \tag{3.11}$$

Solving Eq. (3.11) under the condition that $x=x_B$ at $t=t_B$, we obtain the following expression for the advancing characteristic line:

$$x = 2c_1 t - \frac{1}{2}igt^2 - 3c_1 t_B^{2/3} t^{1/3}. \tag{3.12}$$

Equation (3.12) indicates that if t_B is specified, the advancing characteristic line can be determined and is not parabolic, except at $t_B=0$.

Substituting Eq. (3.5) into Eq. (3.8), the following expression can be obtained:

$$c = \frac{1}{3}\left(2c_1 - \frac{x}{t} - \frac{1}{2}igt\right). \tag{3.13}$$

Equations (3.5), (3.6) and (3.13) lead to the following expression:

$$u = \frac{2}{3}\left(c_1 + \frac{x}{t} - igt\right). \tag{3.14}$$

Equations (3.13) and (3.14) are Peregrine et al.'s solution for dam-break flow over a uniformly sloping bottom. It is not necessary to use Eq. (3.12) in the derivation of their solution.

Equating Eq. (3.8) to Eq. (3.12), the following expression can be obtained:

$$\frac{t_B}{t} = \left(\frac{h_G}{h_1}\right)^{3/4} \equiv \gamma_1, \tag{3.15}$$

where $0 \leqq \gamma_1 \leqq 1$. $\gamma_1=0$ corresponds to the advancing wavefront trajectory and $\gamma_1=1$ to the depression wavefront trajectory. Equation (3.15) indicates that the ratio of t_B to t is constant along a receding characteristic line. The same result can be derived from Ritter's solution.

Equation (3.9) indicates that the extent of the flow region ($= x_s - x_B$) is exactly $3c_1t$. This length is the same as that in Ritter's solution as well as in the spatial distribution forms of the water depth and water particle velocity[3].

4. SOLUTION FOR BASIC EQUATIONS

It is difficult to derive an exact solution for the basic equations described in § **2**. Therefore, let us derive an approximate solution, assuming that each receding characteristic line is parabolic. This assumption seems to be reasonable as a first-order-approximation.

The receding characteristic line can be obtained by integrating the following differential equation:

$$\frac{dx}{dt} = u - c$$

$$= 2c_1 - 3\sqrt{gh_G} + \frac{1}{2}igt_B - igt. \qquad (4.1)$$

The assumption of a receding characteristic line leads to the following approximate expression:

$$\frac{t_B}{t} = \gamma(h_G, t_B)$$

$$\approx \gamma(h_G), \qquad (4.2)$$

where $\gamma(h_G)$ is a dimensionless constant parameter and is likely to be γ_1 in § 3, which is discussed later. Substituting Eq. (4.2) into Eq. (4.1), the differential equation can be rewritten as

$$\frac{dx}{dt} = 2c_1 - 3\sqrt{gh_G} + \left(\frac{\gamma}{2} - 1\right)igt. \qquad (4.3)$$

Solving Eq. (4.3) under the condition that x=0 at t=0, we obtain the following expression for the receding characteristic line:

$$x = \left(2c_1 - 3\sqrt{gh_G}\right)t + \frac{1}{2}\left(\frac{\gamma}{2} - 1\right)igt^2 \qquad (4.4a)$$

or

$$x = \left(2c_1 - 3\sqrt{gh_G}\right)t + \frac{1}{4}igt_Bt - \frac{1}{2}igt^2. \qquad (4.4b)$$

Equation (4.4a) indicates that if h_G is specified, the receding characteristic line can be determined and is parabolic.

The advancing characteristic line can be obtained by integrating the following differential equation:

$$\frac{dx}{dt} = u + c$$

$$= 2c_1 - \sqrt{gh_G} + \frac{3}{2}igt_B - igt. \qquad (4.5)$$

Substituting $\sqrt{gh_G}$ derived from Eq. (4.4b) into Eq. (4.5), the differential equation can be rewritten as

$$\frac{dx}{dt} = \frac{4}{3}c_1 + \frac{17}{12}igt_B + \frac{1}{3}\frac{x}{t} - \frac{5}{6}igt, \qquad (4.6)$$

where it is necessary to note that t_B is constant along the advancing characteristic line. Solving Eq. (4.6) under the condition that $x = x_B = -c_1t_B - igt_B^2/4$ (see Eq. (2.9)) at t=t_B, we obtain the following expression for the advancing characteristic line:

$$x = \left(2c_1 + \frac{17}{8}igt_B\right)t - \frac{1}{2}igt^2 - \left(3c_1 + \frac{15}{8}igt_B\right)t_B^{2/3}t^{1/3}. \qquad (4.7)$$

Equation (4.7) indicates that if t_B is specified, the advancing characteristic line can be determined and is not parabolic, except at t_B=0.

Equating Eq. (4.4b) to Eq. (4.7), the following expression can be obtained:

$$\gamma = \frac{t_B}{t}$$

$$= \gamma_1\left(\frac{1 + 5igt_B/8\sqrt{gh_G}}{1 + 5igt_B/8c_1}\right)^{3/2}, \qquad (4.8)$$

where $0 \leqq \gamma \leqq 1$. The expression of γ in Eq. (4.8) differs from that in Eq. (4.2). This indicates that Eq. (4.2) is clearly an approximate expression, i.e., the receding characteristic line is not parabolic between the advancing and depression wavefront trajectories, and stems from the necessity of satisfying the basic equations under the assumption of Eq. (4.2). Of course, Eq. (4.8) is recommended as γ, rather than, say γ_1 under the present assumption. Equation (4.8) may be used as a clue to derive a more accurate solution.

Equation (4.8) indicates that if h_G is fixed at an arbitrary value and t_B is changed from 0 to t_B, both γ and t are determined along a receding characteristic line, and also that if t_B is fixed at an arbitrary value and h_G is changed from h_1 to 0, both γ and t are determined along an advancing characteristic line. Using these fixed and determined values, the position x on each characteristic line is determined by Eq. (4.4) or (4.7).

Generally, t or x is initially specified, and then the spatial distributions or time histories of c, h, u and so on are requested. When t is initially specified, t_B is merely changed from 0 to t at proper intervals. γ is naturally determined for each assigned t_B and x can be evaluated from Eq. (4.7). When x is initially specified, an arbitrary t is merely chosen. By gradually changing t_B from 0 to the arbitrary t, we try to find t_B that satisfies Eq. (4.7). γ is naturally determined at this stage.

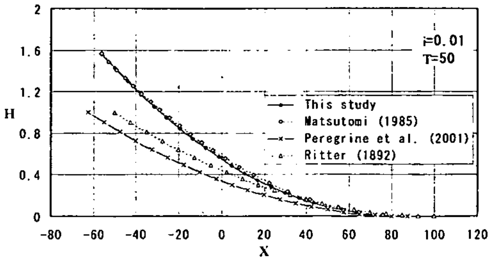

Fig.4 Spatial distributions of water depth at T=50.

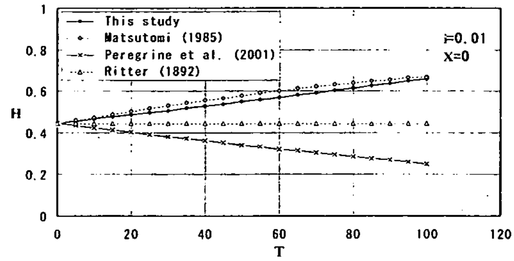

Fig.7 Time histories of water depth at X=0.

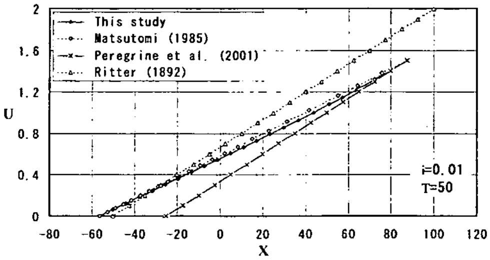

Fig.5 Spatial distributions of water particle velocity at T=50.

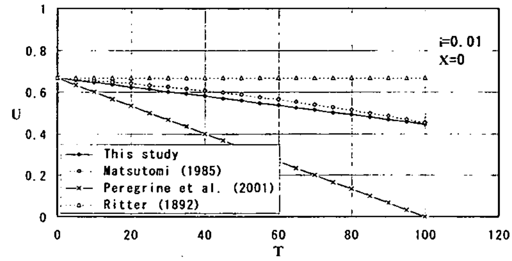

Fig.8 Time histories of water particle velocity at X=0.

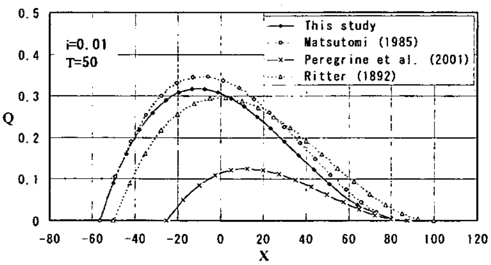

Fig.6 Spatial distributions of discharge per unit width at T=50.

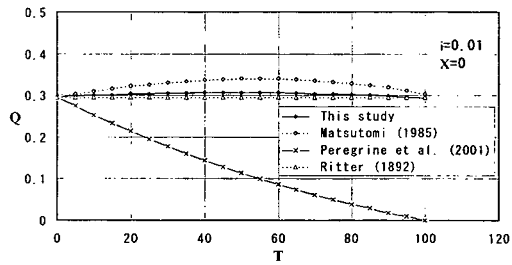

Fig.9 Time histories of discharge per unit width at X=0.

Substituting $\sqrt{gh_G}$ derived from Eq. (2.6) into Eq. (4.4b), the following expression can be obtained:

$$c=\frac{1}{3}\left\{2c_1-\frac{x}{t}-\frac{1}{2}\left(1-\frac{7}{2}\gamma\right)igt\right\}. \qquad (4.9)$$

Equations (2.6), (2.7) and (4.9) lead to

$$u=\frac{2}{3}\left\{c_1+\frac{x}{t}-\left(1-\frac{5}{4}\gamma\right)igt\right\}. \qquad (4.10)$$

Equations (4.9) and (4.10) are the solutions under the flow condition shown in **Fig.1 c)**. It is easy to confirm that Eqs. (4.9) and (4.10) are satisfied with Eqs. (2.3) and (2.5). When γ=0, which also means that the characteristic value along any advancing characteristic line is $2c_1$ (see Eq. (2.5)), Eqs. (4.9) and (4.10) result in Peregrine et al.'s solution. Moreover when i=0, Eqs. (4.9) and (4.10) result in Ritter's solution:

$$c=\frac{1}{3}\left(2c_1-\frac{x}{t}\right), \qquad (4.11)$$

$$u=\frac{2}{3}\left(c_1+\frac{x}{t}\right). \qquad (4.12)$$

Matsutomi's approximate solution[1)] is expressed as follows:

$$c=\frac{4c_1}{12c_1-igt}\left(2c_1-\frac{x}{t}-\frac{igx}{2c_1}+\frac{1}{2}igt-\frac{i^2g^2t^2}{4c_1}\right)$$
$$\approx\frac{1}{3}\left(2c_1-\frac{x}{t}-\frac{7igx}{12c_1}+\frac{2}{3}igt\right)+O(i^2), \qquad (4.13a, b)$$

$$u=\frac{8c_1}{12c_1-igt}\left(c_1+\frac{x}{t}-\frac{igx}{2c_1}-\frac{1}{4}igt-\frac{i^2g^2t^2}{8c_1}\right)$$
$$\approx\frac{2}{3}\left(c_1+\frac{x}{t}-\frac{5igx}{12c_1}-\frac{1}{6}igt\right)+O(i^2). \qquad (4.14a, b)$$

Therefore, it can be understood that all the above solutions have the same expression up to the second term of the solutions. When i is small, Matsutomi's approximate solution is expected to give a slightly large value for both c and u around x=0 compared with Eqs. (4.9) and (4.10).

Not even an approximate solution is yet derived for dam-break flow over a uniformly sloping bottom connected with a horizontal bottom (see **Fig.1 d)**). It is clear that the solution for this case cannot be derived solely by connecting Peregrine et al.'s solution with Ritter's solution at x=0 because of the effect of reflection from the slope, although the basic equations in each region are the same in Peregrine et al.'s and Ritter's solutions.

5. DISCUSSIONS

Examples of the newly derived solution, $H=h/h_1$, $U=u/c_1$ and $Q=hu/h_1c_1$ against $X=x/h_1$ for a fixed time $T=t\sqrt{g/h_1}$ =50 are shown in **Figs.4 ~ 6**, respectively. The bottom slope i is 0.01. In the figures, Matsutomi's approximate solution (Eqs. (4.13a) and (4.14a)), and Peregrine et al.'s and Ritter's solutions are also shown for comparison and to examine the validity of the newly derived solution. Only the positive part is shown for U and Q of Peregrine et al.'s solution. It can be understood from the figures that the newly derived solution is reasonable, i.e., it gives (1) a shallower water depth compared with that of Matsutomi's approximate solution and (2) a deeper water depth and lower water particle velocity compared with those of Ritter's solution. Through a comparison of the water volumes that flowed in and out across the section X=0[1], it has already been confirmed that Matsutomi's approximate solution has a tendency to give a deeper water depth than the actual value.

Examples of H, U and Q against T for a fixed position X=0 are shown in **Figs.7 ~ 9**, respectively, where the correspondence between **Figs.4 ~ 6** and **7 ~ 9** must be considered in terms of a receding characteristic line that can develop in the plus or minus direction of X. It can be understood from **Fig.9** that Q (T) of the newly derived solution has a gentle peak, i.e., a quasi-steady part, which may be more practical in modeling run-up of a relatively long-period wave such as a tsunami over a uniformly sloping beach[10),11)].

6. CONCLUSIONS

Equations (4.9) and (4.10) were derived as a more accurate solution for dam-break flow over a uniformly sloping bottom compared with Matsutomi's approximate solution. The validity and characteristics of the solution were examined through comparisons with Matsutomi's approximate solution, and Ritter's and Peregrine et al.'s solutions. The solution is expected to be useful in modeling run-up of a longer-period wave with a gentle peak of Q (T) over a uniformly sloping bottom, compared with Peregrine et al.'s solution.

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