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THE ELASTIC CATENARY AS A DISPLACEMENT-METHOD ELEMENT

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A variety of cable structures are supposed to be analyzed in the displacement method. In this study, the elastic catenary is considered to be utilized as a displacement-method element. The flexibility relations are studied, in relation with its potential, and especially on the singularity exhibited in a veritically hanging state. A counter treatment of those singular states is presented to deal with any configurations of a cable structure. In the solution procedure, one iteration method is introduced to obtain the tension components in each of the elements at any nodal positions, both theoretically and in numerical examples.

***Key Words** : elastic catenary, singularity, displacement method, iteration*

1. INTRODUCTION

Practically, a pretensioned cable is replaced into the simple tension element. At the same time, for accuracy, there have been presented the sag-embodied elements based on the parabolic cable or the catenary cable.[e.g. 1)–4)] While for a mere cable assembly there exist certain analyses categorized into the force method, the displacement method has a potential capability to deal with a wide variety of "cable structures," such as a mixture of cable and beam members.

In a displacement method, the nonlinear equilibrium equations are iteratively solved for the nodal displacements. If the hanging cable element is employed, an additional difficulty is encountered. The spatial configuration is expressed in a nonlinear parametric form of tension components. That is, the nodal forces of each element are implicit in terms of its nodal coordinates. This compatibility problem has to be solved numerically for all the cable elements, on each cycle of the global iteration.

So far as subjected to a uniform weight per unit natural length, the elastic catenary is exact for any tensions. But, when the cable is vertically hanging, with absence of the horizontal tension component, the characteristic relations become somewhat singular. In this study, after their degrees are clarified, a treatment of those singular states is presented for a complete utilization as a displacement-method element. Finally, the fractional correction method[3)] is verified to hold a consistent convergence in the compatibility problem, even if the singularity is involved.

2. SOLUTION OF A SUSPENDED CABLE

Consider that one end of a flexible cable, length l and extension rigidity EA, is anchored at the origin of spatial rectangular coordinates $\{x, y\}$. With material coordinate s taken along its natural length, as shown in **Fig.1**, the spatial configuration is described by a Lagrangian expression $\{x(s), y(s)\}$. It is assumed that external forces are prescribed with respect to s-coordinate: distributed forces are denoted by $\boldsymbol{q}(s) = \{q_x(s), q_y(s)\}$; and concentrated forces, by $\boldsymbol{P} = \{P_x, P_y\}_\alpha$ with acting point s_α.

In a deformed configuration, an element of initial length ds becomes to a length $d\bar{s} = \sqrt{(dx/ds)^2 + (dy/ds)^2}\, ds$. The magnitude of tension is related to the unit elongation by

$$T(s) = EA\left(\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} - 1\right) \tag{1}$$

Since tension T is acting into the tangent direction on the cable (moment equilibrium), its spatial components are written as

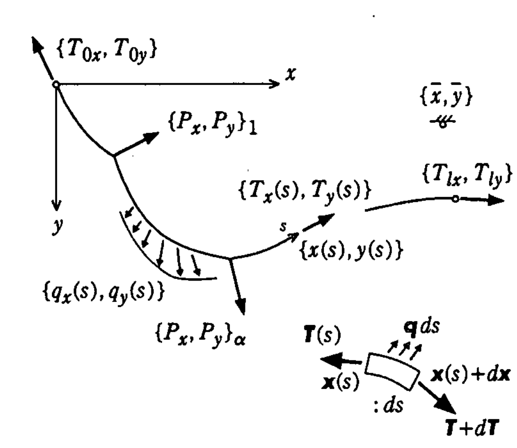

Fig.1 Isolated cable under a general loading

$$\left\{\begin{array}{c} T_x(s) \\ T_y(s) \end{array}\right\} = \frac{T}{\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}} \left\{\begin{array}{c} \frac{dx}{ds} \\ \frac{dy}{ds} \end{array}\right\} \quad (2)$$

On the other hand, the equilibrium equations for element ds are given by

$$\frac{d}{ds}\{T_x,\ T_y\} + \{q_x,\ q_y\} = \{0,\ 0\}$$

By the integration with inclusion of the concentrated $\boldsymbol{P}_\alpha = \{P_x, P_y\}_\alpha$, the tension components are written in the form

$$\left\{\begin{array}{c} T_x(s) \\ T_y(s) \end{array}\right\}$$

$$= \left\{\begin{array}{c} T_{0x} \\ T_{0y} \end{array}\right\} - \int_0^s \left\{\begin{array}{c} q_x \\ q_y \end{array}\right\} ds - \sum_{s_\alpha < s} \left\{\begin{array}{c} P_x \\ P_y \end{array}\right\}_\alpha \quad (3a)$$

or

$$= \left\{\begin{array}{c} T_{lx} \\ T_{ly} \end{array}\right\} + \int_s^l \left\{\begin{array}{c} q_x \\ q_y \end{array}\right\} ds + \sum_{s_\alpha > s} \left\{\begin{array}{c} P_x \\ P_y \end{array}\right\}_\alpha \quad (3b)$$

in which $\{T_{0x}, T_{0y}\}$ and $\{T_{lx}, T_{ly}\}$ are tension components at $s = 0$ and l, respectively.

Introducing $T(s) = \sqrt{T_x(s)^2 + T_y(s)^2}$ and $\sqrt{(dx/ds)^2 + (dy/ds)^2} = 1 + T(s)/EA$ into Eq.(2), we have derivatives $\{dx/ds, dy/ds\}$ expressed in terms of the tension components. After the integration, the equilibrium curve corresponding to tension distribution (3) is written as

$$\left\{\begin{array}{c} x(s) \\ y(s) \end{array}\right\} = \int_0^s \left(\frac{1}{\sqrt{T_x(s)^2 + T_y(s)^2}} + \frac{1}{EA}\right) \left\{\begin{array}{c} T_x(s) \\ T_y(s) \end{array}\right\} ds \quad (4)$$

This solution[3)] is in a parametric form of tension components at one cross-section, $\{T_{0x},\ T_{0y}\}$ or $\{T_{lx},\ T_{ly}\}$. If the cable is supported at one end only, the tension components at the other end are directly known from the mechanical condition. But, if the other end also is fixed at another point, say $\bar{\boldsymbol{x}}(=\{\bar{x},\ \bar{y}\})$, we are faced with a compatibility problem to find $\boldsymbol{T}_l(=\{T_{lx},\ T_{ly}\})$ satisfying $x(\boldsymbol{T}_l,\ l) = \bar{x}$ and $y(\boldsymbol{T}_l,\ l) = \bar{y}$.

In a displacement-method analysis, the latter is the case. It is usual for such algebraic equations to be solved iteratively upon the tangent coefficients. From Eq.(3), $\delta\boldsymbol{T}(s) = \delta\boldsymbol{T}_0 = \delta\boldsymbol{T}_l$. By differentiating Eq.(4) with respect to $\{T_{lx},\ T_{ly}\}$, we have

$$\delta\boldsymbol{x}_l = [Q(\boldsymbol{T}_l)]\,\delta\boldsymbol{T}_l \quad (5a)$$

$$[Q(\boldsymbol{T}_l)] = \left[\int_0^l A(s)\,ds\right] \quad (5b)$$

$$[\boldsymbol{A}(s)] = \left(\frac{1}{|\boldsymbol{T}(s)|} + \frac{1}{EA}\right)[\boldsymbol{I}] - \frac{\boldsymbol{T}(s)\,\boldsymbol{T}(s)^T}{|\boldsymbol{T}(s)|^3} \quad (5c)$$

The quadratic form of integrand $[\boldsymbol{A}(s)]$ is written as

$$\delta\boldsymbol{T}_l^T[\boldsymbol{A}(s)]\,\delta\boldsymbol{T}_l = \frac{1}{EA}(\delta T_{lx}^2 + \delta T_{ly}^2) + \frac{1}{|\boldsymbol{T}(s)|^3}\{T_y(s)\,\delta T_{lx} - T_x(s)\,\delta T_{ly}\}^2 \quad (6)$$

If $\boldsymbol{T}(s) \neq \boldsymbol{0}$, $\delta\boldsymbol{T}_l^T[\boldsymbol{A}(s)]\delta\boldsymbol{T}_l > 0$ for any $\delta\boldsymbol{T}_l(\neq \boldsymbol{0})$. Then, matrix $[Q(\boldsymbol{T}_l)]$ also is positive definite, if $\boldsymbol{T}(s) \neq \boldsymbol{0}$ on the entire length, $0 \leq s \leq l$.

By the symmetry of $[Q(\boldsymbol{T}_l)]$, differential $\delta F = \boldsymbol{x}_l(\boldsymbol{T}_l)^T\,\delta\boldsymbol{T}_l$ is path-independently integrable. Thus, we have a complementary potential

$$F(\boldsymbol{T}_l) = \int_0^l \left[\,|\boldsymbol{T}(s)| + \frac{|\boldsymbol{T}(s)|^2}{2EA}\right] ds \quad (7)$$

By the positive definiteness, this scalar function is convex. Then, by adding linear term $-\bar{\boldsymbol{x}}^T\boldsymbol{T}_l$, we have the total potential

$$W^*(\boldsymbol{T}_l) = F(\boldsymbol{T}_l) - (\bar{x}\,T_{lx} + \bar{y}\,T_{ly}) \quad (8)$$

which has its minimum value at the unique solution of $\boldsymbol{x}_l(\boldsymbol{T}_l) = \bar{\boldsymbol{x}}$.

3. ELASTIC CATENARY

Let the preceding expressions be specified for a uniform self-weight, say w, acting into y-direction (see **Fig.2**). By the actual integration of (4) for

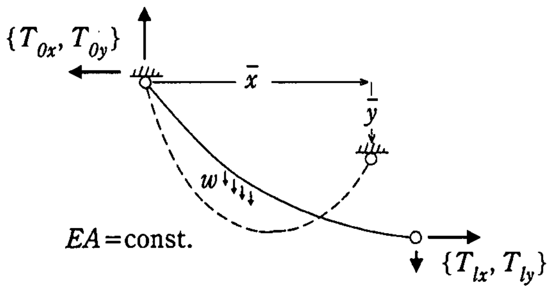

Fig.2 Elastic catenary element

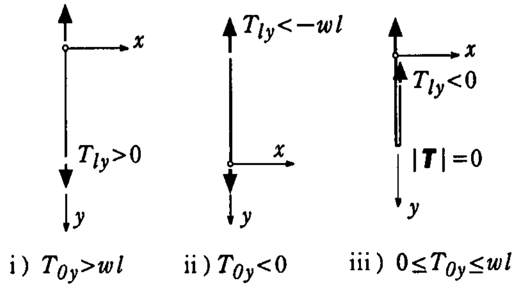

Fig.3 Vertcally hanging cables

$\{T_x(s),\ T_y(s)\} = \{T_{0x},\ T_{0y} - ws\}$, we have

$$x(\boldsymbol{T}_0; s) = \frac{T_{0x}s}{EA} + \frac{T_{0x}}{w}\log\left\{\frac{\sqrt{T_{0x}^2+T_{0y}^2}+T_{0y}}{\sqrt{T_{0x}^2+(T_{0y}-ws)^2}+(T_{0y}-ws)}\right\} \tag{9a}$$

$$y(\boldsymbol{T}_0; s) = \frac{T_{0y}s - \frac{w}{2}s^2}{EA} + \frac{1}{w}\left\{\sqrt{T_{0x}^2+T_{0y}^2} - \sqrt{T_{0x}^2+(T_{0y}-ws)^2}\right\} \tag{9b}$$

This is the "elasic catenary," which has been, according to Ref. 1), presented by Rough[5)] in 1896.

The tangent flexibility matrix and the total complementary potential are eventually obtained as follows :

$$[Q(\boldsymbol{T}_0)] = \begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix} \tag{10}$$

$$Q_{xx} = \frac{l}{EA} + \frac{1}{w}\left\{\log\left(\frac{T_0+T_{0y}}{T_l+T_{ly}}\right) + \frac{T_{ly}}{T_l} - \frac{T_{0y}}{T_0}\right\} \tag{11a}$$

$$Q_{xy} = Q_{yx} = \frac{T_{0x}}{w}\left(\frac{1}{T_0} - \frac{1}{T_l}\right) \tag{11b}$$

$$Q_{yy} = \frac{l}{EA} + \frac{1}{w}\left(\frac{T_{0y}}{T_0} - \frac{T_{ly}}{T_l}\right) \tag{11c}$$

$$W^*(\boldsymbol{T}_0) = \frac{1}{2w}\left\{T_{0x}^2\log\left(\frac{T_0+T_{0y}}{T_l+T_{ly}}\right) + T_{0y}T_0 - T_{ly}T_l\right\} + \frac{l}{2EA}\left\{T_{0x}^2 + \frac{1}{3w}(T_{0y}^3 - T_{ly}^3)\right\} - \left\{\bar{x}\,T_{lx} + \bar{y}\,T_{ly}\right\} \tag{12}$$

where $\{T_{lx}, T_{ly}\} = \{T_{0x}, T_{0y}\} - \{0, wl\}$, $T_0 = \sqrt{T_{0x}^2+T_{0y}^2}$ and $T_l = \sqrt{T_{lx}^2+T_{ly}^2}$.

Since no restrictions have been made upon the sag-to-span ratio, the above expressions are valid to any deep deflections. However, a vertically hanging state with $T_x(s) = 0$ is peculiar. The flexibility relations are here examined for those vertical states. As shown in **Fig.3**, they are separated into three cases: i) $T_{0y} > wl$; ii) $T_{0y} < 0$; and iii) $0 \leq T_{0y} \leq wl$. For $T_x(s) = 0$, expression (9a) directly gives $x(s) = 0$. For simplicity, let $y(s)$ be argued at $s = l$ only. Term $T_0 - T_l = |T_{0y}| - |T_{0y} - wl|$ has different values for the three cases, $T_{0y} - (T_{0y} - wl)$, $-T_{0y} + (T_{0y} - wl)$ and $T_{0y} + (T_{0y} - wl)$. Expression (9b) finally yields the followings :

$$y_l = \begin{cases} l\left\{1 + \frac{1}{EA}\left(T_{0y} - \frac{wl}{2}\right)\right\} \\ l\left\{-1 + \frac{1}{EA}\left(T_{0y} - \frac{wl}{2}\right)\right\} \\ \left(\frac{2T_{0y}}{w} - l\right) + \frac{l}{EA}\left(T_{0y} - \frac{wl}{2}\right) \end{cases} \tag{13}$$

for case i), ii) and iii), respectively. Those y_l are exact for the three states in **Fig.3**. By the actual integration of $\delta W^* = (y_l(T_{ly}) - \bar{y})\,\delta T_{ly}$, we can see potential $W^*(\boldsymbol{T}_0)$ given by (12) also correct.

In the tangent flexibility matrix, (10) and (11), term $\log\{(T_0+T_{0y})/(T_l+T_{ly})\}$ is to be focussed. If $T_x(s) = 0$, this expression is indeterminate for $T_{0y} < 0$ (so $T_{ly} < 0$), for instance. But, the following expansion is possible: since

$$\frac{\sqrt{T_x^2+T_{0y}^2}+T_{0y}}{\sqrt{T_x^2+T_{ly}^2}+T_{ly}} = \frac{\sqrt{T_x^2+T_{0y}^2}+T_{0y}}{\sqrt{T_x^2+T_{ly}^2}+T_{ly}}\cdot\frac{\sqrt{T_x^2+T_{0y}^2}-T_{0y}}{\sqrt{T_x^2+T_{ly}^2}-T_{ly}}$$

$$\frac{\sqrt{T_x^2+T_{ly}^2}-T_{ly}}{\sqrt{T_x^2+T_{0y}^2}-T_{0y}} = \frac{\sqrt{T_x^2+T_{ly}^2}-T_{ly}}{\sqrt{T_x^2+T_{0y}^2}-T_{0y}}$$

we have

$$\log\left(\frac{T_0+T_{0y}}{T_l+T_{ly}}\right) = \log\left(\frac{T_l-T_{ly}}{T_0-T_{0y}}\right) \quad (14)$$

The latter $\log\{(T_l-T_{ly})/(T_0-T_{0y})\}$ yields a definite value for $T_{0y} < 0$. On the contrary, the former expression is valid for $T_{0y} > wl$. But, if $0 \le T_{0y} \le wl$, the cable has a point of $\{T_x(a), T_y(a)\} = \{0, 0\}$. In this case, the derivative $\partial x_l/\partial T_{lx}$ becomes infinite. So, we have

$$\text{i)}: [Q] = \begin{bmatrix} \frac{1}{w}\log\left(\frac{T_{0y}}{T_{0y}-wl}\right)+\frac{l}{EA} & 0 \\ 0 & \frac{l}{EA} \end{bmatrix}$$

$$\text{ii)}: [Q] = \begin{bmatrix} \frac{1}{w}\log\left(\frac{T_{0y}-wl}{T_{0y}}\right)+\frac{l}{EA} & 0 \\ 0 & \frac{l}{EA} \end{bmatrix}$$

$$\text{iii)}: [Q] = \begin{bmatrix} \infty & 0 \\ 0 & \frac{2}{w}+\frac{l}{EA} \end{bmatrix} \quad (15)$$

The singularity is seen only in the element tangent flexibility. Even if the case iii) happens in iteration for the compatibility, only to leave for a normal state, the infinite $\log\{(T_0+T_{0y})/(T_l+T_{ly})\}$ can be temporarily replaced by an unreal large value such as EA/wl. If a cable is found singular in the global displacement method, by giving zero to the corresponding coefficient, we have its tangent stiffness restored. To be noted, such a singular element does not necessarily mean a singularity in the global tangent stiffness.

4. FRACTIONAL CORRECTION METHOD

Our functions, $\{x_l(T_{lx}, T_{ly}), y_l(T_{lx}, T_{ly})\}$, are in a one-to-one correspondence, but are quite nonlinear: the derivatives can vary from a very small $l/(EA)$ to the infinity. So, the simple Newton-Raphson method is not sufficient to solve the compatibility problem. In the below, we consider an advanced Newton-Raphson scheme for the extremal problem of a convex potential,[3] categorized into the so-called "step-length controlled." [6),7)]

Suppose the $[i]$-th trial where $\boldsymbol{x}_{l[i]}$, $[Q]_{[i]}$ and $W^*_{[i]}$ are estimated for $\boldsymbol{T}_{l[i]}$. The Taylor's expansion of potential $W^*(\boldsymbol{T}_l)$ at this $\boldsymbol{T}_{l[i]}$ is written as

$$W^*(\boldsymbol{T}_{l[i]}+\Delta\boldsymbol{T}_l) = W^*_{[i]} + \Delta\boldsymbol{x}^T_{l[i]}\Delta\boldsymbol{T}_l + \frac{1}{2}\Delta\boldsymbol{T}^T_l[Q]_{[i]}\Delta\boldsymbol{T}_l + 0\left(\Delta\boldsymbol{T}_l^3\right) \quad (16)$$

in which $\Delta\boldsymbol{x}_{l[i]}$ indicates error vector $\boldsymbol{x}_{l[i]}-\bar{\boldsymbol{x}}$. Instead of the full correction, $\Delta\boldsymbol{T}_{l[i]} = -[Q]^{-1}_{[i]}\Delta\boldsymbol{x}_{l[i]}$, we now put a fractional $\theta_{[i]}\Delta\boldsymbol{x}_{l[i]}$ upon the tangent correction :

$$\boldsymbol{T}_{l[i+1]} = \boldsymbol{T}_{l[i]} - \theta_{[i]}[Q]^{-1}_{[i]}\Delta\boldsymbol{x}_{l[i]} \quad (17)$$

where $0 < \theta_{[i]} \le 1$. Since $[Q]_{[i]}$ is exact for the differentials, the $\boldsymbol{T}_{l[i+1]}$ shifted with a small enough $\theta_{[i]}$ is improved toward the solution. This feature can be seen upon the potential. By the substitution of (17) into (16), we have

$$W^*_{[i+1]} = W^*_{[i]} - (\theta - \frac{\theta^2}{2})_{[i]}R^2_{[i]} + 0(\theta^3_{[i]}) \quad (18)$$

in which

$$R_{[i]} = \sqrt{\Delta\boldsymbol{x}^T_{l[i]}[Q]^{-1}_{[i]}\Delta\boldsymbol{x}_{l[i]}} \quad (19)$$

This $R_{[i]}$ is a norm of vector $\Delta\boldsymbol{x}_{l[i]}$ weighted by $[Q]^{-1}_{[i]}$. As can be seen from the above (18) for $\theta_{[i]} = 1$, quantity $1/2 \cdot R^2_{[i]}$ represents the first-approximated difference of the current $W^*_{[i]}$ from the minimum at the solution. Even if error $\Delta\boldsymbol{x}_{l[i]}$ is large, by the use of a small $\theta_{[i]}$, the remainder can be made negligible; and $\theta - \theta^2/2 > 0$ for $0 < \theta \le 1$. Hence $W^*_{[i]} > W^*_{[i+1]}$.

To generate the sequenced $\{\theta_{[1]}, \theta_{[2]}, \cdots\}$ in the iteration, the followings are important :

A) If $\theta_{[i]}$ are taken too small, the fractional corrections themselves are kept accurate, but a vainly large number of cycles are needed to attain the solution.

B) Since the accuracy of correction (17) depends upon the magnitude of $\theta_{[i]}\Delta\boldsymbol{x}_{l[i]}$, factor $\theta_{[i]}$ ought to be taken larger with decrease of $\Delta\boldsymbol{x}_{l[i]}$.

Since $R_{[i]}$ is homogeneous to vector $\Delta\boldsymbol{x}_{l[i]}$, scalar $\theta_{[i]}R_{[i]}$ corresponds to $\theta_{[i]}\Delta\boldsymbol{x}_{l[i]}$, and can be regarded as a magnitude of the correction. To carry the above B), we now determine $\theta_{[i]}$ such that $\theta_{[i]}R_{[i]}$ are kept constant on the earlier cycles. With decrease of $R_{[i]}$ in $\theta_{[i]}R_{[i]}$ =const., factor $\theta_{[i]}$ becomes infinitely large, but is to be bounded by 1 (the full correction). Eventually, we have

$$\theta_{[i]} = \min.\left(\frac{\theta_0 R_{[0]}}{R_{[i]}}, 1\right) \quad (20)$$

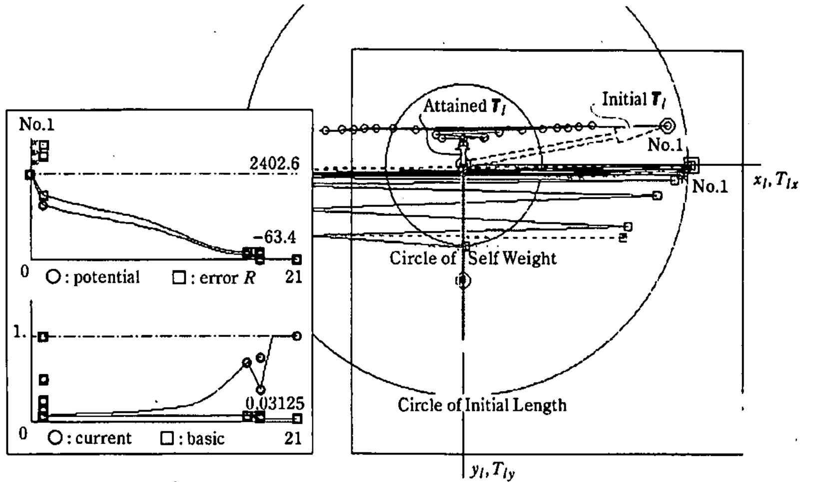

Fig.4 Convergence to a vertical support

in which θ_0 is a quantity called "basic correction factor," initially assumed within $0 < \theta_0 \leq 1$.

Inequality $W^*_{[i]} > W^*_{[i+1]}$ means that $\boldsymbol{T}_{l[i+1]}$ is relatively improved from $\boldsymbol{T}_{l[i]}$. But, after the solution is almost approached, this comparison becomes numerically difficult: for, difference $W^*_{[i+1]} - W^*_{[i]}$ is consisting of the quadratic and higher-order terms of $\Delta \boldsymbol{x}_{l[i]}$. On the other hand, $R_{[i]}$ becomes accurate to measure smaller $\Delta \boldsymbol{x}_{l[i]}$. Then, we examine each correction by

$$W^*_{[i]} > W^*_{[i+1]} \qquad \text{for} \quad \theta_{[i]} < 1 \qquad (21\text{a})$$

$$R_{[i]} > R_{[i+1]} \qquad \text{for} \quad \theta_{[i]} = 1 \qquad (21\text{b})$$

If this inequality is not true, factor θ_0 is found still too large. The false $[i+1]$-th result is abandoned, and, before resuming the iteration from the previous $[i]$-th, we change θ_0 into a smaller one, for instance, such that the new $\theta_{[i]}$ becomes half of the false $\theta_{[i]}$:

$$\theta_{0\,\text{new}} = \frac{\theta_{[i]\,\text{false}}}{2} \cdot \frac{R_{[i]}}{R_{[0]}} \qquad (22)$$

(alternatively, a simple $\theta_0 \to \theta_0/2$ is acceptable)

Sequence $\{W^*_{[0]}, W^*_{[1]}, \cdots, W^*_{[I]}\}$ and $\{R_{[I+1]}, R_{[I+2]}, \cdots\}$ are bounded from below, and are regulated by the re-iteration routine to be monotone decreasing. Those two conditions say the iteration is convergent. Since $W^*(\boldsymbol{T}_l)$ is properly convex, there exists such a positive $\bar{\theta}$ that, for $\theta_0 \leq \bar{\theta}$, $W^*_{[i]} > W^*_{[i+1]}$ at any $\boldsymbol{T}_{l[i]}$. In other words, after its change by several times at most, factor θ_0 remains a certain fraction (does not vanish into 0). The convergence is a result of $\Delta \boldsymbol{x}_{l[i]} \to \mathbf{0}$. In the above iteration upon (17), (19), (20), (21) and (22), the unique solution is attained from any initial $\boldsymbol{T}_{l[0]}$ and θ_0.

5. NUMERICAL EXAMPLE

(1) Isolated Cable

$l = 100.$ [L], $EA = 1000.$ [F] and $w = 0.1$ [F/L] are assumed for the cable shown in **Fig.2**. For a leveled support, $\{\bar{x}, \bar{y}\} = \{102., \cdot 0.\}$ [L], the tension components are beforehand obtained: $\{T_{lx}, T_{ly}\} = \{26.04, -5.\}$ [F].

For a vertical support, $\{\bar{x}, \bar{y}\} = \{0., 50.\}$, the fractional correction method is executed from the former state with $\theta_0 = 1$, until $(R/\sqrt{F})_{[i]} < 0.2 \times 10^{-4}$. After the fifth change of factor θ_0 ($\to 0.03125$), $\{T_{lx}, T_{ly}\} = \{-0.212 \times 10^{-4}, -2.512\}$ is attained at the 21-st cycle. The process is shown in **Fig.4** : in a common $\{x, y\}$-plane with a relative scaling, $\{T_{lx}, T_{ly}\}_{[i]}$ and $\{x_l, y_l\}_{[i]}$ are plotted by the lines with symbol ○ and □, respectively; and the changes of $W^*_{[i]}$ (○: potential), $R_{[i]}$ (□: error R), $\theta_{[i]}$ (○: current) and θ_0 (□: basic) are shown in the left box. This final state is not treated as singular, for the attained T_{lx} is infinitesimally small but not exact zero. The reversed computation is also executed: from $\{T_{lx}, T_{ly}\} = \{0., -2.512\}$ (singular) with $\theta_0 = 1$, the leveled state is approached with no change of θ_0 after the ninth cycle.

(2) Cable Assembly

The three members shown in **Fig.5** are chained into one cable, and so can be dealt with by the force method, directly. But, the displacement method is here applied to the nodal freedom: on each correction of the nodal positions, the members are computed for their tensions. By the use of the strain energy (also given by the second term in (12)) and the self-weight potential in each elastic catenary,[8] we can have the total potential

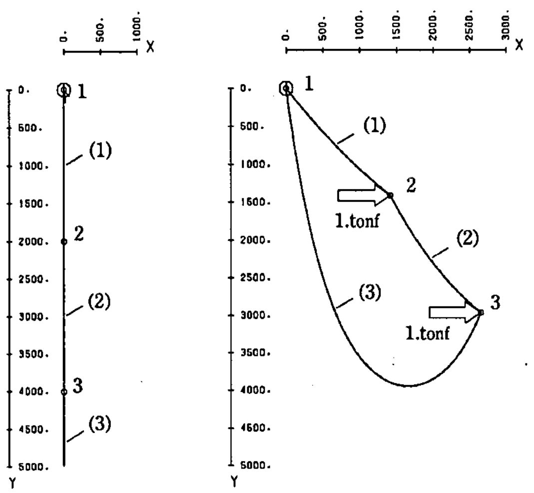

Fig.5 An assembly of three cables

energy. To minimize that potential toward the solution, the fractional correction method is applied to this equilibrium problem also.

The members are assumed to have w = 0.0395 tonf/m, EA = 92000.tonf and natural length $l_{(1),(2),(3)}$ = 20., 20. and 60.m. In the vertical hanging with the self-weight, joint 2 and 3 are located at y_2 = 20.00034 and y_3 = 40.00052 m. In this state, the horizontal stiffnesses at the nodes are extremely small (but not zero). Shown on the right side is another equilibrium for two horizaontal 1.tonf added at the free nodes, in which $\{x, y\}_2 = \{14.12088, 14.10464\}$ and $\{x, y\}_3 = \{26.52301, 29.62051\}$ m. The iteration is executed between those two: in the left $\rightarrow$ ($\leftarrow$) the right, the basic correction factor is changed from 1 into 0.03125 (0.015625), and the other one is approached after 78 (48) cycles. This fractional correction is substantially load-incremental, and so, in the interior iteration on each step, the compatible tensions in the members are obtained after less than ten cycles from their previous values.

6. CONCLUDING REMARKS

The present analysis is enabled by the double iterations: the force-method iteration on each of the catenary cables; and the displacement-method iteration for the global equilibrium. In such a multilayered scheme, the consisting routines are required to leave no uncertainty in their operations. Our numerical examples are involved with the singular cables. If iterated within a regular domain, the fractional correction method yields much fast convergences, and is, of course, applicable to a general convex-potential problem.

REFERENCES

1) Jennings, A.: discussion of "Cable movements under two-dimensional loads" by W.T. O'Brien and A.J. Francis, *J. Str. Div.*, ASCE, Vol.91, No.ST1, pp.307–311, Feb., 1965.
2) Maeda, Y., Hayashi, M. and Maeda, K.: Analytical calculation of sagging cable, *Proc. JSCE*, No.257, Jan., 1977 (in Japanese).
3) Ai, M., Nishioka, T. and Okumura, T.: A theoretical analysis of cable assemblies, *Proc. JSCE*, No.260, pp.17–32, Apr., 1977 (in Japanese).
4) Goto, S.: Tangent stiffness equation of flexible cable and some considerations, *Proc. JSCE*, No.270, pp.41–49, Feb., 1978 (in Japanese).
5) Routh, E.J.: *A Treatise on Analytical Statics*, Cambridge Univ. Press, Cambridge, England, Vol.1, 2nd Edition, 1896.
6) Buchholdt, H.A. and McMillan, B.R.: Iterative methods for the solution of pretensioned cable structures and pinjointed assemblies having significant geometrical displacements, *Proc. IASS Pacific Symposium-Part II*, Tokyo and Kyoto, pp.305–316, 1972.
7) Kar, A.K. and Okazaki, C.Y.: Convergence in highly nonlinear cable net problems, *J. of Str. Div.*, ASCE, Vol.99, No.ST3, pp.321–334, Mar., 1973.
8) Ai, M., Nakano, T. and Masuda, S.: A consideration on the slakened and tightened cables, *Structural Eng./Earthquake Eng.*, JSCE, Vol.11, No.3, pp.141s–144s, Oct., 1994.

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変位法要素としての弾性カテナリー

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構造用ケーブルは，はり部材等と組合わせて用いられるのが普通であり，汎用的なケーブル構造解析としては，やはり，変位法が考えられる．本文では，弾性カテナリーによる変位法ケーブル要素の展開とその適用を示している．その表現は，任意のたわみに対して正確であるが，水平張力成分がなく鉛直に垂れ下がる状態で特異となる．特異性の程度を明らかにしながら，それを除去する数値処理の方法を提案しており，一貫した構造解析を可能としている．弾性カテナリー要素では，両端節点位置に対する張力成分は陽な表現にはない．変位法要素としてその張力成分が，特異状態をも含めて，収束値として常に得られる１つの数値計算法も示している．