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EFFECTIVE COLUMN LENGTHS IN A RECTANGULAR RIGID FRAME*

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There is already presented a quantitative method to estimate the stablitity states of frame members in a global buckling. In that analysis applied to a rectangular rigid frame for the individual members, however, the resulting effective lengths of columns are different from those required in the design scheme. In this study, a rectangular frame is decomposed into those structural units which represent the axial strengths of columns stiffened with their adjacent beams. The effective column lengths are estimated upon the critical axial forces in those respective units. At the same time, to observe their interactions in a global buckling, the preceding analysis for members is generalized to deal with the stability of structural units.

Key Words : *rectangular frame, buckling, effective length, column unit*

1. INTRODUCTION

In the design of framed structures, the axial strengths of members are dealt with by the concept of *effective length.* At the same time, there has been a controversy on how those lengths be determined accurately in an actual structure. [3)–8),10)] In this study, the attention is focussed on *rectangular frames with rigid nodes.* In the existing design codes, [e.g.1),2)] the effects of elastic restraint by the neighboring beams are certainly reflected onto the buckling strengths of column members. But, since the treatments are not based on an exact analysis, their effective lengths could be much conservative.

In a global buckling, recently in Ref.10), the stability states of frame members are quantitatively estimated by the use of their displacement modes and axial forces. The expansions themselves are correct as a stability analysis of discretized structures, and the effective lengths are defined for the individual members. However, in case of a rectangular frame analyzed for the separated column and beam members, the numerical results are much trivial: in a global buckling with vertical loading, the columns are compressed enough into their instable ranges to balance with the stable deformations of beams. On the other hand, what is required in our design procedure is the eventual axial strength of each column member after stiffened with its adjacent beam members.

In this study, instead of the columns and beams, another segmentation is presented in a rectangular frame to estimate the stiffened strengths of columns. First, the column mem-

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bers are classified by their locations and joinings to beam members. Then, it is shown that, of each class, the equal columns and their adjoining beams can be reassembled into another rectangular frame. Since they are joined to each other in the same manner, the resulting frame is uniform in structure. Thus, its buckling under uniform compression in columns is such that one buckling pattern is repeated over all its panels. The critical axial force can be found upon the unit structure cut out from the repeated buckling mode. Different reassemblages stand for the respective kinds of columns. The critical forces in their structural units can be regarded as the columns' own stiffened strengths. On the other hand, a usual rectangular frame is made of the different units. Their coupling states in a global buckling are also observed in this study, by a generalization of the preceding stability analysis of frame members to the structural units.

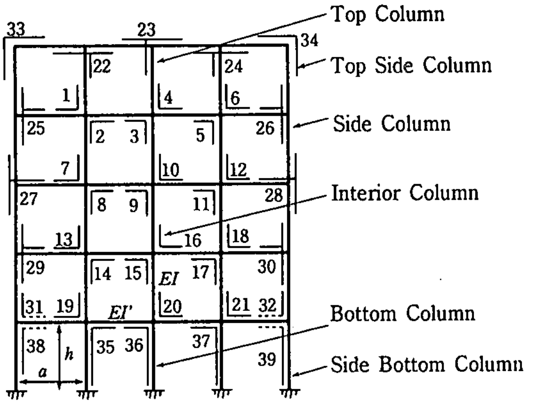


Fig. 1 A simple rectangular rigid frame

2. COLUMN UNITS

Our analysis is developed upon a simple rectangular frame shown in **Fig.1**. As the first case, let the frame be unbraced. Then, first of all, our attention is concentrated upon the interior columns and beams in the intermediate stories. We now want to see their own behaviors in a uniform vertical loading. For that purpose, it is rational to consider such a structure as made of those columns and beams only. As the result of reassembling, we have the infinite frame shown in **Fig.2**. In its sidesway buckling, inflections form at the mid points of members. The subassemlages bounded by those inflection points have the same buckling mode. Further, by the symmetry of the mode with respect to the center node, the buckling of the infinite frame is finally reduced to that of the one-column-and-one-beam structure shown in the figure. Let this substructure be called *unit of interior column*. By the use of the slope-deflection relations in both the simple bending and the beam-column theory, the buckling equation for axial force P is eventually obtained as

$$\beta \tan\beta = 3\kappa \qquad (1\cdot a)$$

$$\beta = \sqrt{\frac{P}{EI}} \cdot \frac{h}{2}, \quad \kappa = \frac{hEI'}{aEI} \qquad (1\cdot b)$$

where a and h are width and height of a panel; and EI and EI' are bending rigidities of column and beam, respectively.

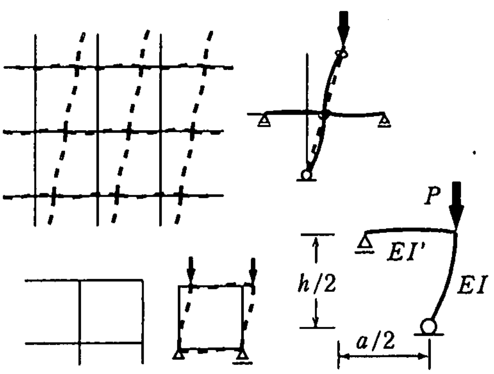


Fig. 2 Unit of interior column

We proceed to the side edge columns in the intermediate stories. By reassembling those side columns and the adjoining beams, we have the vertically infinite ladder-frame shown in **Fig.3**. On the other hand, as for the mid columns in the uppermost story, we have the one-story infinite frame. By the similar treatments in their repeated buckling patterns, we have the *units of side column* and *top column*. Their buckling equations are written as

$$\beta \tan\beta = \frac{3}{2}\kappa \qquad \text{for side column} \qquad (2\cdot a)$$

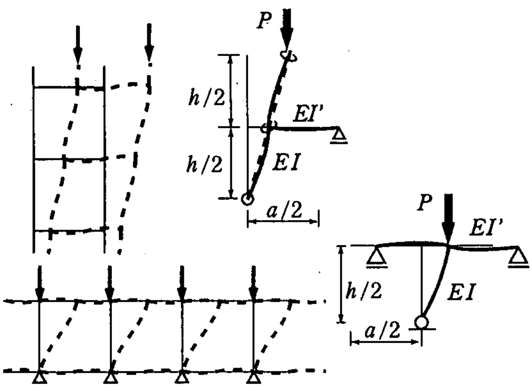


Fig. 3 Units of side column and top column

$$\beta \tan \beta = 6\,\kappa \qquad \text{for top column} \qquad (2\text{-b})$$

As for the side edge columns in the uppermost story, we have the one-panel frame shown in **Fig.2**. This frame has the same buckling with the unit of interior column.

In the bottom story, the columns are supported at their lower ends, with the other ends attached to beams. To extract their essential behaviors in each support condition, again, those columns and beams are reassembled. The two-story frames shown in (a) and (b) of **Fig.4** stand for the mid columns with clamped and hinged supports, respectively. They have the sidesway modes. As for the roller support, the mode shown in (c) has the lowest buckling load. The *units of bottom column with fixed, hinged* and *roller supports* are found in the repeated buckling modes. Their buckling equations are obtained as follows :

$$\beta' \cot \beta' = -6\,\kappa \qquad \text{for fixed support} \qquad (3\text{-a})$$

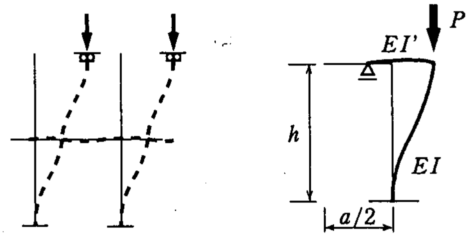
$$\beta' \tan \beta' = 6\,\kappa \qquad \text{for hinged support} \qquad (3\text{-b})$$

$$\beta' \tan \beta' = 2\,\kappa \qquad \text{for roller support} \qquad (3\text{-c})$$

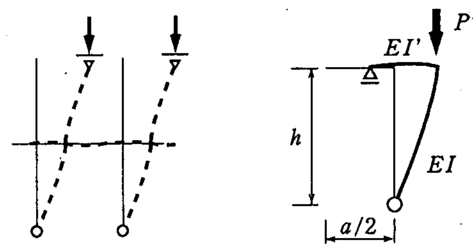
where

$$\beta' = \sqrt{\frac{P}{EI}} \cdot h \qquad (3\text{-d})$$

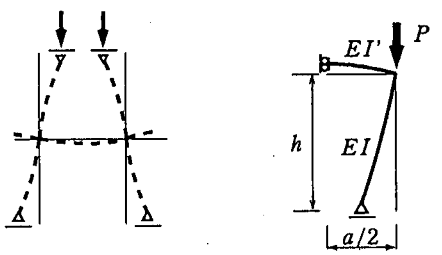
For the side edge columns in the bottom story, the numbers of attached beams are halved from the corresponding units of mid bottom column. In case of a clamped side column, the reassembling is shown in **Fig.5**. The buckling equations for those *units of side bottom column* are written as



(a) fixed support



(b) hinged support



(c) roller support

Fig. 4 Units of bottom column

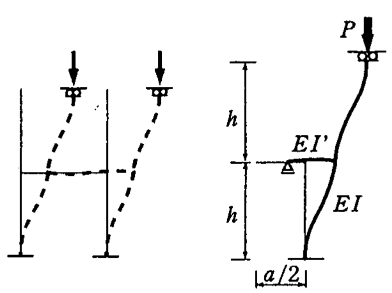


Fig. 5 Unit of side bottom column with a fixed support

$$\beta' \cot \beta' = -3\,\kappa \qquad \text{for fixed support} \qquad (4\text{-a})$$

$$\beta' \tan \beta' = 3\,\kappa \qquad \text{for hinged support} \qquad (4\text{-b})$$

$$\beta' \tan \beta' = \kappa \qquad \text{for roller support} \qquad (4\text{-c})$$

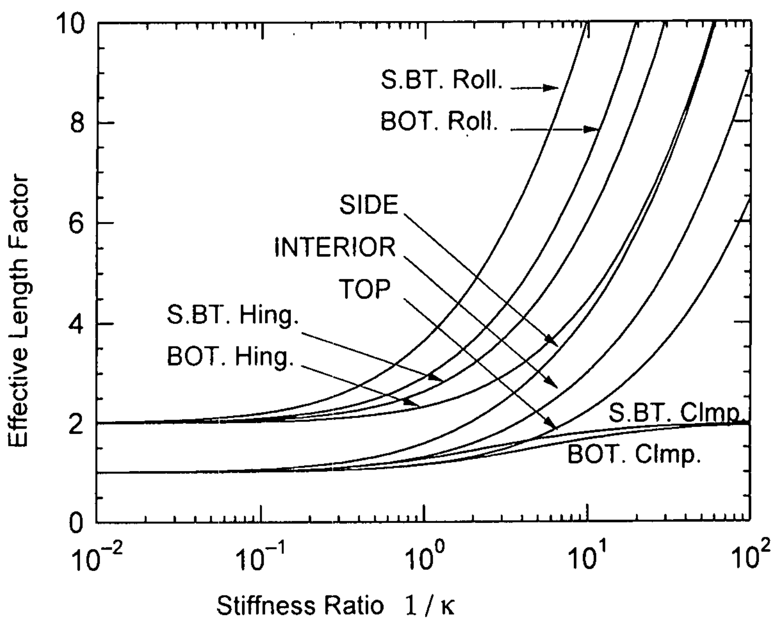


Fig. 6 Effective length factor of column units

The above buckling equations, (1) to (4), are solved numerically for the critical axial forces, say $P_{cr.}$. And, associated with the column in a simple support, the effective lengths are given by

$$l_{\text{eff.}} = \pi \sqrt{\frac{EI}{P_{cr.}}} \qquad (5)$$

For range $10^{-2} \leq 1/\kappa (= aEI/hEI') \leq 10^{2}$, the effective length factors ($K = l_{\text{eff.}}/h$) are numerically obtained as shown in **Fig.6**.

In case the frame is braced, there are obtained also nine column units of the same members with the above unbraced ones, but different supports are placed at the member edges. The actual treatments are given in **APPENDIX–A**.

3. STABILITY OF COLUMN UNITS IN A GLOBAL BUCKLING

A usual rectangular frame is made of the different column units. We next consider the stability states of those units in a global buckling. For simplicity, the expansions are developed upon the linearlized stability theory. Let the frame be subjected to a linear loading :

$$\{ P_N \} = \rho \{ P_N^* \} \qquad (6)$$

where $\{P_N^*\}$ and ρ are a prescribed mode of loading and the load factor. The magnitude of buckling load, ρ^S, and the buckling mode, $\{X_N^S\}$, are determined by the eigenvalue problem :

$$\Big([K_{ON}] + \rho^S [K_{GN}^*] \Big) \Big\{ X_N^S \Big\} = \{ 0 \} \qquad (7)$$

where $[K_{ON}]$ is the initial stiffness matrix; and $[K_{GN}^*]$ is the geometrical stiffness matrix per unit of loading factor ρ. In other words, at this load ρ^S, tangent stiffness $[K_N(\rho)] (= [K_{ON}] + \rho[K_{GN}^*])$ turns from properly positive definite into semipositive definite: its quadratic form has positive values for any other modes, but, for the buckling mode

$$\Gamma = \Big\{ X_N^S \Big\}^T \Big[K_N(\rho^S) \Big] \Big\{ X_N^S \Big\} = 0 \qquad (8)$$

The followings are a summary of the stability analysis of frame members in Ref.10): Since global stiffness $[K_N]$ is a result of superposition of member stiffness matrices $[k]_{(m)}$, the above Γ can be rewritten into the sum of the quadratic forms of members :

$$\Gamma = \sum_{m=1}^{M} I_{(m)} \qquad (9\text{-a})$$

$$I_{(m)} = \Big\{ \boldsymbol{X}^S \Big\}_{(m)}^T \Big[k(\rho^S) \Big]_{(m)} \Big\{ \boldsymbol{X}^S \Big\}_{(m)} \qquad (9\text{-b})$$

where $\{X^S\}_{(m)}$ are the displacement modes of members, collected from $\{X_N^S\}$. The global buckling state ($\Gamma = 0$) usually consists of positive and negative $I_{(m)}$ of members. The sign of $I_{(m)}$ indicates the stability of each member in its mode $\{X^S\}_{(m)}$ with the exerted axial force, say $N_{(m)}^S$. Besides the actual $N_{(m)}^S$ in the global buckling, there can be considered a critical axial force, $N_{(m)}^C$, for each member to make the quadratic work into zero :

$$\left\{X^S\right\}_{(m)}^T \left([k_O] + N^C [\bar{k}_G]\right)_{(m)} \left\{X^S\right\}_{(m)} = 0 \quad (10)$$

where $[k_O]_{(m)}$ is the member initial stiffness matrix; and $[\bar{k}_G]_{(m)}$ is the geometrical stiffness per unit axial compression.

We now generalize the above expansions to a subassemblage of several members, in a frame structure. Let subassemblages be denoted by $[i] = [1], [2] \ldots$, and the consisting members of each $[i]$, by $\{(m)\}_{[i]}$. It is straightforward from expression (9) that the sum of the consisting members' quadratic works

$$\Lambda_{[i]} = \sum_{[i]} \left\{X^S\right\}_{(m)}^T \left[k(\rho^S)\right]_{(m)} \left\{X^S\right\}_{(m)} \quad (11)$$

characterizes the stability of $[i]$, in the global buckling: if $\Lambda_{[i]} > 0 \quad (< 0)$, the subassemblage is stable (instable), acting to restrain (accelerate) the global buckling.

To define the critical state of $[i]$ in the displacement modes $\{X^S\}_{(m)}$, let a main member, say (m°), be selected among $\{(m)\}_{[i]}$. The remainings are denoted by $\{(m')\}$. Then, it is here assumed that the axial forces in $\{(m)\}$ are induced by their relative ratio at the global buckling: with the use of the axial force of (m°) as unknown N°

$$N_{(m')} = \alpha_{(m')} N^\circ , \quad \alpha_{(m')} = \frac{N_{(m')}^S}{N_{(m^\circ)}^S} \quad (12)$$

By introducing this expression into the quadratic work, we have the following buckling equation :

$$\sum_{[i]} \left\{X^S\right\}_{(m)}^T \left([k_O] + N^{\circ C} \alpha [\bar{k}_G]\right)_{(m)} \left\{X^S\right\}_{(m)} = 0 \quad (13)$$

Since modes $\{X^S\}_{(m)}$ and ratios $\alpha_{(m)}$ have been known, the critical axial force is given by

$$N^{\circ C} = - \frac{\sum \left\{X^S\right\}_{(m)}^T [k_O]_{(m)} \left\{X^S\right\}_{(m)}}{\sum \alpha_{(m)} \left\{X^S\right\}_{(m)}^T [\bar{k}_G]_{(m)} \left\{X^S\right\}_{(m)}} \quad (14)$$

In the global buckling, the members are regulated to move into $\{X^S\}_{(m)}$. $N^{\circ C}$ is a critical force of each subassembly in that displacement modes. With regard to the cross section of main member (m°), this axial force is now converted into the effective length :

$$l_{\mathrm{eff.}[i]}^{\circ C} = \pi \sqrt{\left(\frac{EI}{N^C}\right)_{(m^\circ)}} \quad (15)$$

When the actual $N_{[i]}^{\circ S}$ are plotted on those $l_{\mathrm{eff.}[i]}^{\circ C}$, correspondingly to the former sayings for $\Lambda_{[i]} > 0$ or < 0, their states subjected to $N_{[i]}^{\circ S}$ are separated by the Euler curve into stable or instable. Through the above procedure, the former column units can be examined for their stability states in a global buckling.

4. NUMERICAL EXAMPLES

In **Sec.3.**, for simplicity, the stability relations are described upon the linearlized stiffness matrices after subjected to the axial forces. In this section, for accuracy, the numerical analyses are carried out similarly to Ref.10): in the method of separation-into-rigid-displacement-and-deformation, 9) where the nonlinear terms of deformation parameters are taken up to their third order.

(1) In case of unbraced

The rectangular frame shown in **Fig.1** is numerically analyzed for $a = h = 500.$cm, $E = 2100.$tonf/cm^2, $I = I' = 3650.$cm^4, cross-section area $A = A' = 101.5\,$cm^2 and yield stress $\sigma_Y = 2.4\,$tonf/cm^2. The lowermost side-edge beams (indicated by dotted line) are, to be noted, taken twice into the column units, and so their quadratic works are each halved into the relevant two units. In vertical P applied equally at the uppermost nodes, the global buckling is determined at $P = 158.76\,$tonf. The quadratic works of the column units are distributed as shown in **Fig.7**, in which the solid-lined and broken-lined widths indicate magnitudes of the negative and positive works (instable and stable), respectively.

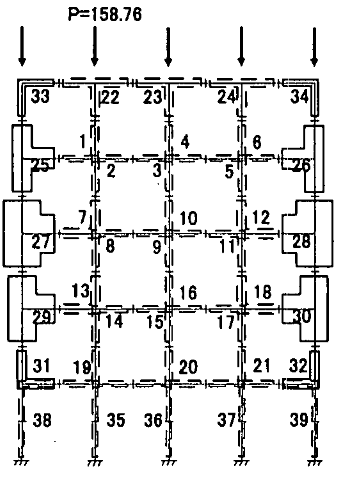


Fig. 7 Quadratic work of units in Example (1)

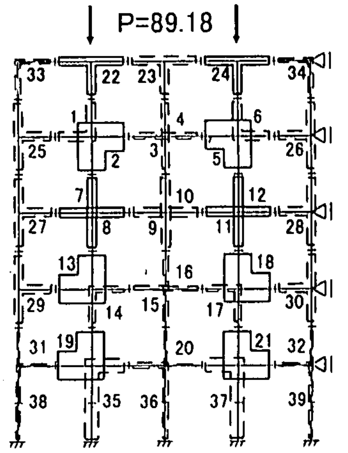


Fig. 9 Quadratic work of units in Example (2)

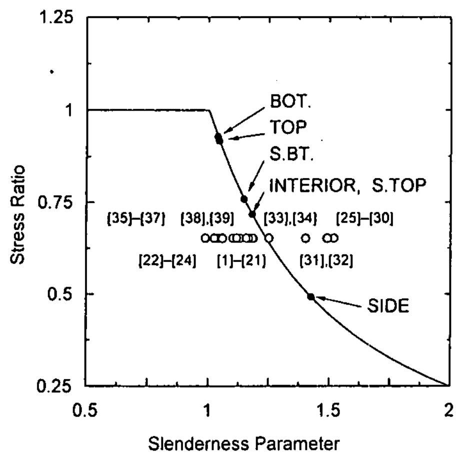


Fig. 8 Stability state of units in Example (1)

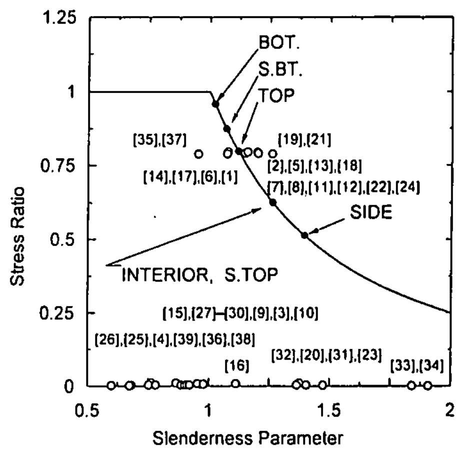


Fig. 10 Stability state of units in Example (2)

At the same time, their effective lengths are derived by the use of Eqs.(14) and (15). Their stability states are shown with circle points in **Fig.8**, in which the bullet points on the Euler curve indicate the units' own critical states determined by the buckling equations in **Sec.2.**

(2) In case of braced

Next, the former rectangular frame is braced. After the cross sections are changed to $I = I' = 508.\mathrm{cm}^4$ and $A = A' = 46.78\,\mathrm{cm}^2$, vertical P are applied to the two top nodes shown in **Fig.9**. In this loading, the global buckling is determined at $P = 89.18\,\mathrm{tonf}$. The quadratic works of units are shown in the figure. And, their axial force–effective length states are plotted in **Fig.10**.

5. CONCLUDING REMARKS

The global buckling is a result of interaction between the stable and instable members. The present study is preceded by an estimation method of those member stability states. [10] An expected contribution of those analyses is to give a rational proposal on the design strengths of members. The preceding method can deal with a rectangular frame also for the separated columns and beams. But, the treatments of member axial strengths are different from those in the design procedure. In this study, the column units are employed as an alternative decomposition to estimate the axial strengths of the column members stiffened with their adjacent beams. For the following two, the column units can be called "unit": a usual rectangular frame can be assembled by

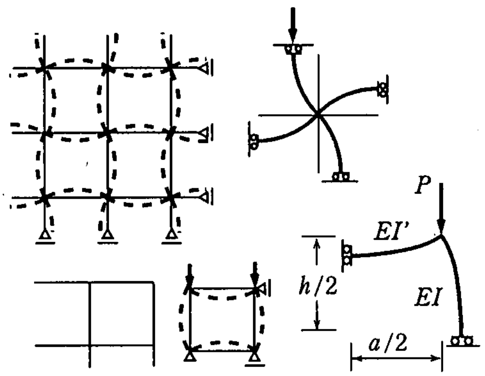


Fig. 11 Unit of interior column / braced

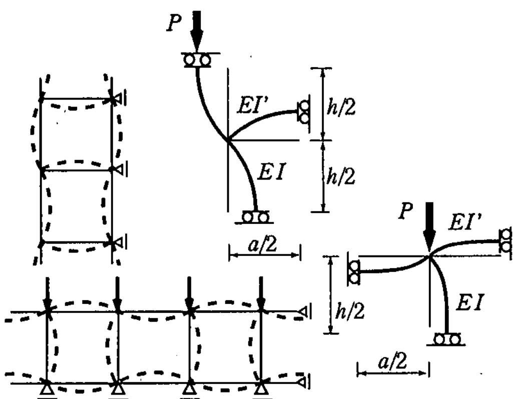
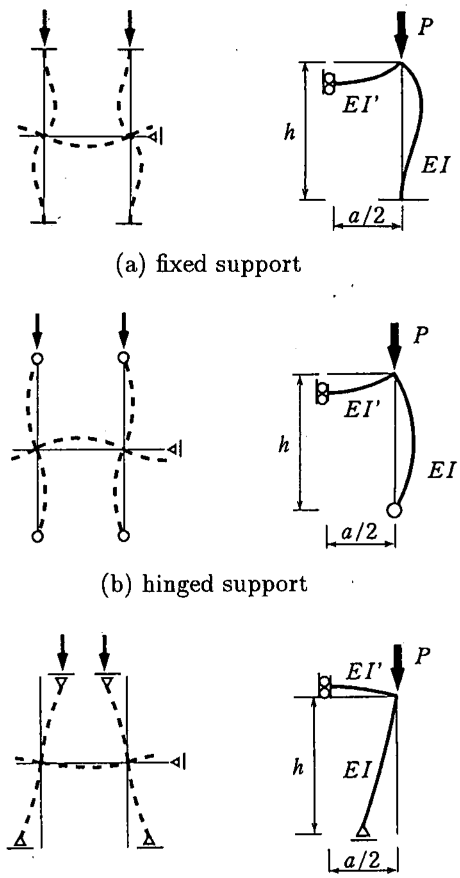


Fig. 12 Units of side column and top column / braced



(a) fixed support

(b) hinged support

(c) roller support

Fig. 13 Units of bottom column / braced

the use of those different units; and, for each class, there exists a uniform rectangular frame consisting of the same units only. The column units have each their own critical axial strengths. The global buckling of an actual rectangular frame is a result of interaction between those different units in various stability states.

In **Fig.8**, the units' own critical states are plotted on the Euler curve. Their strengths are ordered: side column → interior/top-side columns → bottom-side column → top column → bottom column. Under the uniform loading, the global-buckling load is placed at an averaged value of those units' strengths, or might be between the units of side column and interior column. On that level of axial force, to be noted, the effective lengths of the 39 units estimated in the global buckling are lying in the same order to their own strengths. On the other hand, **Fig.9** and **10** are a result in a partial loading. The distribution of the quadratic works seems natural for that loading, but their scattered plots in **Fig.10** are difficult to be understood for their interactions.

APPENDIX–A. COLUMN UNITS IN A BRACED RECTANGULAR FRAME

Suppose the frame shown in **Fig.1** is braced. The interior columns and beams are reassembled into the infinite structure shown in **Fig.11**. By the symmetries of its buckling mode with respect to both the vertical and horizontal lines passing through mid points of members, we have the *unit of interior column* shown in the figure. The buck-

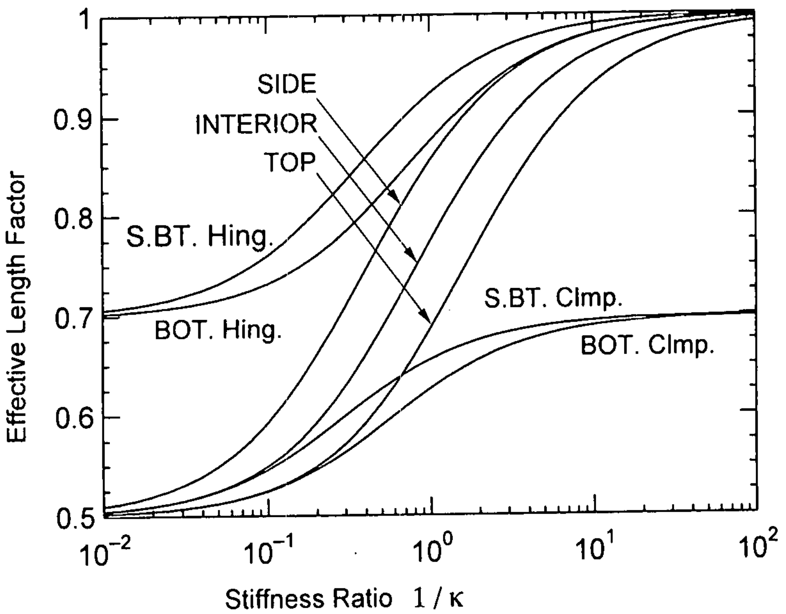


Fig. 14 Effective length factor of column units / braced

ling equation is eventually obtained as

$$\beta \cot \beta = -\kappa \qquad (16)$$

Also for the remaining columns and beams differently placed in the frame, eventually, we have those structural units consisting of the same members with the unbraced units in **Sec.2**. But, since their sidesways are constrained, the member-edge conditions are changed to represent the axially symmetric modes. Those units are shown in **Fig.12** and **Fig.13**. In the below, their buckling equations are written.

Side column : $\beta \cot \beta = -\frac{\kappa}{2}$ (17·a)

Top column : $\beta \cot \beta = -2\kappa$ (17·b)

Bottom column :

$$(\beta' \sin \beta' - \beta'^2 \cos \beta') + 2\kappa (2 - 2\cos \beta' - \beta' \sin \beta') = 0 \quad \text{for fixed support} \quad (18\cdot a)$$

$$\beta'^2 \sin \beta' + 2\kappa(\sin \beta' - \beta' \cos \beta') = 0 \quad \text{for hinged support} \quad (18\cdot b)$$

$$\beta' \tan \beta' = 2\kappa \quad \text{for roller support} \quad (18\cdot c)$$

Side bottom column :

$$(\beta' \sin \beta' - \beta'^2 \cos \beta') + \kappa (2 - 2\cos \beta' - \beta' \sin \beta') = 0 \quad \text{for fixed support} \quad (19\cdot a)$$

$$\beta'^2 \sin \beta' + \kappa(\sin \beta' - \beta' \cos \beta') = 0 \quad \text{for hinged support} \quad (19\cdot b)$$

$$\beta' \tan \beta' = \kappa \quad \text{for roller support} \quad (19\cdot c)$$

The effective length factors for the above column units are numrically obtained as shown in **Fig.14**. The curves for the bottom columns with roller support are given in **Fig.6** : Eqs.(18·c) and (19·c) are the same to the unbraced (3·c) and (4·c).

REFERENCES

1) Japan Road Association: *Specifications for highway bridges, Part II, Steel Bridges*, Maruzen, Feb., 1990 (in Japanese).

2) AISC: *Load and resistance factor design, 2nd edition*, Manual of steel construction, American Institute of Steel Construction, Chicago, 1994.

3) Galambos, T.V.: *Structural Members and Frames*, Englewood Cliffs, N.J., 1968.

4) Yura, J.A.: The effective length of columns in unbraced frames, *AISC Engineering Journal*, Vol.8, No.2, pp.37–42, Apr., 1971.

5) Nishino, F., Miki, C. and Suzuki, A: Background of revision of specifications for highway bridges (Japan road association) – design of rigid frames, *Bridge*

and Foundation Engineering, Vol.15, No.10, pp.10–13, 1981–10 (in Japanese).

6) Kuranishi, S. (ed.): *Ultimate strength and design of steel structures*, JSCE, pp.130–133, 1994 (in Japanese).

7) Nishino, F. and Attia, W.: A proposal for in-plane stability design of steel framed structures, *Structural Eng./Earthquake Eng.*, Vol.8, No.4, pp.169s–178s, Jan., 1992.

8) Nogami, K. and Yamamoto, K.: On the effective buckling length of framed columns using eigenvalue analysis *J. Struct. Mech. Earthquake Eng.*, JSCE, No.489/I-27, pp.157-166, 1994–4 (in Japanese).

9) Ai, M. and Nishino, F.: Mechanics in geometrically nonlinear problem of discrete system and application to plane frame-works, *Proceedings of JSCE*, No.304, pp.17–32, 1980–12 (in Japanese).

10) Nishino, F., Ai, M. and Nakano, T.: On the stability of frame members in a global buckling, *Structural Eng./Earthquake Eng.*, JSCE, Vol.14, No.2, pp.175s–184s, Oct., 1997.

11) Tamura, K., Miyake, A., Ai, M. and Nishino, F.: Buckling units in a braced rectangular frame, *Proc. of the Annual Conference*, JSCE, Vol.I(A), pp.478–479, 1998–10 (in Japanese).

12) Ohsumi, Y., Imai, H., Ai, M. and Nishino, F.: Effective column lengths in an unbraced rectangular frame, *Proc. of the Annual Conference*, JSCE, Vol.I(A), pp.480–481, 1998–10 (in Japanese).

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矩形ラーメン柱の有効座屈長

西野 文雄・阿井 正博・田村 健太郎・今井 裕敬

矩形ラーメン柱の有効座屈長が，隣接するはり部材の弾性拘束をも含めた座屈挙動によって考えられるべきことは周知の処であるが，理論的に確立された具体的な手法についてこれまで必ずしも明確ではない．本文では，各種類毎の柱とそれに接続するはり部材に注目したとき，それらの部材のみより成る一様なラーメン構造が存在し，その座屈モードの中に構造単位が見出せることを最初に示している．それらの構造単位の座屈軸力は，はりによって弾性拘束された後の各柱の軸力強度であり，目的の有効長に変換することができる．一般の矩形ラーメンは異る構造単位を組み合わせた構造であるが，全体座屈時でのそれらの連成状態を評価することも行っている．