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# BASIC THREE-DIMENSIONAL THEORY OF LARGE DEFORMATION AND ITS APPLICATION TO THE TREATMENT OF MOIRÉ FRINGES

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## ABSTRACT

The treatment and the theory appeared in the past papers on Moiré method which uses stripping film sticked on a surface of two-dimensional model were not necessarily correct. The correct and universal treatment of Moiré method are mentioned on the basis of precise three-dimensional theory of stress-strain relation of large deformation.

## LIST OF SYMBOLS

$\boldsymbol{D}$ = displacement vector of a point in an elastic body

$E$ = Young's modulus

$(e_{XX}, e_{XY}, e_{XZ}, \cdots)$ = strain tensor

$(e_1, e_2, e_3)=(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ = Principal form of a strain-tensor

$p$ = pitch of master grid (undeformed grid)

subscript, $i$ or $f$ = denotes initial or final length before or after deformation respectively.

subscript, 0 = denotes a value before deformation.

subscript, 1 or 2 = denotes a value after deformation.

superscript, $E$ or $L$ = denotes a value by Eulerian or Lagrangian description respectively.

$U$, $V$, $W$ = resultant displacements in $x$, $y$ and $z$ directions respectively

$u$, $v$, $w$ = displacements in $x$, $y$ and $z$ directions respectively

$\alpha$ = angle between $x$-axis and a direction of principal strain ($\alpha<90°$)

$\gamma_{xy}$ = shear strain for the right angle between $x$ and $y$ directions

$\delta_x$, $\delta_y$ = Moiré fringe intervals measured along $x$ and $y$ directions respectively

$\varepsilon$ = longitudinal strain of a line segment

$\varepsilon_1$, $\varepsilon_2$ or $\varepsilon_3$ = principal strain respectively

$\overline{\varepsilon_1{}^E}=\overline{e_{XX}{}^E}$ or $\bar{u}_1$ = uniform virtual strain or virtual displacement caused by mismatching of grids in a direction of principal section respectively

$\nu$ = Poisson's ratio

$\xi$, $\eta$, $\zeta$ = three orthogonal directions of principal strains

$\sigma_1$, $\sigma_2$ or $\sigma_3$ = principal stress respectively

$\varphi$ = angle of rigid body rotation

## I. INTRODUCTION AND THE GENERAL REPRESENTATION OF TRUE STRAIN

It is the aim of this paper to develop the basic static theory of large deformation and also to mention as one of its application the correct method for the treatment of Moiré fringes.

Up today it was generally common that we had dealt with the errors of mismatching and misalignment of Moiré experiments according to graphic data obtained only by experimental calibration having almost no theoretical basis, and moreover that we could not know beforehand, for instance, the rigid body rotation angle (both their uniform and partial values) necessary to apply the graphic data in actual case.

On the other hand, it is also by all means necessary in finite elastic theory that we should discriminate clearly the absolute coordinate from the intrinsic coordinate (convected coordinate) and should not mix up or identify the two coordinate systems.

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### (1) The general representation of true-strain

(a) *The general representation of true-strain by Lagrangian description.*

The longitudinal strain in Lagrangian description is

$$\varepsilon^L = \frac{l_f - l_i}{l_i} \quad \cdots\cdots(1)$$

We can see from Fig. 1,

$$\left.\begin{aligned}
\varepsilon_x{}^L &= \sqrt{\left(1+\frac{\partial u_0}{\partial x_0}\right)^2+\left(\frac{\partial v_0}{\partial x_0}\right)^2+\left(\frac{\partial w_0}{\partial x_0}\right)^2}-1\\
\varepsilon_y{}^L &= \sqrt{\left(\frac{\partial u_0}{\partial y_0}\right)^2+\left(1+\frac{\partial v_0}{\partial y_0}\right)^2+\left(\frac{\partial w_0}{\partial y_0}\right)^2}-1\\
\varepsilon_z{}^L &= \sqrt{\left(\frac{\partial u_0}{\partial z_0}\right)^2+\left(\frac{\partial v_0}{\partial z_0}\right)^2+\left(1+\frac{\partial w_0}{\partial z_0}\right)^2}-1
\end{aligned}\right\} \quad \cdots\cdots(2)$$

or

$$\left.\begin{aligned}
\varepsilon_x{}^L &= \left(\frac{\partial u_0}{\partial x_0}\right)+\frac{1}{2}\left\{\left(\frac{\partial v_0}{\partial x_0}\right)^2+\left(\frac{\partial w_0}{\partial x_0}\right)^2\right\}\\
&\quad -\frac{1}{2}\left(\frac{\partial u_0}{\partial x_0}\right)\left\{\left(\frac{\partial v_0}{\partial x_0}\right)^2+\left(\frac{\partial w_0}{\partial x_0}\right)^2\right\}+\cdots\\
\varepsilon_y{}^L &= \left(\frac{\partial v_0}{\partial y_0}\right)+\frac{1}{2}\left\{\left(\frac{\partial w_0}{\partial y_0}\right)^2+\left(\frac{\partial u_0}{\partial y_0}\right)^2\right\}\\
&\quad -\frac{1}{2}\left(\frac{\partial v_0}{\partial y_0}\right)\left\{\left(\frac{\partial w_0}{\partial y_0}\right)^2+\left(\frac{\partial u_0}{\partial y_0}\right)^2\right\}+\cdots\\
\varepsilon_z{}^L &= \left(\frac{\partial w_0}{\partial z_0}\right)+\frac{1}{2}\left\{\left(\frac{\partial u_0}{\partial z_0}\right)^2+\left(\frac{\partial v_0}{\partial z_0}\right)^2\right\}\\
&\quad -\frac{1}{2}\left(\frac{\partial w}{\partial z_0}\right)\left\{\left(\frac{\partial u_0}{\partial z_0}\right)^2+\left(\frac{\partial v_0}{\partial z_0}\right)^2\right\}+\cdots
\end{aligned}\right\} \quad \cdots\cdots(3)$$

On the other hand, we can see from Fig. 2,

$$\gamma_{xy}{}^L = \theta_{XX}{}^L + \theta_{YY}{}^L = 90° - \angle ABC \quad \cdots\cdots(4)$$

$$\begin{aligned}
\therefore\ \sin\gamma_{xy}{}^L &= \sin(90° - \angle ABC) = \cos\angle ABC\\
&= \cos(\widehat{l_f, x})\cos(\widehat{m_f, x})\\
&\quad + \cos(\widehat{l_f, y})\cos(\widehat{m_f, y})\\
&\quad + \cos(\widehat{l_f, z})\cos(\widehat{m_f, z})
\end{aligned}$$

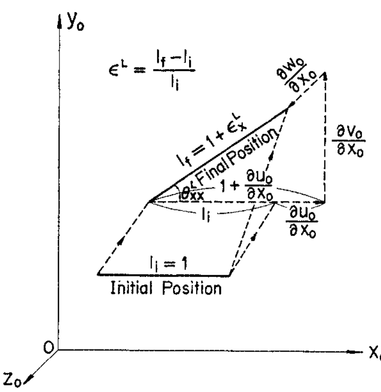


**Fig. 1** Lagrangian description of logitudinal strain.

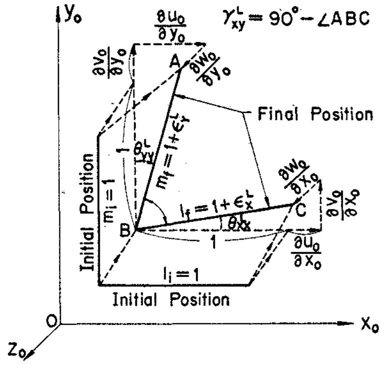


**Fig. 2** Lagrangian description of shear strain.

$$\left.\begin{aligned}
&= \frac{\left(1+\frac{\partial u_0}{\partial x_0}\right)\left(\frac{\partial u_0}{\partial y_0}\right)}{(1+\varepsilon_x{}^L)(1+\varepsilon_y{}^L)}
+ \frac{\frac{\partial v_0}{\partial x_0}\left(1+\frac{\partial v_0}{\partial y_0}\right)}{(1+\varepsilon_x{}^L)(1+\varepsilon_y{}^L)}
+ \frac{\left(\frac{\partial w_0}{\partial x_0}\right)\left(\frac{\partial w_0}{\partial y_0}\right)}{(1+\varepsilon_x{}^L)(1+\varepsilon_y{}^L)}\\
\therefore\ \sin\gamma_{xy}{}^L &= \frac{\left(\frac{\partial v_0}{\partial x_0}+\frac{\partial u_0}{\partial y_0}\right)+\left(\frac{\partial u_0}{\partial x_0}\right)\left(\frac{\partial u_0}{\partial y_0}\right)+\left(\frac{\partial v_0}{\partial x_0}\right)\left(\frac{\partial v_0}{\partial y_0}\right)+\left(\frac{\partial w_0}{\partial x_0}\right)\left(\frac{\partial w_0}{\partial y_0}\right)}{(1+\varepsilon_x{}^L)\cdot(1+\varepsilon_y{}^L)}\\
\sin\gamma_{yz}{}^L &= \frac{\left(\frac{\partial w_0}{\partial y_0}+\frac{\partial v_0}{\partial z_0}\right)+\left(\frac{\partial u_0}{\partial y_0}\right)\left(\frac{\partial u_0}{\partial z_0}\right)+\left(\frac{\partial v_0}{\partial y_0}\right)\left(\frac{\partial v_0}{\partial z_0}\right)+\left(\frac{\partial w_0}{\partial y_0}\right)\left(\frac{\partial w_0}{\partial z_0}\right)}{(1+\varepsilon_y{}^L)\cdot(1+\varepsilon_z{}^L)}\\
\therefore\ \sin\gamma_{zx}{}^L &= \frac{\left(\frac{\partial u_0}{\partial z_0}+\frac{\partial w_0}{\partial x_0}\right)+\left(\frac{\partial u_0}{\partial z_0}\right)\left(\frac{\partial u_0}{\partial x_0}\right)+\left(\frac{\partial v_0}{\partial z_0}\right)\left(\frac{\partial v_0}{\partial x_0}\right)+\left(\frac{\partial w_0}{\partial z_0}\right)\left(\frac{\partial w_0}{\partial x_0}\right)}{(1+\varepsilon_z{}^L)\cdot(1+\varepsilon_x{}^L)}
\end{aligned}\right\} \quad \cdots\cdots(5)$$

(b) *The general representation of true-strain by Eulerian description.*

The longitudinal strain in Eulerian description is

$$\varepsilon^E = \frac{l_f - l_i}{l_f} \quad \cdots\cdots(6)$$

From Fig. 3, we can see

$$\left.\begin{aligned}
\varepsilon_x{}^E &= 1-\sqrt{\left(1-\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial v_1}{\partial x_1}\right)^2+\left(\frac{\partial w_1}{\partial x_1}\right)^2}\\
\varepsilon_y{}^E &= 1-\sqrt{\left(\frac{\partial u_1}{\partial y_1}\right)^2+\left(1-\frac{\partial v_1}{\partial y_1}\right)^2+\left(\frac{\partial w_1}{\partial y_1}\right)^2}\\
\varepsilon_z{}^E &= 1-\sqrt{\left(\frac{\partial u_1}{\partial z_1}\right)^2+\left(\frac{\partial v_1}{\partial z_1}\right)^2+\left(1-\frac{\partial w_1}{\partial z_1}\right)^2}
\end{aligned}\right\} \quad \cdots\cdots(7)$$

or

$$\begin{aligned}
\varepsilon_x{}^E &= \frac{\partial u_1}{\partial x_1}-\frac{1}{2}\left\{\left(\frac{\partial v_1}{\partial x_1}\right)^2+\left(\frac{\partial w_1}{\partial x_1}\right)^2\right\}\\
&\quad -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)\left\{\left(\frac{\partial v_1}{\partial x_1}\right)^2+\left(\frac{\partial w_1}{\partial x}\right)^2\right\}+\cdots
\end{aligned}$$

$$\left.\begin{aligned}\varepsilon_y{}^E&=\frac{\partial v_1}{\partial y_1}-\frac{1}{2}\left\{\left(\frac{\partial w_1}{\partial y_1}\right)^2+\left(\frac{\partial u_1}{\partial y_1}\right)^2\right\}\\&\quad-\frac{1}{2}\left(\frac{\partial v_1}{\partial y_1}\right)\left\{\left(\frac{\partial w_1}{\partial y_1}\right)^2+\left(\frac{\partial u_1}{\partial y_1}\right)^2\right\}+\cdots\\\varepsilon_z{}^E&=\frac{\partial w_1}{\partial z_1}-\frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial z_1}\right)^2+\left(\frac{\partial v_1}{\partial z_1}\right)^2\right\}\\&\quad-\frac{1}{2}\left(\frac{\partial w_1}{\partial z_1}\right)\left\{\left(\frac{\partial u_1}{\partial z_1}\right)^2+\left(\frac{\partial v_1}{\partial z_1}\right)^2\right\}+\cdots\end{aligned}\right\}\qquad\cdots\cdots(8)$$

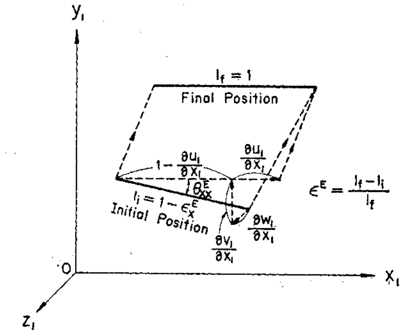


**Fig. 3** Eulerian description of longitudinal strain.

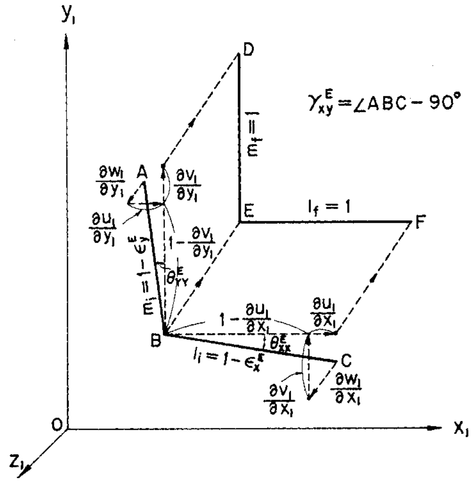


**Fig. 4** Eulerian description of shear strain.

Further we can see from Fig. 4,

$$\begin{aligned}\gamma_{xy}{}^E&=\sin(\theta_{XX}{}^E+\theta_{YY}{}^E)=\sin(\angle\mathrm{ABC}-90°)\\&=-\cos\angle\mathrm{ABC}\\&=-\{\cos(\widehat{l_i,x})\cos(\widehat{m_i,x})\\&\quad+\cos(\widehat{l_i,y})\cos(\widehat{m_i,y})+\cos(\widehat{l_i,z})\cos(\widehat{m_i,z})\}\\&=-\left\{\frac{\left(1-\frac{\partial u_1}{\partial x_1}\right)}{1-\varepsilon_x{}^E}\cdot\frac{\left(-\frac{\partial u_1}{\partial y_1}\right)}{1-\varepsilon_y{}^E}+\frac{\left(-\frac{\partial v_1}{\partial x_1}\right)}{1-\varepsilon_x{}^E}\right.\\&\quad\left.\cdot\frac{\left(1-\frac{\partial v_1}{\partial y_1}\right)}{1-\varepsilon_y{}^E}+\frac{\left(-\frac{\partial w_1}{\partial x_1}\right)}{1-\varepsilon_x{}^E}\cdot\frac{\left(-\frac{\partial w_1}{\partial y_1}\right)}{1-\varepsilon_y{}^E}\right\}\end{aligned}$$

$$\left.\begin{aligned}\therefore\ \sin\gamma_{xy}{}^E&=\frac{\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial u_1}{\partial y_1}\right)-\left(\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial y_1}\right)-\left(\frac{\partial v_1}{\partial x_1}\right)\left(\frac{\partial v_1}{\partial y_1}\right)-\left(\frac{\partial w_1}{\partial x_1}\right)\left(\frac{\partial w_1}{\partial y_1}\right)}{(1-\varepsilon_x{}^E)\cdot(1-\varepsilon_y{}^E)}\\\sin\gamma_{yz}{}^E&=\frac{\left(\frac{\partial w_1}{\partial y_1}+\frac{\partial v_1}{\partial z_1}\right)-\left(\frac{\partial u_1}{\partial y_1}\right)\left(\frac{\partial u_1}{\partial z_1}\right)-\left(\frac{\partial v_1}{\partial y_1}\right)\left(\frac{\partial v_1}{\partial z_1}\right)-\left(\frac{\partial w_1}{\partial y_1}\right)\left(\frac{\partial w_1}{\partial z_1}\right)}{(1-\varepsilon_y{}^E)\cdot(1-\varepsilon_z{}^E)}\\\therefore\ \sin\gamma_{zx}{}^E&=\frac{\left(\frac{\partial u_1}{\partial z_1}+\frac{\partial w_1}{\partial x_1}\right)-\left(\frac{\partial u_1}{\partial z_1}\right)\left(\frac{\partial u_1}{\partial x_1}\right)-\left(\frac{\partial v_1}{\partial z_1}\right)\left(\frac{\partial v_1}{\partial x_1}\right)-\left(\frac{\partial w_1}{\partial z_1}\right)\left(\frac{\partial w_1}{\partial x_1}\right)}{(1-\varepsilon_z{}^E)\cdot(1-\varepsilon_x{}^E)}\end{aligned}\right\}\qquad\cdots\cdots(9)$$

Eqs. (2)～(9) are the equations which hold strictly in any conditions of strain accompanied by rotation (uniform and partial) and also even in the case when the amounts of rotation and strains are not small.

The above theory was developed originally I. S. Sokolnikoff in 1946 and it is based on absolute coordinate, but it is not the theory which states the relation between the partial derivatives of displacement and strain tensor.

On the other hand, the theory of traditional elasticity states the relation between the partial derivatives of displacement and strain tensor, but it is the theory based on intrinsic coordinate and it is only true when the strain is infinitesimal.

Then it is necessary to develop the universal theory about the relation between the partial derivatives of displacement and strain tensor, which is based on absolute coordinate and holds strictly also in any finite deformation.

The author has considered that it is not suitable in a universal theory, which is based on absolute coordinate and holds in any finite deformation, to define strain tensor elements as

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

as traditional elasticity; and he has defined strain tensor elements as partial derivatives of displacement in the case when rigid body rotation angle is equal to zero, i.e.,

$$e_{ij}=\frac{\partial u_i}{\partial x_j}=\frac{\partial u_j}{\partial x_i},$$

and for the general case of partial derivatives of displacement when rigid body rotation angle is not equal to zero, he has introduced another vector quantity of rigid body rotation.

## II. THE REPRESENTATION OF PARTIAL DERIVATIVES OF DISPLACEMENT *u*, *v* or *w* BY STRAIN TENSOR AND RIGID BODY ROTATION

### (1) Relation between rigid body rotation and rotation angle.

If we assume any fixed coordinate axes $x_0$, $y_0$ and $z_0$ in a vector field of displacement $\boldsymbol{D}^L=(iu_0+jv_0+kw_0)$, then

$$\text{rot. } \boldsymbol{D}^L=\left(i\frac{\partial}{\partial x_0}+j\frac{\partial}{\partial y_0}+k\frac{\partial}{\partial z_0}\right)\times(iu_0+jv_0+k\,w_0)$$

$$=\left\{k\left(\frac{\partial v_0}{\partial x_0}-\frac{\partial u_0}{\partial y_0}\right)+i\left(\frac{\partial w_0}{\partial y_0}-\frac{\partial v_0}{\partial z_0}\right)+j\left(\frac{\partial u_0}{\partial z_0}-\frac{\partial w_0}{\partial x_0}\right)\right\}$$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_0} & \frac{\partial}{\partial y_0} & \frac{\partial}{\partial z_0} \\ u_0 & v_0 & w_0 \end{vmatrix} \quad\cdots\cdots(10)$$

where $i$, $j$ and $k$ denote the base unit-vectors in $x_0$, $y_0$ and $z_0$ directions respectively.

In a special case of rigid body rotation, we can see from Fig. 5 that the following relation exists.

$$\frac{\partial V_0}{\partial X_0}=-\frac{\partial U_0}{\partial Y_0}=\sin\varphi$$

$$\therefore\ \frac{\partial V_0}{\partial X_0}-\frac{\partial U_0}{\partial Y_0}=2\sin\varphi$$

and

$$\frac{\partial W_0}{\partial Y_0}=0,\ \frac{\partial V_0}{\partial Z_0}=0,\ \frac{\partial U_0}{\partial Z_0}=0,\ \frac{\partial W_0}{\partial X_0}=0.$$

$$\therefore\ |\text{rot. } \boldsymbol{D}^L|=\sqrt{\left(\frac{\partial V_0}{\partial X_0}-\frac{\partial U_0}{\partial Y_0}\right)^{2*}+\left(\frac{\partial W_0}{\partial Y_0}-\frac{\partial V_0}{\partial Z_0}\right)^2+\left(\frac{\partial U_0}{\partial Z_0}-\frac{\partial W_0}{\partial X_0}\right)^2}$$

$$=|2\sin\varphi|$$

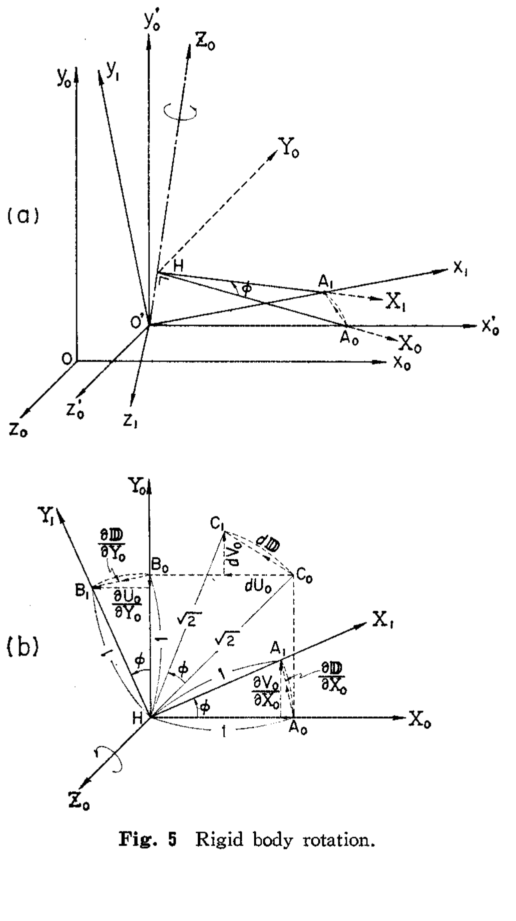


**Fig. 5** Rigid body rotation.

where $\varphi$ denote the rigid body rotation angle.

It is already known in vector analysis that the quantity $|\text{rot. } \boldsymbol{D}^L|$ is independent to the selection of coordinate axes, then in the case of rigid body rotation the following relation must be generally true for any coordinate axes $x_0$, $y_0$ and $z_0$.

$$|\text{rot. } \boldsymbol{D}^L|=\sqrt{\left(\frac{\partial v_0}{\partial x_0}-\frac{\partial u_0}{\partial y_0}\right)^2+\left(\frac{\partial w_0}{\partial y_0}-\frac{\partial v_0}{\partial z_0}\right)^{2*}+\left(\frac{\partial u_0}{\partial z_0}-\frac{\partial w_0}{\partial x_0}\right)^2}$$

$$=|2\sin\varphi|$$

And in this case the vector quantity, rot. $\boldsymbol{D}^L$, has a direction of rigid body rotation axis.

### (2) The representation of partial derivatives of displacement by strain-tensor when rigid body rotation angle $\varphi=0$.

(a) *The case of Lagrangian description.*

We call a set of derivatives $\left(\frac{\partial u_0}{\partial x_0},\frac{\partial u_0}{\partial y_0},\frac{\partial u_0}{\partial z_0},\cdots\right)$ in the case of $\varphi=0$, as a strain-tensor in Lagrangian description $(e_{XX}{}^L, e_{XY}{}^L, e_{XZ}{}^L, \cdots)$.

When rigid body rotation angle $\varphi=0$, the direction of principal strain before deformation remains unchanged also after deformation. Then

$$\begin{bmatrix} \frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial y_0} & \frac{\partial u_0}{\partial z_0} \\ \frac{\partial v_0}{\partial x_0} & \frac{\partial v_0}{\partial y_0} & \frac{\partial v_0}{\partial z_0} \\ \frac{\partial w_0}{\partial x_0} & \frac{\partial w_0}{\partial y_0} & \frac{\partial w_0}{\partial z_0} \end{bmatrix}=\begin{bmatrix} e_{XX}{}^L & e_{XY}{}^L & e_{XZ}{}^L \\ e_{YX}{}^L & e_{YY}{}^L & e_{YZ}{}^L \\ e_{ZX}{}^L & e_{ZY}{}^L & e_{ZZ}{}^L \end{bmatrix}$$

$$=\begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}\begin{bmatrix} \varepsilon_1{}^L & & 0 \\ & \varepsilon_2{}^L & \\ 0 & & \varepsilon_3{}^L \end{bmatrix}\begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}\cdots(11)$$

where $(l_x, m_x, n_x)$, $(l_y, m_y, n_y)$ and $(l_z, m_z, n_z)$ denote the direction cosines* of $x_0$-, $y_0$- and $z_0$- axis referred to the rectangular coordinate axes $(\xi, \eta, \zeta)$ respectively. We can see in the above eq. (11), $e_{XY}{}^L=e_{YX}{}^L$, $e_{YZ}{}^L=e_{ZY}{}^L$, $e_{ZX}{}^L=e_{XZ}{}^L$.

(b) *The case of Eulerian description.*

Consider about deformation of a parallelepiped which is to become a cube having each side of unit length in a direction of coordinate axes $(x_1', y_1', z_1')$ in final condition.

When rigid body rotation angle $\varphi=0$, the direction of principal strain before deformation remains

* These direction cosines are not quite independent from each other. There are following six relations.

$$\left.\begin{array}{l} l_x^2+m_x^2+n_x^2=1 \\ l_y^2+m_y^2+n_y^2=1 \\ l_z^2+m_z^2+n_z^2=1 \end{array}\right\},\ \left.\begin{array}{l} l_xl_y+m_xm_y+n_xn_y=0 \\ l_yl_z+m_ym_z+n_yn_z=0 \\ l_zl_x+m_zm_x+n_zn_x=0 \end{array}\right\}.$$

Therefore only three of these direction cosines are independent variables.

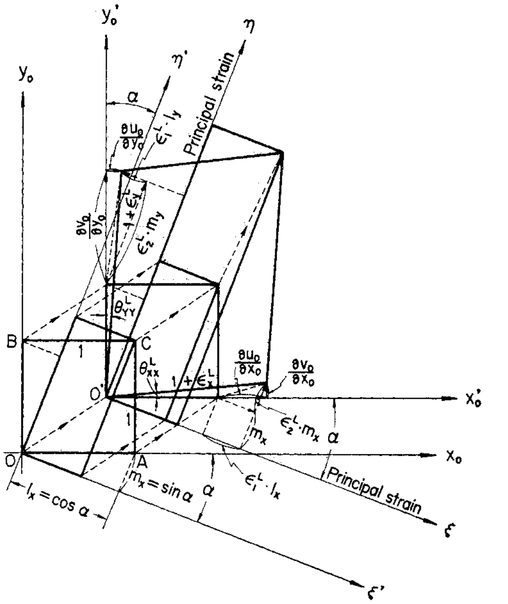


**Fig. 6** Two-dimensional illustration about the Lagrangian description of strains when rigid body rotation angle $\varphi=0$.

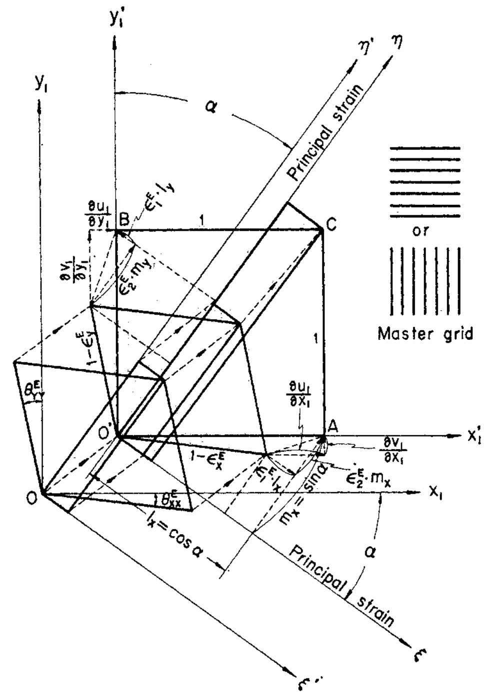


**Fig. 7** Two-dimensional illustration about the Eulerian description of strains when rigid body rotation angle $\varphi=0$.

unchanged also after deformation, then similarly as in the case (a),

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial z_1} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial z_1} \\ \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial z_1} \end{bmatrix} = \begin{bmatrix} e_{XX}{}^E & e_{XY}{}^E & e_{XZ}{}^E \\ e_{YX}{}^E & e_{YY}{}^E & e_{YZ}{}^E \\ e_{ZX}{}^E & e_{ZY}{}^E & e_{ZZ}{}^E \end{bmatrix}$$

$$= \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} \begin{bmatrix} \varepsilon_1{}^E & & 0 \\ & \varepsilon_2{}^E & \\ 0 & & \varepsilon_3{}^E \end{bmatrix} \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \cdots(12)$$

and

$$e_{XY}{}^E=e_{YX}{}^E,\quad e_{YZ}{}^E=e_{ZY}{}^E,\quad e_{ZX}{}^E=e_{XZ}{}^E,$$

where $(l_x, m_x, n_x)$, $(l_y, m_y, n_y)$ and $(l_z, m_z, n_z)$ denote the direction consines of $x_1$-, $y_1$- and $z_1$-axis referred to the rectangular coordinate axes ($\xi$, $\eta$, $\zeta$) respectively.

In a particular case of two dimensional strain, the above eq. (12) become (ref. Fig. 7),

$$\left.\begin{aligned} \frac{\partial u_1}{\partial x_1} &= e_{XX}{}^E=\varepsilon_1{}^E \cos^2\alpha+\varepsilon_2{}^E \sin^2\alpha \\ &= \frac{(\varepsilon_1{}^E+\varepsilon_2{}^E)}{2}+\frac{(\varepsilon_1{}^E-\varepsilon_2{}^E)}{2}\cos 2\alpha \\ \frac{\partial v_1}{\partial x_1} &= e_{YX}{}^E=-\varepsilon_1{}^E \cos\alpha\sin\alpha+\varepsilon_2{}^E \sin\alpha\cos\alpha \\ &= -\frac{(\varepsilon_1{}^E-\varepsilon_2{}^E)}{2}\sin 2\alpha \\ \frac{\partial u_1}{\partial y_1} &= e_{XY}{}^E=-\varepsilon_1{}^E \sin\alpha\cos\alpha+\varepsilon_2{}^E \cos\alpha\sin\alpha \\ &= -\frac{(\varepsilon_1{}^E-\varepsilon_2{}^E)}{2}\sin 2\alpha \\ \frac{\partial v_1}{\partial y_1} &= e_{YY}{}^E=\varepsilon_1{}^E \sin^2\alpha+\varepsilon_2{}^E \cos^2\alpha \\ &= \frac{(\varepsilon_1{}^E+\varepsilon_2{}^E)}{2}-\frac{(\varepsilon_1{}^E-\varepsilon_2{}^E)}{2}\cos 2\alpha \end{aligned}\right\} \cdots\cdots(12)'$$

(c) *The relation between strain-tensor and stress-tensor.*

In strictly speaking, each element of a strain-tensor is not equal generally to the corresponding one of true-strains

$$\left(\varepsilon_x{}^L,\ \frac{1}{2}\gamma_{xy}{}^L,\ \frac{1}{2}\gamma_{xz}{}^L,\ \cdots\right)$$

or

$$\left(\varepsilon_x{}^E,\ \frac{1}{2}\gamma_{xy}{}^E,\ \frac{1}{2}\gamma_{xz}{}^E,\ \cdots\right)$$

Any form of a strain-tensor can be induced from the principal form of the strain-tensor $(e_1{}^L, e_2{}^L, e_3{}^L)$ or $(e_1{}^E, e_2{}^E, e_3{}^E)$ (in this case any element of a strain-tensor is also a true-strain) only by the transformation of coordinate axes.* If we assume that the Hooke's law holds originally between principal strains and principal stresses within an elastic limit in Lagrangian description, then the following equations hold strictly (notice $\varphi=0$).

* It is to be noticed that the partial derivatives of displacement in general case are not merely transformed quantities of strains only by the transformation of coordinate axes on account of the existence of rigid body rotation angle $\varphi$. Refer eq. (18).

$$\begin{bmatrix} e_1^L & & 0 \\ & e_2^L & \\ 0 & & e_3^L \end{bmatrix} = \frac{(1+\nu)}{E} \begin{bmatrix} \sigma_1^L & & 0 \\ & \sigma_2^L & \\ 0 & & \sigma_3^L \end{bmatrix} - \frac{\nu}{E}(\sigma_1^L + \sigma_2^L + \sigma_3^L) \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \sigma_x^L & \tau_{xy}^L & \tau_{xz}^L \\ \tau_{yx}^L & \sigma_y^L & \tau_{yz}^L \\ \tau_{zx}^L & \tau_{zy}^L & \sigma_z^L \end{bmatrix} = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} \cdot \begin{bmatrix} \sigma_1^L & & 0 \\ & \sigma_2^L & \\ 0 & & \sigma_3^L \end{bmatrix} \cdot \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}$$

Since $\sigma_1^L + \sigma_2^L + \sigma_3^L = \sigma_x^L + \sigma_y^L + \sigma_z^L$, we can see

$$\begin{bmatrix} e_{XX}^L & e_{XY}^L & e_{XZ}^L \\ e_{YX}^L & e_{YY}^L & e_{YZ}^L \\ e_{ZX}^L & e_{ZY}^L & e_{ZZ}^L \end{bmatrix} = \frac{(1+\nu)}{E} \begin{bmatrix} \sigma_x^L & \tau_{xy}^L & \tau_{xz}^L \\ \tau_{yz}^L & \sigma_y^L & \tau_{yz}^L \\ \tau_{zx}^L & \tau_{zy}^L & \sigma_z^L \end{bmatrix} - \frac{\nu}{E}(\sigma_x^L + \sigma_y^L + \sigma_z^L) \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \quad \cdots\cdots(13)$$

The above eq. (13) is the generalized form of Hooke's law in Lagrangian description. In a particular case of two dimensional stress, the original form of Hooke's law becomes

$$e_1^L = \frac{\sigma_1^L}{E} - \nu\frac{\sigma_2^L}{E}, \quad e_2^L = \frac{\sigma_2^L}{E} - \nu\frac{\sigma_1^L}{E}$$

$$e_3^L = -\nu\frac{(\sigma_1^L + \sigma_2^L)}{E}$$

and the direction cosines become

$$\begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then the above eq. (13) becomes

$$e_{XX}^L = \frac{(\varepsilon_1^L + \varepsilon_2^L)}{2} + \frac{(\varepsilon_1^L - \varepsilon_2^L)}{2}\cos 2\alpha = \frac{(\sigma_1^L + \sigma_2^L)}{2}\cdot\frac{(1-\nu)}{E} + \frac{(\sigma_1^L - \sigma_2^L)}{2}\cdot\frac{(1+\nu)}{E}\cos 2\alpha = \frac{\sigma_x^L}{E} - \nu\frac{\sigma_y^L}{E}$$

$$e_{XY}^L = e_{YX}^L = -\frac{(\varepsilon_1^L - \varepsilon_2^L)}{2}\sin 2\alpha = -\frac{(\sigma_1^L - \sigma_2^L)}{2}\cdot\frac{(1+\nu)}{E}\sin 2\alpha = \frac{(1+\nu)}{E}\cdot\tau_{xy}^L = \frac{(1+\nu)}{E}\cdot\tau_{yx}^L$$

$$e_{YY}^L = \frac{(\varepsilon_1^L + \varepsilon_2^L)}{2} - \frac{(\varepsilon_1^L - \varepsilon_2^L)}{2}\cos 2\alpha = \frac{(\sigma_1^L + \sigma_2^L)}{2}\cdot\frac{(1-\nu)}{E} - \frac{(\sigma_1^L - \sigma_2^L)}{2}\cdot\frac{(1+\nu)}{E}\cos 2\alpha = \frac{\sigma_y^L}{E} - \nu\frac{\sigma_x^L}{E}$$

$$e_{XZ}^L = e_{ZX}^L = 0, \quad e_{YZ}^L = e_{ZY}^L = 0$$

$$e_{ZZ}^L = -\frac{\nu}{E}(\sigma_x^L + \sigma_y^L)$$

On the other hand, in the case of two dimensional strain, eq. (13) becomes

$$e_{XX}^L = \frac{\sigma_x^L}{E} - \nu\frac{(\sigma_y^L + \sigma_z^L)}{E}$$

$$e_{XY}^L = e_{YX}^L = \frac{(1+\nu)}{E}\tau_{xy}^L = \frac{(1+\nu)}{E}\tau_{yx}^L$$

$$e_{YY}^L = \frac{\sigma_y^L}{E} - \nu\frac{(\sigma_z^L + \sigma_x^L)}{E}$$

$$e_{XZ}^L = e_{ZX}^L = e_{YZ}^L = e_{ZY}^L = 0$$

$$e_{ZZ}^L = 0 = \frac{\sigma_z^L}{E} - \nu\frac{(\sigma_x^L + \sigma_y^L)}{E}$$

If we assume that the Hooke's law holds in Eulerian description instead of Lagrangian description, then similar equation as eq. (13) also holds in Eulerian description. But if we assume that the Hooke's law holds in Lagrangian description as commonly used, then the Hooke's law in Eulerian description does not hold strictly*.

**(3) The representation of partial derivatives of displacement when only the rigid body rotation angle $\varphi$ exists and all strains are equal to zero.**

(a) *The case of Lagrangian description.*

In this case, we can see from Fig. 8(a),

$$\begin{bmatrix} \frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial y_0} & \frac{\partial u_0}{\partial z_0} \\ \frac{\partial v_0}{\partial x_0} & \frac{\partial v_0}{\partial y_0} & \frac{\partial v_0}{\partial z_0} \\ \frac{\partial w_0}{\partial x_0} & \frac{\partial w_0}{\partial y_0} & \frac{\partial w_0}{\partial z_0} \end{bmatrix} = \begin{bmatrix} l_x - 1 & m_x & n_x \\ l_y & m_y - 1 & n_y \\ l_z & m_z & n_z - 1 \end{bmatrix} \quad \cdots\cdots(14)$$

and

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \frac{1}{K}\begin{pmatrix} \frac{\partial w_0}{\partial y_0} - \frac{\partial v_0}{\partial z_0} \\ \frac{\partial u_0}{\partial z_0} - \frac{\partial w_0}{\partial x_0} \\ \frac{\partial v_0}{\partial x_0} - \frac{\partial u_0}{\partial y_0} \end{pmatrix} = \frac{1}{K}\begin{pmatrix} m_z - n_y \\ n_x - l_z \\ l_y - m_x \end{pmatrix} \quad \cdots\cdots(15)$$

where

$$K = |\text{rot. } \boldsymbol{D}^L| = \sqrt{(m_z - n_y)^2 + (n_x - l_z)^2 + (l_y - m_x)^2} = |2\sin\varphi|$$

(b) *The case of Eulerian description.*

We can seen from Fig. 8(b),

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial z_1} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial z_1} \\ \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial z_1} \end{bmatrix} = \begin{bmatrix} 1 - l_x & -m_x & -n_x \\ -l_y & 1 - m_y & -n_y \\ -l_z & -m_z & 1 - n_z \end{bmatrix} \quad \cdots(16)$$

* There are differences between the strains in Lagrangian description and Eulerian description from each other, not only in the different base quantities assumed as a denominator but also in the assumed initial position of a line segment before deformation. Therefore, in strictly speaking, the relations $\varepsilon_x^E = \frac{\varepsilon_x^L}{1+\varepsilon_x^L}$, $\varepsilon_x^L = \frac{\varepsilon_x^E}{1-\varepsilon_x^E}$ hold only in one particular case of principal strain. Further see eq. (21)~eq. (22).

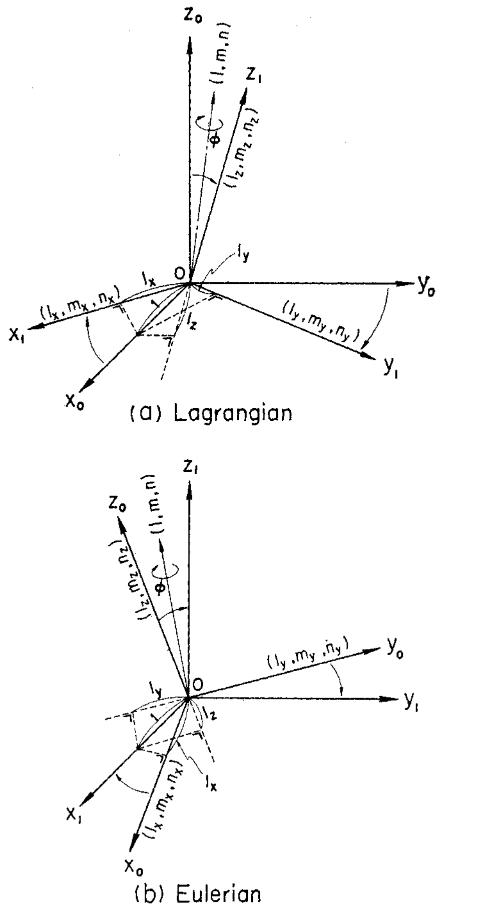


**Fig. 8** Relation between two sets of rectangular coordinate.

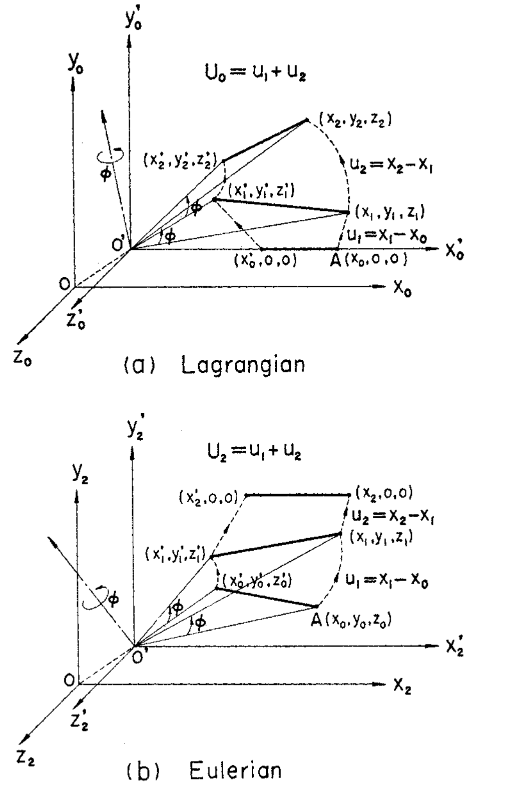


**Fig. 9** The displacement of a point A when both rigid body rotation angle $\varphi$ and strains exist.

and

$$\begin{pmatrix} n \\ m \\ l \end{pmatrix} = \frac{1}{K}\begin{pmatrix} \dfrac{\partial w_1}{\partial y_1}-\dfrac{\partial v_1}{\partial z_1} \\ \dfrac{\partial u_1}{\partial z_1}-\dfrac{\partial w_1}{\partial x_1} \\ \dfrac{\partial v_1}{\partial x_1}-\dfrac{\partial u_1}{\partial y_1} \end{pmatrix} = \frac{1}{K}\begin{pmatrix} -m_z+n_y \\ -n_x+l_z \\ -l_y+m_x \end{pmatrix} \quad \cdots(17)$$

where $(l, m, n)$ and $(l_x, m_x, n_x)$ or $(l_y, m_y, n_y)$ or $(l_z, m_z, n_z)$ are the direction cosines of rotating axis (in left hand system) and of $x_0$- or $y_0$- or $z_0$-axis, all referred to the final coordinate axes $(x_1, y_1, z_1)$ respectively, and

$$K = |\text{rot. } \boldsymbol{D}^E| = \sqrt{(m_z-n_y)^2+(n_x-l_z)^2+(l_y-m_x)^2}$$
$$= |2\sin\varphi|$$

In two dimensional case, the above eq. (16) becomes

$$\begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial y_1} \\ \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial y_1} \end{bmatrix} = \begin{bmatrix} 2\sin^2\dfrac{\varphi}{2} & -\sin\varphi \\ \sin\varphi & 2\sin^2\dfrac{\varphi}{2} \end{bmatrix}$$

**(4) The representation of partial derivatives of displacement when both rigid body rotation angle $\varphi$ and strains exist at the same time.**

(a) *The case of Lagrangian description.*

As shown in Fig. 9(a), if we assume that abscissa $x_0$ of a point A moves at first from $x_0$ to $x_1$ on account of strains and then moves from $x_1$ to $x_2$ on account of a rigid body rotation angle $\varphi$, and the resultant movement of abscissa $x_0$ has become to $U_0$,* then

$$U_0 = u_1 + u_2$$
$$u_1 = x_1 - x_0$$
$$u_2 = x_2 - x_1$$

From eq. (11),

$$\begin{pmatrix} \dfrac{\partial u_1}{\partial x_0} & \dfrac{\partial u_1}{\partial y_0} & \dfrac{\partial u_1}{\partial z_0} \\ \dfrac{\partial v_1}{\partial x_0} & \dfrac{\partial v_1}{\partial y_0} & \dfrac{\partial v_1}{\partial z_0} \\ \dfrac{\partial w_1}{\partial x_0} & \dfrac{\partial w_1}{\partial y_0} & \dfrac{\partial w_1}{\partial z_0} \end{pmatrix} = \begin{pmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{pmatrix} \cdot \begin{pmatrix} \varepsilon_1{}^L & & 0 \\ & \varepsilon_2{}^L & \\ 0 & & \varepsilon_3{}^L \end{pmatrix} \begin{pmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{pmatrix}$$

and

* In this case we may consider the total displacement $U_0$ free from the arising order of actual strains and rotation, if the initial and final positions of a point A are the same.

$$\frac{\partial u_2}{\partial x_0}=\frac{\partial u_2}{\partial x_1}\cdot\frac{\partial x_1}{\partial x_0}+\frac{\partial u_2}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_0}+\frac{\partial u_2}{\partial z_1}\cdot\frac{\partial z_1}{\partial x_0}$$
$$=\frac{\partial u_2}{\partial x_1}\cdot\left(\frac{\partial u_1}{\partial x_0}+1\right)+\frac{\partial u_2}{\partial y_1}\left(\frac{\partial v_1}{\partial x_0}\right)+\frac{\partial u_2}{\partial z_1}\left(\frac{\partial w_1}{\partial x_0}\right)$$
$$\frac{\partial u_2}{\partial y_0}=\frac{\partial u_2}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_0}+\frac{\partial u_2}{\partial y_1}\cdot\frac{\partial y_1}{\partial y_0}+\frac{\partial u_2}{\partial z_1}\cdot\frac{\partial z_1}{\partial y_0}$$
$$=\frac{\partial u_2}{\partial x_1}\cdot\frac{\partial u_1}{\partial y_0}+\frac{\partial u_2}{\partial y_1}\cdot\left(\frac{\partial v_1}{\partial y_0}+1\right)+\frac{\partial u_2}{\partial z_1}\cdot\left(\frac{\partial w_1}{\partial y_0}\right)$$
$$\frac{\partial u_2}{\partial z_0}=\frac{\partial u_2}{\partial x_1}\cdot\frac{\partial u_1}{\partial z_0}+\frac{\partial u_2}{\partial y_1}\cdot\frac{\partial v_1}{\partial z_0}+\frac{\partial u_2}{\partial z_1}\left(\frac{\partial w_1}{\partial z_0}+1\right)$$

and so on.

Therefore

$$\begin{bmatrix}\frac{\partial u_2}{\partial x_0} & \frac{\partial u_2}{\partial y_0} & \frac{\partial u_2}{\partial z_0}\\ \frac{\partial v_2}{\partial x_0} & \frac{\partial v_2}{\partial y_0} & \frac{\partial v_2}{\partial z_0}\\ \frac{\partial w_2}{\partial x_0} & \frac{\partial w_2}{\partial y_0} & \frac{\partial w_2}{\partial z_0}\end{bmatrix}=\begin{bmatrix}\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial z_1}\\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial z_1}\\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial y_1} & \frac{\partial w_2}{\partial z_1}\end{bmatrix}\cdot\begin{bmatrix}\left(\frac{\partial u_1}{\partial x_0}+1\right) & \frac{\partial u_1}{\partial y_0} & \frac{\partial u_1}{\partial z_0}\\ \frac{\partial v_1}{\partial x_0} & \left(\frac{\partial v_1}{\partial y_0}+1\right) & \frac{\partial v_1}{\partial z_0}\\ \frac{\partial w_1}{\partial x_0} & \frac{\partial w_1}{\partial y_0} & \left(\frac{\partial w_1}{\partial z_0}+1\right)\end{bmatrix}$$

On the other hand, we see from eq. (14)

$$\begin{bmatrix}\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial z_1}\\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial z_1}\\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial y_1} & \frac{\partial w_2}{\partial z_1}\end{bmatrix}=\begin{bmatrix}(l_x'-1) & m_x' & n_x'\\ l_y' & (m_y'-1) & n_y'\\ l_z' & m_z' & (n_z'-1)\end{bmatrix}$$

Thus we obtain

$$\begin{bmatrix}\frac{\partial u_0}{\partial x_0} & \frac{\partial u_2}{\partial y_0} & \frac{\partial u_2}{\partial z_0}\\ \frac{\partial v_2}{\partial x_0} & \frac{\partial v_2}{\partial y_0} & \frac{\partial v_2}{\partial z_0}\\ \frac{\partial w_2}{\partial x_0} & \frac{\partial w_2}{\partial y_0} & \frac{\partial w_2}{\partial z_0}\end{bmatrix}=\begin{bmatrix}(l_x'-1) & m_x' & n_x'\\ l_y' & (m_y'-1) & n_y'\\ l_z' & m_z' & (n_z'-1)\end{bmatrix}\cdot\begin{bmatrix}\left(\frac{\partial u_1}{\partial x_0}+1\right) & \frac{\partial u_1}{\partial y_0} & \frac{\partial u_1}{\partial z_0}\\ \frac{\partial v_1}{\partial x_0} & \left(\frac{\partial v_1}{\partial y_0}+1\right) & \frac{\partial v_1}{\partial z_0}\\ \frac{\partial w_1}{\partial x_0} & \frac{\partial w_1}{\partial y_0} & \left(\frac{\partial w_1}{\partial z_0}+1\right)\end{bmatrix}$$

Therefore

$$\begin{bmatrix}\frac{\partial U_0}{\partial x_0} & \frac{\partial U_0}{\partial y_0} & \frac{\partial U_0}{\partial z_0}\\ \frac{\partial V_0}{\partial x_0} & \frac{\partial V_0}{\partial y_0} & \frac{\partial V_0}{\partial z_0}\\ \frac{\partial W_0}{\partial x_0} & \frac{\partial W_0}{\partial y_0} & \frac{\partial W_0}{\partial z_0}\end{bmatrix}=\begin{bmatrix}\frac{\partial u_1}{\partial x_0} & \frac{\partial u_1}{\partial y_0} & \frac{\partial u_1}{\partial z_0}\\ \frac{\partial v_1}{\partial x_0} & \frac{\partial v_1}{\partial y_0} & \frac{\partial v_1}{\partial z_0}\\ \frac{\partial w_1}{\partial x_0} & \frac{\partial w_1}{\partial y_0} & \frac{\partial w_1}{\partial z_0}\end{bmatrix}+\begin{bmatrix}\frac{\partial u_2}{\partial x_0} & \frac{\partial u_2}{\partial y_0} & \frac{\partial u_2}{\partial z_0}\\ \frac{\partial v_2}{\partial x_0} & \frac{\partial v_2}{\partial y_0} & \frac{\partial v_2}{\partial z_0}\\ \frac{\partial w_2}{\partial x_0} & \frac{\partial w_2}{\partial y_0} & \frac{\partial w_2}{\partial z_0}\end{bmatrix}=\begin{bmatrix}l_x' & m_x' & n_x'\\ l_y' & m_y' & n_y'\\ l_z' & m_z' & n_z'\end{bmatrix}$$
$$\cdot\begin{bmatrix}l_x & m_x & n_x\\ l_y & m_y & n_y\\ l_z & m_z & n_z\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_1^L & & 0\\ & \varepsilon_2^L & \\ 0 & & \varepsilon_3^L\end{bmatrix}\cdot\begin{bmatrix}l_x & l_y & l_z\\ m_x & m_y & m_z\\ n_x & n_y & n_z\end{bmatrix}+\begin{bmatrix}(l_x'-1) & m_x' & n_x'\\ l_y' & (m_y'-1) & n_y'\\ l_z' & m_z' & (n_z'-1)\end{bmatrix}\quad\cdots\cdots\cdots(18)$$

where $l_x$, $m_x$, $\cdots$, are the direction cosines of $x_0$-, $y_0$- and $z_0$-axis referred to the directions of principal strains $(\xi, \eta, \zeta)$ respectively and $l_x'$, $m_x'$, $\cdots$, are, referring to initial coordinate axes $(x_0, y_0, z_0)$, the direction cosines of three coordinate axes which are obtained by rotating the initial coordinate axes by an angle $\varphi$ around an axis in left hand system respectively. The right side of eq. (18), is not a set of merely transformed quantities of principal strains by a transformation of coordinate axes, namely is not a strain-tensor.

(b) *The case of Eulerian description.*

In this case, we obtain as similarly as in the case (a) in this article the following results.

$$\begin{bmatrix}\frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial y_2} & \frac{\partial U_2}{\partial z_2}\\ \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial y_2} & \frac{\partial V_2}{\partial z_2}\\ \frac{\partial W_2}{\partial x_2} & \frac{\partial W_2}{\partial y_2} & \frac{\partial W_2}{\partial z_2}\end{bmatrix}=\begin{bmatrix}l_x' & m_x' & n_x'\\ l_y' & m_y' & n_y'\\ l_z' & m_z' & n_z'\end{bmatrix}\cdot\begin{bmatrix}l_x & m_x & n_x\\ l_y & m_y & n_y\\ l_z & m_z & n_z\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_1^E & & 0\\ & \varepsilon_2^E & \\ 0 & & \varepsilon_3^E\end{bmatrix}$$
$$\cdot\begin{bmatrix}l_x & l_y & l_z\\ m_x & m_y & m_z\\ n_x & n_y & n_z\end{bmatrix}-\begin{bmatrix}(l_x'-1) & m_x' & n_x'\\ l_y' & (m_y'-1) & n_y'\\ l_z' & m_z' & (n_z'-1)\end{bmatrix}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(18)'$$

and

$$\begin{pmatrix}l\\ m\\ n\end{pmatrix}=\frac{1}{K}\begin{pmatrix}-m_z'+n_y'\\ -n_x'+l_z'\\ -l_y'+m_x'\end{pmatrix}$$
$$K=\sqrt{(m_z'-n_y')^2+(n_x'-l_z')^2+(l_y'-m_x')^2}$$
$$=|2\sin\varphi|$$

where $l_x$, $m_x$, $\cdots$ denote the direction cosines of $x_2$-, $y_2$- and $z_2$-axis referred to the directions of principal strains $(\xi, \eta, \zeta)$ respectively and $l_x'$, $m_x'$, $\cdots$ denote, referring to the final coordinate axes $(x_2, y_2, z_2)$, the direction cosines of three coordinate axes which are to become in the final positions only after rotating them by an angle $\varphi$ around an axis in left hand system respectively and further $(l, m, n)$ denote the direction cosines of the rotation axis referred to the final coordinate axes. The right side of eq. (18)′ is not a set of merely transformed quantities of principal strains by the transformation of coordinate axes, namely is not a strain-tensor as similarly as in the case (a) in this article.

In a special case of two dimensional strain, the above eq. (18)′ becomes

$$\begin{bmatrix} \frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial y_2} & \frac{\partial U_2}{\partial z_2} \\ \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial y_2} & \frac{\partial V_2}{\partial z_2} \\ \frac{\partial W_2}{\partial x_2} & \frac{\partial W_2}{\partial y_2} & \frac{\partial W_2}{\partial z_2} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_{XX}{}^E & e_{XY}{}^E & 0 \\ e_{YX}{}^E & e_{YY}{}^E & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\sin^2\frac{\varphi}{2} & -\sin\varphi & 0 \\ \sin\varphi & 2\sin^2\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In a special case of two dimensional stress, eq. (18)′ becomes

$$\begin{bmatrix} \frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial y_2} & \frac{\partial U_2}{\partial z_2} \\ \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial y_2} & \frac{\partial V_2}{\partial z_2} \\ \frac{\partial W_2}{\partial x_2} & \frac{\partial W_2}{\partial y_2} & \frac{\partial W_2}{\partial z_2} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_{XX}{}^E & e_{XY}{}^E & 0 \\ e_{YX}{}^E & e_{YY}{}^E & 0 \\ 0 & 0 & e_{ZZ}{}^E \end{bmatrix} + \begin{bmatrix} 2\sin^2\frac{\varphi}{2} & -\sin\varphi & 0 \\ \sin\varphi & 2\sin^2\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \cdots\cdots (18)''$$

and $\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

where $e_{ZZ}{}^E = e_3{}^E$ and $e_{XY}{}^E = e_{YX}{}^E$.

In this special case of two dimensional stress, we can see from eq. (18)″,

$$\text{rot. } \boldsymbol{D}^E = k\left(\frac{\partial V_2}{\partial x_2} - \frac{\partial U_2}{\partial y_2}\right) = k(2 - e_{XX}{}^E - e_{YY}{}^E)\sin\varphi^*,$$

then we get

$$\varphi = \mathrm{Sin}^{-1} \frac{\left(\frac{\partial V_2}{\partial x_2} - \frac{\partial U_2}{\partial y_2}\right)}{(2 - e_{XX}{}^E - e_{YY}{}^E)} \qquad \cdots\cdots (19)$$

(c) *The relation between partial derivatives of displacement and stress-tensor.*

In Lagrangian description, from eqs. (13) and (18) we get,

$$\begin{bmatrix} \frac{\partial U_0}{\partial x_0} & \frac{\partial U_0}{\partial y_0} & \frac{\partial U_0}{\partial z_0} \\ \frac{\partial V_0}{\partial x_0} & \frac{\partial V_0}{\partial y_0} & \frac{\partial V_0}{\partial z_0} \\ \frac{\partial W_0}{\partial x_0} & \frac{\partial W_0}{\partial y_0} & \frac{\partial W_0}{\partial z_0} \end{bmatrix} = \frac{(1+\nu)}{E} \cdot \begin{bmatrix} l_x' & m_x' & n_x' \\ l_y' & m_y' & n_y' \\ l_z' & m_z' & n_z' \end{bmatrix}' \cdot \begin{bmatrix} \sigma_x{}^L & \tau_{xy}{}^L & \tau_{xz}{}^L \\ \tau_{yx}{}^L & \sigma_y{}^L & \tau_{yz}{}^L \\ \tau_{zx}{}^L & \tau_{zy}{}^L & \sigma_z{}^L \end{bmatrix} - \frac{\nu}{E}(\sigma_x{}^L + \sigma_y{}^L + \sigma_z{}^L) \begin{bmatrix} l_x' & m_x' & n_x' \\ l_y' & m_y' & n_y' \\ l_z' & m_z' & n_z' \end{bmatrix} + \begin{bmatrix} (l_x'-1) & m_x' & n_x' \\ l_y' & (m_y'-1) & n_y' \\ l_z' & m_z' & (n_z'-1) \end{bmatrix} \qquad \cdots\cdots (20)$$

where $(\sigma_x{}^L, \tau_{xy}{}^L, \cdots)$ denote stress-tensor referred to the coordinate axes $(x_0, y_0, z_0)$, and $l_x', m_x', \cdots,$ denote the same direction cosines as denoted in the case (a) in this article.

In Eulerian description, as similarly as above we get the following results.

$$\begin{bmatrix} \frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial y_2} & \frac{\partial U_2}{\partial z_2} \\ \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial y_2} & \frac{\partial V_2}{\partial z_2} \\ \frac{\partial W_2}{\partial x_2} & \frac{\partial W_2}{\partial y_2} & \frac{\partial W_2}{\partial z_2} \end{bmatrix} = \frac{(1+\nu)}{E} \cdot \begin{bmatrix} l_x' & m_x' & n_x' \\ l_y' & m_y' & n_y' \\ l_z' & m_z' & n_z' \end{bmatrix}' \cdot \begin{bmatrix} \sigma_x{}^E & \tau_{xy}{}^E & \tau_{xz}{}^E \\ \tau_{yx}{}^E & \sigma_y{}^E & \tau_{yz}{}^E \\ \tau_{zx}{}^E & \tau_{zy}{}^E & \sigma_z{}^E \end{bmatrix} - \frac{\nu}{E}(\sigma_x{}^E + \sigma_y{}^E + \sigma_z{}^E) \begin{bmatrix} l_x' & m_x' & n_x' \\ l_y' & m_y' & n_y' \\ l_z' & m_z' & n_z' \end{bmatrix} - \begin{bmatrix} (l_x'-1) & m_x' & n_x' \\ l_y' & (m_y'-1) & n_y' \\ l_z' & m_z' & (n_z'-1) \end{bmatrix} \qquad \cdots\cdots (20)'$$

* It is to be noticed that in this case rot. $\boldsymbol{D}^E$ is different from the pure rigid body rotation, $i(-m_z'+n_y') + j(-n_x'+l_z') + k(-l_y'+m_x') = k \cdot 2\sin\varphi$, where $i$, $j$ and $k$denote the base unit-vectors in the directions of $x_2$-, $y_2$-and $z_2$-axis respectively.

where $(\sigma_x{}^E, \tau_{xy}{}^E, \cdots)$ denote the stress-tensor referred to the directions of final line segments after deformation and rigid body rotation which are in the directions of coordinate axes $(x_2, y_2, z_2)$ and $l_x', m_x', \cdots$ denote the same direction cosines as denoted in the case (b) in this article. It is to be noticed that both the left sides of eqs. (20) and (20)′ are not merely strain-tensors and only the sets of partial derivatives of displacements.

## III. TRANSFORMATION FROM EULERIAN DESCRIPTION TO LAGRANGIAN DESCRIPTION

### (1) Transformation of partial derivatives of displacement

$$\begin{cases} x_0 = x_1 - u \\ y_0 = y_1 - v \\ z_0 = z_1 - w \end{cases}$$

and $\boldsymbol{D}^E = iu_1 + jv_1 + kw_1 = iu_0 + jv_0 + kw_0 = \boldsymbol{D}^L$,

$$\therefore \frac{\partial u_1}{\partial x_1} = \frac{\partial u_0}{\partial x_0}\cdot\frac{\partial x_0}{\partial x_1} + \frac{\partial u_0}{\partial y_0}\cdot\frac{\partial y_0}{\partial x_1} + \frac{\partial u_0}{\partial z_0}\cdot\frac{\partial z_0}{\partial x_1}$$
$$= \frac{\partial u_0}{\partial x_0}\cdot\left(1 - \frac{\partial u_1}{\partial x_1}\right) + \frac{\partial u_0}{\partial y_0}\cdot\left(-\frac{\partial v_1}{\partial x_1}\right) + \frac{\partial u_0}{\partial z_0}\cdot\left(-\frac{\partial w_1}{\partial x_1}\right)$$

Similarly,

$$\frac{\partial u_1}{\partial y_1} = \frac{\partial u_0}{\partial x_0}\cdot\frac{\partial x_0}{\partial y_1} + \frac{\partial u_0}{\partial y_0}\cdot\frac{\partial y_0}{\partial y_1} + \frac{\partial u_0}{\partial z_0}\cdot\frac{\partial z_0}{\partial y_1}$$
$$= \frac{\partial u_0}{\partial x_0}\cdot\left(-\frac{\partial u_1}{\partial y_1}\right) + \frac{\partial u_0}{\partial y_0}\cdot\left(1 - \frac{\partial v_1}{\partial y_1}\right) + \frac{\partial u_0}{\partial z_0}\cdot\left(-\frac{\partial w_1}{\partial y_1}\right)$$

$$\frac{\partial u_1}{\partial z_1} = \frac{\partial u_0}{\partial x_0}\cdot\left(-\frac{\partial u_1}{\partial z_1}\right) + \frac{\partial u_0}{\partial y_0}\cdot\left(-\frac{\partial v_1}{\partial z_1}\right) + \frac{\partial u_0}{\partial z_0}\cdot\left(1 - \frac{\partial w_1}{\partial z_1}\right)$$

and so on.

$$\therefore \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial z_1} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial z_1} \\ \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial z_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial y_0} & \frac{\partial u_0}{\partial z_0} \\ \frac{\partial v_0}{\partial x_0} & \frac{\partial v_0}{\partial y_0} & \frac{\partial v_0}{\partial z_0} \\ \frac{\partial w_0}{\partial x_0} & \frac{\partial w_0}{\partial y_0} & \frac{\partial w_0}{\partial z_0} \end{bmatrix} \cdot \begin{bmatrix} \left(1-\frac{\partial u_1}{\partial x_1}\right) & -\frac{\partial u_1}{\partial y_1} & -\frac{\partial u_1}{\partial z_1} \\ -\frac{\partial v_1}{\partial x_1} & \left(1-\frac{\partial v_1}{\partial y_1}\right) & -\frac{\partial v_1}{\partial z_1} \\ -\frac{\partial w_1}{\partial x_1} & -\frac{\partial w_1}{\partial y_1} & \left(1-\frac{\partial w_1}{\partial z_1}\right) \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial y_0} & \frac{\partial u_0}{\partial z_0} \\ \frac{\partial v_0}{\partial x_0} & \frac{\partial v_0}{\partial y_0} & \frac{\partial v_0}{\partial z_0} \\ \frac{\partial w_0}{\partial x_0} & \frac{\partial w_0}{\partial y_0} & \frac{\partial w_0}{\partial z_0} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial z_1} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial z_1} \\ \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial z_1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \cdot \frac{1}{\Delta} \qquad \cdots\cdots(21)$$

where

$$\Delta = \begin{vmatrix} \left(1-\frac{\partial u_1}{\partial x_1}\right) & -\frac{\partial u_1}{\partial y_1} & -\frac{\partial u_1}{\partial z_1} \\ -\frac{\partial v_1}{\partial x_1} & \left(1-\frac{\partial v_1}{\partial y_1}\right) & -\frac{\partial v_1}{\partial z_1} \\ -\frac{\partial w_1}{\partial x_1} & -\frac{\partial w_1}{\partial y_1} & \left(1-\frac{\partial w_1}{\partial z_1}\right) \end{vmatrix}$$

and $A_{ij}$ denotes the cofactor of the element at $i$ row and $j$ column in the determinant $\Delta$.

### (2) Transformation of strain-tensor

As stated already, strain-tensor is a special case of partial derivatives of displacement when there is no rigid body rotation. In this case of no rigid body rotation, eq. (21) becomes to the following form.

$$\begin{bmatrix} \frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial y_0} & \frac{\partial u_0}{\partial z_0} \\ \frac{\partial v_0}{\partial x_0} & \frac{\partial v_0}{\partial y_0} & \frac{\partial v_0}{\partial z_0} \\ \frac{\partial w_0}{\partial x_0} & \frac{\partial w_0}{\partial y_0} & \frac{\partial w_0}{\partial z_0} \end{bmatrix} = \begin{bmatrix} e_{XX}{}^L & e_{XY}{}^L & e_{XZ}{}^L \\ e_{YX}{}^L & e_{YY}{}^L & e_{YZ}{}^L \\ e_{ZX}{}^L & e_{ZY}{}^L & e_{ZZ}{}^L \end{bmatrix}$$
$$= \begin{bmatrix} e_{XX}{}^E & e_{XY}{}^E & e_{XZ}{}^E \\ e_{YX}{}^E & e_{YY}{}^E & e_{YZ}{}^E \\ e_{ZX}{}^E & e_{ZY}{}^E & e_{ZZ}{}^E \end{bmatrix} \cdot \begin{bmatrix} A_{11}{}^E & A_{21}{}^E & A_{31}{}^E \\ A_{12}{}^E & A_{22}{}^E & A_{32}{}^E \\ A_{13}{}^E & A_{23}{}^E & A_{33}{}^E \end{bmatrix} \cdot \frac{1}{\Delta^E} \qquad \cdots\cdots(22)$$

where

$$\Delta^E = \begin{vmatrix} (1-e_{XX}{}^E) & -e_{XY}{}^E & -e_{XZ}{}^E \\ -e_{YX}{}^E & (1-e_{YY}{}^E) & -e_{YZ}{}^E \\ -e_{ZX}{}^E & -e_{ZY}{}^E & (1-e_{ZZ}{}^E) \end{vmatrix}$$

and $A_{ij}{}^E$ denotes the cofactor of the element at $i$ row and $j$ column in the determinant $\Delta^E$.

## IV. AN APPLICATION OF THE GENERAL THEORY OF LARGE DEFORMATION TO THE CORRECT TREATMENT OF MOIRÉ FRINGES

### (1) Relation between Moiré fringes and partial derivatives of displacement in Eulerian description

If we assume the orders of Moiré fringes formed by a master grid, the perpendicular direction of which (it is called a principal section and the parallel direction is called a secondary section) is the direction of $x_1$ coordinate-axis, as the numbers of $p$ in

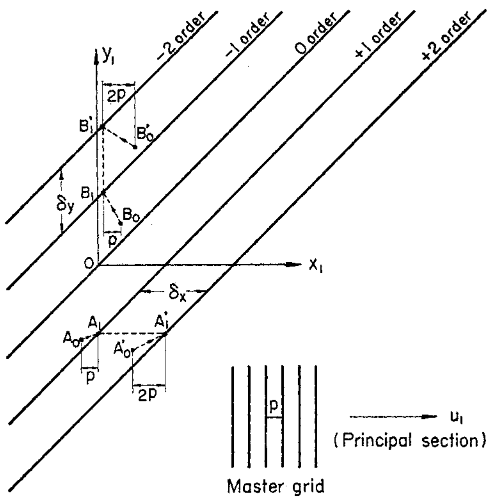


**Fig. 10** $u$-Moiré fringe pattern.

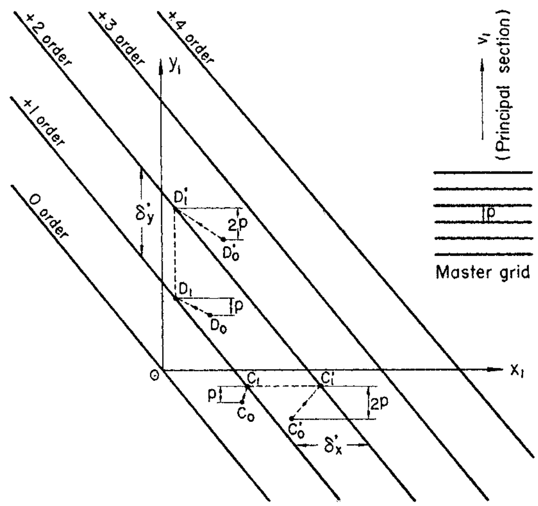


**Fig. 11** $v$-Moiré fringe pattern.

$u_1$-displacements and also take the origin of coordinate-axes at a point on the Moiré fringe of 0-order as shown in Fig. 10 (such a Moiré pattern is called $u$-Moiré pattern), we can see from Fig. 10 the following facts.

The displacement $u_1$ by $A_0 \to A_1$ or $B_0 \to B_1$ is equal to $p$, and the displacement $u_1$ by $A_0' \to A_1'$ or $B_0' \to B_1'$ is equal to $2p$, then

$$\left.\begin{aligned} \frac{\partial u_1}{\partial x_1} &= \frac{p}{\delta_x} \\ \frac{\partial u_1}{\partial y_1} &= \frac{p}{\delta_y} \end{aligned}\right\} \qquad \text{(23)*}$$

Similarly in Fig. 11 (such a Moiré pattern is called $v$-Moiré pattern),

$$\left.\begin{aligned} \frac{\partial v_1}{\partial x_1} &= \frac{p}{\delta'_x} \\ \frac{\partial v_1}{v y_1} &= \frac{p}{\delta'_y} \end{aligned}\right\} \qquad \text{(24)}$$

Namely, we can say that $u$- or $v$-Moiré fringes are the curves of equal displacement $u_1$ in $x_1$-direction or equal displacement $v_1$ in $y_1$-direction represented in Eulerian description respectively.

### (2) Determination of sign of displacement-derivatives

As to the determinations of sign of displacement-derivatives and the order of Moiré fringes, the author only cites an example (Figs. 12, 13) and some notes about it as follows.

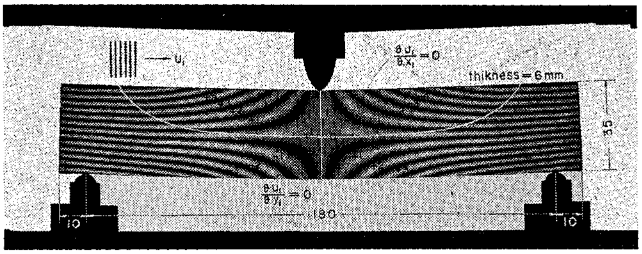


**Fig. 12** $u$-Moré fringe photograph of a prismatical simple beam (material=Lucite, pitch of grid=300/in, central load=88.9 kg).

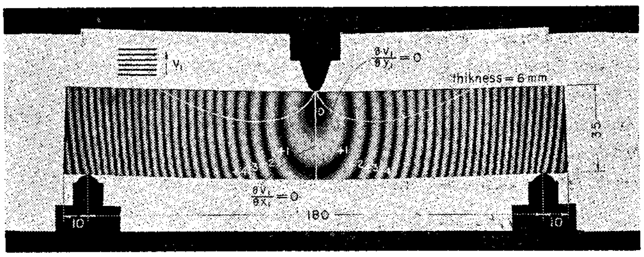


**Fig. 13** $v$-Moiré fringe photograph of a prismatical simple beam (material=Lucite, pitch of grid=300/in, central load=87.9 kg).

(a) If $\left(\frac{\partial u_1}{\partial x_1}\right)$ changes its sign as a point $x_1$ moves, then due to the continuity of the value of $\left(\frac{\partial u_1}{\partial x_1}\right)$, it is necessary to cross the curve $\left(\frac{\partial u_1}{\partial x_1}=0\right)$ in the moving process of that point $x_1$. Therefore in the region which is in one side of the curve $\left(\frac{\partial u_1}{\partial x_1}=0\right)$, the sign of $\left(\frac{\partial u_1}{\partial x_1}\right)$ is always invariable. However, the reverse rule that the derivative $\left(\frac{\partial u_1}{\partial x_1}\right)$ should always change its sign in the region of another side across the curve $\left(\frac{\partial u_1}{\partial x_1}=0\right)$, is not

---

* In order to obtain more accurately the value of $\left(\frac{\partial u_1}{\partial x_1}\right)$ or $\left(\frac{\partial u_1}{\partial y_1}\right)$, it is recommended to draw the relation between $u_1$ and $x_1$ in a principal section or $u_1$ and $y_1$ in a secondary section into a graph and then to get the value of a direction coefficient of a tangential line at any point on the curve with an abscissa $x_1$ or $y_1$.

necessarily self-evident. But we may check whether it is correct or not from the orders of Moiré fringes.

(b) From Fig. 14 we see

$$\frac{\left(\dfrac{\partial u_1}{\partial y_1}\right)}{\left(\dfrac{\partial u_1}{\partial x_1}\right)}=\frac{\left(\dfrac{p}{\delta_y}\right)}{\left(\dfrac{p}{\delta_x}\right)}=\frac{\delta_x}{\delta_y}$$

and

$$\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_1}{\partial y_1}\right)^2=\left(\frac{p}{\delta_x}\right)^2+\left(\frac{p}{\delta_y}\right)^2=p^2\frac{\delta_x{}^2+\delta_y{}^2}{\delta_x{}^2\delta_y{}^2}$$
$$=p^2\frac{\overline{AB}^2}{\overline{OA}^2\cdot\overline{OB}^2}=p^2\left\{\frac{1}{\overline{OB}\cdot\left(\dfrac{\overline{OA}}{\overline{AB}}\right)}\right\}^2=\frac{p^2}{\overline{OH}^2}$$

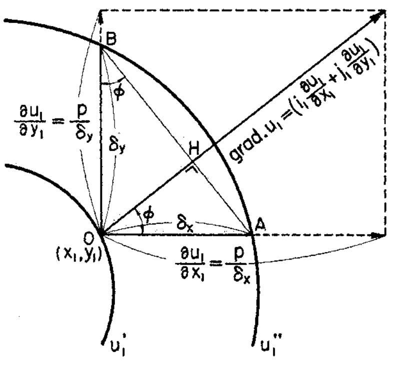


**Fig. 14** Gradient of surface $u$.

Namely, grad. $u_1=\left(i\dfrac{\partial u_1}{\partial x_1}+j\dfrac{\partial u_1}{\partial y_1}\right)$ is a derivative of $u_1$ in the direction perpendicular to the curve of $u$-Moiré fringe, where $i$ and $j$ denote the base unit-vectors in $x_1$ and $y_1$ directions respectively.

If we denote further the base unit-vector in $z_1$ direction as $k$ and assume a displacement vector $\boldsymbol{D}^E=iu_1+jv_1+kw_1$ at a point $(x_1, y_1, z_1)$, then grad. $\boldsymbol{D}^E$ is expressed more generally as follows.

$$\text{grad. }\boldsymbol{D}^E=\begin{bmatrix}\text{grad. } u_1\\ \text{grad. } v_1\\ \text{grad. } w_1\end{bmatrix}=\begin{bmatrix}\dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial y_1} & \dfrac{\partial u_1}{\partial z_1}\\ \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial y_1} & \dfrac{\partial v_1}{\partial z_1}\\ \dfrac{\partial w_1}{\partial x_1} & \dfrac{\partial w_1}{\partial y_1} & \dfrac{\partial w_1}{\partial z_1}\end{bmatrix}$$

### (3) Virtual strain by mismatching and misalignment

(a) *The case due to only a difference between the pitches of master and model grids. (mismatching)*

If we assume that the virtual strains by mismatchings in $x_1$ and $y_1$ directions are equal, then

$$\left.\begin{aligned}&\frac{\partial \bar{u}_1}{\partial x_1}=\bar{\varepsilon}_1{}^E=\bar{\varepsilon}_2{}^E=\frac{\partial \bar{v}_1}{\partial y_1}=\bar{e}_{XX}{}^E=\bar{e}_{YY}{}^E\\&\frac{\partial \bar{u}_1}{\partial y_1}=\frac{\partial \bar{v}_1}{\partial x_1}=\bar{e}_{XY}{}^E=\bar{e}_{YX}{}^E=0\end{aligned}\right\}\cdots(25)$$

Since the values of partial derivatives in the above equation (25) are known from the experimental results, we are always able to know the value of $\bar{\varepsilon}_1{}^E$.

(b) *The case where both mismatching and misalignment (uniform rigid body rotation angle $\varphi$) exist at the same time.*

From the expansion of eq. (18)″, we get

$$\left.\begin{aligned}&\frac{\partial \bar{u}_1}{\partial x_1}=\bar{e}_{XX}{}^E\cos\varphi+2\sin^2\frac{\varphi}{2}=1-(1-\bar{\varepsilon}_1{}^E)\cos\varphi\\&\frac{\partial \bar{u}_1}{\partial y_1}=\bar{e}_{YY}{}^E\sin\varphi-\sin\varphi=-(1-\bar{\varepsilon}_1{}^E)\sin\varphi\\&\frac{\partial \bar{v}_1}{\partial x_1}=-\bar{e}_{XX}{}^E\sin\varphi+\sin\varphi=(1-\bar{\varepsilon}_1{}^E)\sin\varphi\\&\frac{\partial \bar{v}_1}{\partial y_1}=\bar{e}_{YY}{}^E\cos\varphi+2\sin^2\frac{\varphi}{2}=1-(1-\bar{\varepsilon}_1{}^E)\cos\varphi\end{aligned}\right\}\cdots\cdots(26)$$

(c) *The case of composite virtual strains with true-strains.*

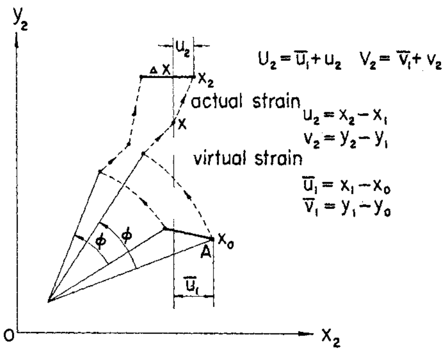


**Fig. 15** The configuration of a point A, when rigid body rotation angle $\varphi$, virtual principal strain $\bar{\varepsilon}^E$, and also actual strain exist at the same time.

From Fig. 15 and eq. (26) we can see

$$\frac{\partial \bar{u}_1}{\partial x_2}=\frac{\partial \bar{u}_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial x_2}+\frac{\partial \bar{u}_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2}$$
$$=\frac{\partial \bar{u}_1}{\partial x_1}\left(1-\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial \bar{u}_1}{\partial y_1}\left(-\frac{\partial v_2}{\partial x_2}\right)$$
$$=\{1-(1-\bar{\varepsilon}_1{}^E)\cos\varphi\}(1-e_{XX}{}^E)+(1-\bar{\varepsilon}_1{}^E)\sin\varphi\cdot e_{XY}{}^E$$

$$\frac{\partial u_2}{\partial x_2}=e_{XX}{}^E$$

$$\therefore\ \frac{\partial U_2}{\partial x_2}=(1-\bar{\varepsilon}_1{}^E)\left(e_{XX}{}^E\cos\varphi+e_{XY}{}^E\sin\varphi+2\sin^2\frac{\varphi}{2}\right)+\bar{\varepsilon}_1{}^E$$

Similarly

$$\frac{\partial U_2}{\partial y_2}=(1-\bar{\varepsilon}_1{}^E)(e_{XY}{}^E\cos\varphi+e_{YY}{}^E\sin\varphi-\sin\varphi)$$

$$\frac{\partial V_2}{\partial x_2}=(1-\bar{\varepsilon}_1{}^E)(e_{XY}{}^E\cos\varphi-e_{XX}{}^E\sin\varphi+\sin\varphi)$$

$$\frac{\partial V_2}{\partial y_2}=(1-\bar{\varepsilon}_1{}^E)\Big((e_{YY}{}^E\cos\varphi-e_{XY}{}^E\sin\varphi$$

$$+2\sin^2\frac{\varphi}{2}\Big)+\bar{\varepsilon}_1{}^E \qquad \cdots\cdots(27)$$

where $\varphi$ denotes the value of an angle including not only the misalignment (uniform rigid body rotation angle) but also the partial rigid body rotation angle which is varying at each point in a model as a accumulative result of actual strains.

**(4) The correct procedure to obtain the stress value from Moiré fringes.**

1° To calculate the virtual strain $\bar{\varepsilon}_1{}^E$ from an experiment of no load condition.

2° To calculate the values of partial derivatives $\left(\frac{\partial U_2}{\partial x_2}, \frac{\partial U_2}{\partial y_2}, \frac{\partial V_2}{\partial x_2}, \frac{\partial V_2}{\partial y_2}\right)$ at each point on a cross section from the experimental results of Moiré fringes under loading.

3° To calculate actual strain tensor ($e_{XX}{}^E$, $e_{XY}{}^E$, $e_{YY}{}^E$) and $\varphi$ by solving the eq. (27) as a simultanenus equation. (In this procedure, we may also calculate the approximate values of $e_{XX}{}^E$, $e_{XY}{}^E$, $e_{YY}{}^E$ and $\varphi$ by neglecting the terms of higher orders in eq. (27))

4° From eq. (12)′, we see

$$\left.\begin{aligned}\frac{(\varepsilon_1{}^E+\varepsilon_2{}^E)}{2}+\frac{(\varepsilon_1{}^E-\varepsilon_2{}^E)}{2}\cos 2\alpha&=e_{XX}{}^E\\ \frac{(\varepsilon_1{}^E+\varepsilon_2{}^E)}{2}-\frac{(\varepsilon_1{}^E-\varepsilon_2{}^E)}{2}\cos 2\alpha&=e_{YY}{}^E\\ -\frac{(\varepsilon_1{}^E-\varepsilon_2{}^E)}{2}\sin 2\alpha&=e_{XY}{}^E\end{aligned}\right\}$$

We can calculate the true principal strains $\varepsilon_1{}^E$, $\varepsilon_2{}^E$ and the angle $\alpha$ from the above equations.

5° If we denote the true principal strains in Lagrangian description as $\varepsilon_1{}^L$, $\varepsilon_2{}^L$, then in this case of principal strains there are relations between Eulerian and Lagrangian descriptions as follows

$$\left.\begin{aligned}\frac{\varepsilon_1{}^E}{1-\varepsilon_1{}^E}&=\varepsilon_1{}^L\\ \frac{\varepsilon_2{}^E}{1-\varepsilon_2{}^E}&=\varepsilon_2{}^L\end{aligned}\right\}\cdots\cdots(28)$$

Then we can get $\varepsilon_1{}^L$, $\varepsilon_2{}^L$ from the above eq. (28).

6° To calculate the values of principal stresses $\sigma_1{}^L$ and $\sigma_2{}^L$ from the following equations.

$$\left.\begin{aligned}\varepsilon_1{}^L&=\frac{\sigma_1{}^L}{E}-\nu\frac{\sigma_2{}^L}{E}\\ \varepsilon_2{}^L&=\frac{\sigma_2{}^L}{E}-\nu\frac{\sigma_1{}^L}{E}\end{aligned}\right\}$$

or

$$\left.\begin{aligned}\sigma_1{}^L&=\frac{E}{(1-\nu^2)}(\varepsilon_1{}^L+\nu\varepsilon_2{}^L)\\ \sigma_2{}^L&=\frac{E}{(1-\nu^2)}(\nu\varepsilon_1{}^L+\varepsilon_2{}^L)\end{aligned}\right\}$$

## V. CONCLUSION

The treatment about the sticking Moiré fringes was examined from a general theory of large deformation, and as a result the following matters have been clarified.

(1) Basic theory of large deformation was developed precisely in both Lagrangian and Eulerian descriptions.

(2) The distinction between displacement-derivatives and strain tensor and the distinction between true strains and strain tensor, and further the relation between strain tensor and stress tensor, all in finite deformation, were clearly described.

(3) General equation of transformation from Eulerian to Lagrangian description was developed.

(4) The errors in sticking Moiré method caused by mismatching and misalignment were clearly shown by analytical method.

(5) Correct procedure amending the contents in the past papers on the treatment of sticking Moiré fringes was developed. As a result, it was clarified that the values of true strains as I.S. Sokolnikoff (article I) are not required at all to obtain the values of stresses and it is always enough only to know the value of strain tensor for that purpose.

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