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LARGE STRAIN, ELASTIC-PLASTIC NUMERICAL ANALYSIS BY MEANS OF FINITE ELEMENT METHOD

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ABSTRACT

A method is presented for a large strain elastic-plastic analysis by the finite element method. The constitutive equations of elastic-plastic bodies can be obtained as the incremental relations between stress and strain. Thus, the finite element formulations result in the incremental relations between displacement and external load. In order to compute the incremental equations, several techniques hav to be introduced.

1. INTRODUCTION

Elastic-plastic analysis under the assumption of small deformation by the finite element method has been studied by many research workers such as Yamada, Yhshimura and Sakurai[1], Marcal et. al.[2)~5], Armen, Isakson and Pifko[6),7], Zienkiewicz, Valliappan and King[8] and others[9)~12]. Survey papers presented by Yamada[13], Yamaguchi[14], Marcal[15] and Oden[16] clarified the outline of the theories. Finite element applications to continuous media on the basis of finite strain theory are recently contributed by Oden[17),18], Kawai[19] and others. Especially, the analysis of a finite strain elastic body is already published by Oden and Sato[20], Oden and Kubitzua[21], Haltz and Nather[22], Seguchi and Shindo[23] and others.

Elastic-plastic constitutive equations using finite strain theory have been developed on the basis of the thermodynamic considerations. Green and Naghdi[24),25] derived the constitutive equations which were stated as the relations between the Green's strain tensor and the Kirchhoff's stress tensor. Yoshimura[26] studied the relations based on the strain increment, and others[27] led to the relations between the deformation rate tensor and stress flux tensor.

Elastic-plastic bodies treated in this paper are defined using the Green's strain tensor and the Kirchhoff's stress tensor. Let it be assumed that the Green's strain tensor consists of the elastic and plastic strain tensor, and that the Helmholtz free energy is a function only of an elastic strain and an absolute temperature. The constitutive equations are led to be expressed by the incremental relations between strain and stress tensor.

Analytical solution of these media is much complicated and seems to be difficult in practical use. Yamada, Yoshimura and Sakurai[1], Oden and Kubitzua[21], and Hofmeister, Greenbaum and Evensen[28] presented an approach by means of the finite element method. Since the constitutive equations are given as the incremental relations, the governing equations become also incremental form in terms of the increments of load and displacement in the elastic-plastic problem mensioned above.

Those equations are generally called the incremental equations. In the numerical computation of the incremental equations, it is noted that the different load increment sometimes leads to the different solution. Consequently, the several practical techniques have been introduced to calculate the incremental governing equations.

This paper treats the method of solution summarized as follows : The governing equations are transformed into the relations between the total displacement and external load, and the transformed equations are similar to the equations in the method of direct formulation. However, these are actually the equations of the incremental formulation, because the above governing equations include the stress and displacement in the reference configuration.

The Newton-Raphson method using the first and the second order derivatives of the objective equations is applied to the nonlinear simultaneous formulations, which is generally solved by Newton-

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Raphson method only with the first order derivatives. This second order method brings the better convergency in the numerical solution than the first order method.

2. KINEMATICAL EQUATIONS

Throughout this paper, the material description is employed and the summation convention with repeated indices is introduced. Material and spatial coordinates are denoted by X_K (K=1, 2, 3) and x_k (k=1, 2, 3) respectively, and in the reference state, both are chosen to be coincident each other and to be rectangular Cartesian coordinates as shown in Fig. 1. The components of the displacement vector with respect to X_K are denoted by U_K and can be expressed by equation (2.1).

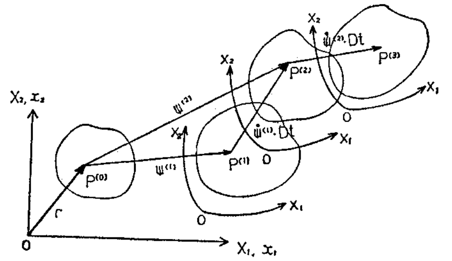

Fig. 1 Reference frames and motion of a body (two dimensional case)

$$x_k=\delta_{kK}U_K+\delta_{kK}X_K \quad \cdots\cdots(2.1)$$

in which δ_{kK} is a shifter. This paper deals with the case that δ_{kK} can be regarded to be Kronecker's delta function.

$$\left.\begin{array}{l}\delta_{kK}=1 \text{ if } k=K\\ \delta_{kK}=0 \text{ if } k\neq K\end{array}\right\}\cdots\cdots(2.2)$$

Velocity vector v_k at x_k can be written as,

$$v_k=\dot{x}_k \quad \cdots\cdots(2.3)$$

where superposed dot denotes the material differentiation with respect to time. Using equation (2.1), equation (2.3) is transformed into equation (2.4).

$$v_k=\delta_{kK}\dot{U}_K \quad \cdots\cdots(2.4)$$

The material time differentiation can be reduced to the usual partial differentiation in the material coordinates system. The deformation gradient tensor $x_{k,K}$ is obtained from equation (2.1) as in the following :

$$x_{k,K}=\delta_{kK}+\delta_{kL}U_{L,K}=\delta_{kM}(\delta_{MK}+U_{M,K}) \quad \cdots\cdots(2.5)$$

in which, K denotes the differention $\partial/\partial X_K$. Using $x_{k,K}$, Green's strain tensor e_{KL} is defined by the following equation :

$$2e_{KL}=x_{k,K}x_{k,L}-\delta_{KL} \quad \cdots\cdots(2.6)$$

in which δ_{KL} is the Kronecker's delta function. Making use of equation (2.5), equation (2.6) can be transformed into the form expressed in terms of U_K.

$$2e_{KL}=U_{K,L}+U_{L,K}+U_{M,K}U_{M,L} \quad \cdots\cdots(2.7)$$

The rate of strain tensor $\dot{e}_{KL}$ is obtained by differentiating both sides of equation (2.6) with respect to time as in equation (2.8).

$$\dot{e}_{KL}=d_{kl}x_{l,K}x_{k,L} \quad \cdots\cdots(2.8)$$

where

$$d_{kl}=\frac{1}{2}(v_{k,l}+v_{l,k}) \quad \cdots\cdots(2.9)$$

in which d_{kl} is named the deformation rate tensor. In the derivation of equation (2.8), equation (2.10) is used.

$$\dot{x}_{k,K}=v_{k,l}x_{l,K}=\delta_{kL}\dot{U}_{L,K} \quad \cdots\cdots(2.10)$$

Substitution of equation (2.4) and equation (2.5) into equation (2.8) and use of equation (2.10) lead to equation (2.11).

$$2\dot{e}_{KL}=\dot{U}_{K,L}+\dot{U}_{L,K}+\dot{U}_{M,K}U_{M,L}+U_{M,K}\dot{U}_{M,L} \quad \cdots\cdots(2.11)$$

3. EQUILIBRIUM EQUATIONS

It is assumed that the strain tensor e_{KL} can be decomposed into elastic strain tensor e_{KL}' and plastic strain tensor e_{KL}'', i.e.

$$e_{KL}=e_{KL}'+e_{KL}'' \quad \cdots\cdots(3.1)$$

Local from of energy balance equation can be expressed in the form of equation (3.2).

$$\rho_0 r-\rho_0\dot{U}+S_{KL}\dot{e}_{KL}-Q_{K,K}=0 \quad \cdots\cdots(3.2)$$

in which ρ_0, r, U and Q_K are the mass density, the heat sourse per unit mass, the internal energy per unit mass and the heat flux vector, respectively. These are the quantities measured in the reference state. S_{KL} is the Kirchhoff's stress tensor, which is the stress measured per unit undeformed area referred to the deformed X_K coordinate. Denoting the stress vector by $\boldsymbol{t}$,

$$\boldsymbol{t}=S_{KL}N_K\boldsymbol{G}_L \quad \cdots\cdots(3.3)$$

in which N_K is the outward unit normal of the underformed surface area, and $\boldsymbol{G}_L$ is the base vectors of the deformed material coordinates i.e.,

$$\boldsymbol{G}_L=(\delta_{KL}+U_{K,L})\boldsymbol{I}_K \quad \cdots\cdots(3.4)$$

where $\boldsymbol{I}_K$ is the base vectors of the undeformed material coordinates.

Clausius-Duhem inequality in terms of the quantities in the reference state can be described in the following form :

$$\rho_0 T\dot{S}-\rho_0 r+Q_{K,K}-\frac{Q_K T_K}{T}\geqq 0 \quad \cdots\cdots(3.5)$$

in which S is the specific entropy and T denotes the absolute temperature. Stress equilibrium equations can be written as equation (3.6), using the body force F_k.

$$(x_{k,K}S_{KL})_{,L}+\rho_0 F_k=0 \quad \cdots\cdots(3.6)$$

In this stage, the Gibbs equation (3.7) is introduced.

$$\rho_0 \dot{S} T = \rho_0 \dot{U} - S_{KL} \dot{e}_{KL}' \cdots\cdots (3.7)$$

With the aid of equation (3.1) and equation (3.7), equation (3.2) and (3.5) can be transformed into equation (3.8) and equation (3.9), respectively.

$$\rho_0 r - \rho_0 T \dot{S} - Q_{K,K} + S_{KL} \dot{e}_{KL}'' = 0 \cdots\cdots (3.8)$$

$$S_{KL} \dot{e}_{KL}'' - \frac{Q_K T_K}{T} \geqq 0 \cdots\cdots (3.9)$$

In the case that $Q_K=0$ and $r=0$, equation (3.8) and equation (3.9) can be reduced to the followings.

$$\rho_0 T \dot{S} = S_{KL} \dot{e}'' \geqq 0 \cdots\cdots (3.10)$$

The first part of equation (3.10) means that the specific entropy is caused by the rate of plastic work $S_{KL}\dot{e}_{KL}''$, and the second part of equation (3.10) states that the rate of plastic work has to be positive or equal to zero.

Introducing the Helmholtz free energy,

$$F = U - TS \cdots\cdots (3.11)$$

and substituting this into equation (3.7), equation (3.12) associated with elastic strain can be obtained.

$$\rho_0 \dot{F} = -\rho_0 S \dot{T} + S_{KL} e_{KL}' \cdots\cdots (3.12)$$

Assuming that the free energy F is the function of only e_{KL}' and T, as in equation (3.13):

$$F = F(e_{KL}', T) \cdots\cdots (3.13)$$

equation (3.14) is followed.

$$\rho_0 \dot{F} = \rho_0 \frac{\partial F}{\partial e_{KL}'} \cdot \dot{e}_{KL}' + \rho_0 \frac{\partial F}{\partial T} \dot{T} \cdots\cdots (3.14)$$

Comparison between equation (3.14) and equation (3.12) leads to the following equation associated with the specific entropy S and the stress S_{KL}.

$$S = -\frac{\partial F}{\partial T} \cdots\cdots (3.15)$$

$$S_{KL} = \rho_0 \frac{\partial F}{\partial e_{KL}'} \cdots\cdots (3.16)$$

4. EQUATIONS OF VIRTUAL WORK

The formulation of the finite element method requires the equations of the virtual work. Both the direct and incremental forms of the virtual work equations are derived with the aid of the stress equilibrium equations. Multiplying both sides of stress equilibrium equation (3.6) by v_k and integrating throughout the whole volume of a body, equation (4.1) can be obtained.

$$\int_{V_0} (x_{k,K} S_{KL})_{,L} v_k dV_0 + \int_{V_0} \rho_0 F_k v_k dV_0 = 0 \cdots\cdots (4.1)$$

where V_0 and A_0 denote the undeformed volume of the body and its surrounding surface area.

Application of the Green-Gauss theorem to the first term of the left hand side of equation (4.1) leads to the following direct type virtual work equations.

$$\int_{V_0} (S_{KL} \dot{e}_{KL}) dV_0 = \Omega \cdots\cdots (4.2)$$

where

$$\Omega = \int_{A_0} P_N \dot{U}_N dA_0 + \int_{V_0} \rho_0 F_N \dot{U}_N dV_0 \cdots\cdots (4.3)$$

in which P_N is the surface force relating to the stress shown in equation (4.4).

$$P_N = S_{KL}(\delta_{NK} + U_{N,K}) N_L \cdots\cdots (4.4)$$

In the elastic-plastic problems, the incremental type virtual work equations are required, because the constitutive equations are introduced in term of $\dot{S}_{KL}$ and $\dot{e}_{KL}$. Differentiating both sides of equation (3.6) with respect to time, equation (4.5) is obtaind.

$$(\dot{x}_{k,K} S_{KL} + \dot{S}_{KL} x_{k,K})_{,L} + \rho_0 \dot{F}_k = 0 \cdots\cdots (4.5)$$

Multiplying both sides of equation (4.5) by v_k, integrating throughout the whole volume of the body, and using the Green-Gauss theorem leads to equation (4.6).

$$\int_{V_0} (\dot{S}_{KL} \dot{e}_{KL} + S_{KL} \dot{x}_{kL} \delta_{kN} \dot{U}_{N,K}) dV_0 = \int_{A_0} (\dot{S}_{KL} x_{k,K} + S_{KL} \dot{x}_{k,N}) N_L \delta_{kM} \dot{U}_M dA_0 + \int_{V_0} \rho_0 \dot{F}_N \dot{U}_N dV_0 \cdots\cdots (4.6)$$

Substituting equation (2.5) and equation (2.10) into equation (4.6), the incremental type virtual work equation can be described in the following form.

$$\int_{V_0} (\dot{S}_{KL} \dot{e}_{KL} + S_{KL} \dot{U}_{M,K} \dot{U}_{M,L}) dV_0 = \dot{\Omega} \cdots\cdots (4.7)$$

where

$$\dot{\Omega} = \int_{A_0} \dot{P}_N \dot{U}_N dA_0 + \int_{V_0} \rho_0 \dot{F}_N \dot{U}_N dV_0 \cdots\cdots (4.8)$$

in which $\dot{P}_N$ is the rate of the surface force relating to the stress and stress rate shown in equation (4.9).

$$\dot{P}_N = [\dot{S}_{KL}(\delta_{NK} + U_{N,K}) + S_{KL} \dot{U}_{N,K}] N_L \cdots (4.9)$$

Equation (4.7) can also be derived by differentiating both sides of equation (4.2) and by using the following relation neglecting accelelation.

$$\ddot{U}_K = 0 \cdots\cdots (4.10)$$

$$\ddot{e}_{KL} = \dot{U}_{M,K} \dot{U}_{M,L} \cdots\cdots (4.11)$$

Namely, equation (4.7) can be rewritten as,

$$\int_{V_0} (\dot{S}_{KL} \dot{e}_{KL} + S_{KL} \ddot{e}_{KL}) dV_0 = \dot{\Omega} \cdots\cdots (4.12)$$

In the practical computation of the elastic-plastic problems, it is convenient to assume that the stress S_{KL} is constant during the small increment of the external load. Therefore, equation (4.7) and equation (4.12) can be transformed into equation (4.13) and equation (4.14), respectively.

$$\int_{V_0} (\dot{S}_{KL} \dot{e}_{KL} + S_{KL}(0) \dot{U}_{M,K} \dot{U}_{M,L}) dV_0 = \dot{\Omega} \cdots\cdots (4.13)$$

$$\int_{V_0} (\dot{S}_{KL} \dot{e}_{KL} + S_{KL}(0) \ddot{e}_{KL}) dV_0 = \dot{\Omega} \cdots\cdots (4.14)$$

where S_{KL} (0) denotes the reference stress in the state before the small increment of the external load is applied and is assumed to be constant during the application of the increment.

5. CONSTITUTIVE EQUATIONS

For simplicity, the thermal effects in the yielding function of elastic-plastic bodies are neglected in this paper. The constitutive equations of elastic-plastic bodies are given by the relation between stress increment $\dot{S}_{KL}$ and strain increment $\dot{e}_{KL}$. Assumming that the yielding function f is the function of stress S_{KL} and plastic strain e_{KL}'', equation (5.1) can be obtained with work hardening parameter κ.

$$f(S_{KL}, e_{KL}'')=\kappa \qquad (5.1)$$

Using the function f, the following four conditions can be defined :

i Elastic state ($\dot{e}_{KL}''=0$)

$$f<\kappa \text{ with } \dot{\kappa}=0 \qquad (5.2)$$

ii Loading from an elastic-plastic state ($\dot{e}_{KL}''\neq 0$)

$$f=\kappa \text{ with } \dot{\kappa}\neq 0 \text{ and } \frac{\partial f}{\partial S_{KL}}\dot{S}_{KL}>0 \qquad (5.3)$$

iii Unloading from an elastic-plastic state ($\dot{e}_{KL}''=0$)

$$f=\kappa \text{ with } \dot{\kappa}=0 \text{ and } \frac{\partial f}{\partial S_{KL}}\dot{S}_{KL}<0 \qquad (5.4)$$

iv Neutral state ($\dot{e}_{KL}''=0$)

$$f=\kappa \text{ with } \dot{\kappa}=0 \text{ and } \frac{\partial f}{\partial S_{KL}}\dot{S}_{KL}=0 \qquad (5.5)$$

With respect to plastic strain increment $\dot{e}_{KL}''$, the flow rule is introduced as in equation (5.6).

$$\dot{e}_{KL}''=\Lambda\frac{\partial f}{\partial S_{KL}} \qquad (5.6)$$

where Λ is a scalar. Elastic strain increment is related to stress increment $\dot{S}_{KL}$ as shown in equation (5.7) :

$$\dot{S}_{KL}=A_{KLMN}\dot{e}_{MN}'+B_{KL}\dot{T} \qquad (5.7)$$

where

$$A_{KLMN}=\rho_0\frac{\partial^2 F}{\partial e_{KL}'\partial e_{MN}'}$$

$$B_{KL}=\rho_0\frac{\partial^2 F}{\partial e_{KL}'\partial T}$$

Equation (5.7) can be derived by differentiating both sides of equation (3.16) with respect to time and by using equation (3.14). Substituting equation (3.1) into equation (5.7) leads to equation (5.8).

$$\dot{S}_{KL}=A_{KLMN}\dot{e}_{MN}-A_{KLMN}\dot{e}_{MN}''+B_{KL}\dot{T} \qquad (5.8)$$

Differentiating both sides of equation (5.1) with respect to time and considering that κ is the function of e_{KL}'' only, equation (5.9) can be obtained.

$$\frac{\partial f}{\partial S_{KL}}\dot{S}_{KL}+\frac{\partial f}{\partial e''_{KL}}\dot{e}_{KL}''=\frac{\partial \kappa}{\partial e_{KL}''}\cdot\dot{e}_{KL}'' \qquad (5.9)$$

Combining equation (5.6), equation (5.8) and equation (5.9), Λ can be expressed in terms of $\dot{e}_{KL}$ and $\dot{T}$. Using the resulted Λ and using again equation (5.6) and equation (5.8), the constitutive equations of elastic-plastic bodies can be written as the relation between stress increment $\dot{S}_{KL}$, strain increment $\dot{e}_{KL}$ and temperature increment $\dot{T}$.

$$\dot{S}_{KL}=C_{KLMN}\dot{e}_{MN}+D_{KL}\dot{T} \qquad (5.10)$$

where

$$C_{KLMN}=A_{KLMN}-\frac{A_{KLPQ}\frac{\partial f}{\partial S_{PQ}}\cdot\frac{\partial f}{\partial S_{RS}}A_{RSMN}}{\left(\frac{\partial f}{\partial S_{MN}}\right)\cdot\left(\frac{\partial \kappa}{\partial e_{MN}''}-\frac{\partial f}{\partial e_{MN}''}+A_{PQMN}\frac{\partial f}{\partial S_{PQ}}\right)}$$

$$D_{KL}=B_{KL}-\frac{A_{KLPQ}\frac{\partial f}{\partial S_{PQ}}\cdot\frac{\partial f}{\partial S_{RS}}B_{RS}}{\left(\frac{\partial f}{\partial S_{MN}}\right)\cdot\left(\frac{\partial \kappa}{\partial e_{MN}''}-\frac{\partial f}{\partial e_{MN}''}+A_{PQMN}\frac{\partial f}{\partial S_{PQ}}\right)}$$

In the loading state, i.e. under the condition of equation (5.3), the constitutive equation is given in equation (5.10). Under the other conditions expessed by equations (5.2), (5.4) and (5.5), the constitutive equation is reduced to equation (5.7). Equation (5.10) is a generalized formulation of the relation developed by Yamada, Yoshimura and Sakurai[1], Marcal[2], and Oden[16] using Green's strain tensor and Kirchhoff's stress tensor. Equation (5.10) is valid under the restriction expressed by equation (3.9), and it is clear that C_{KLMN} has to be positive definite.

In order to illustrate the method of solution, one of the special case of equation (5.10) is described in the followings, based on the assumption of free energy F given in equation (5.11).

$$\rho_0 F=E_{KLMN}e_{KL}'e_{MN}'-\beta\delta_{KL}e_{KL}'(T-T_0)-\frac{C_\varepsilon}{2T_0}(T-T_0)^2 \qquad (5.11)$$

where

$$E_{KLMN}=\frac{\nu E}{(1-2\nu)(1+\nu)}\delta_{KL}\delta_{MN}+\frac{E}{2(1+\nu)}(\delta_{KM}\delta_{LN}+\delta_{KN}\delta_{LM})$$

$$\beta=\frac{\alpha E}{1-2\nu}$$

in which E, ν, α, C_ε and T_0 is elastic modulus, Poisson's ratio, thermal expansion coefficient, specific heat and temperature in natural state, respectively. In this case, equation (5.7) becomes,

$$\dot{S}_{KL}=E_{KLMN}\dot{e}_{MN}'-\beta\delta_{KL}\dot{T} \qquad (5.12)$$

Von Mieses yielding criterion is supposed as :

$$f=\sqrt{3J_2'} \qquad (5.13)$$

where

$$J_2'=\frac{1}{2}S_{KL}'S_{KK}',\quad S_{KL}'=S_{KL}-\frac{S_{MM}}{3}\delta_{KL}$$

Using equation (5.11) and equation (5.12), equation (5.10) can be reduced into equation (5.14).

$$\dot{S}_{KL}=C_{KLMN}\dot{e}_{MN}+D_{KL}\dot{T} \cdots\cdots (5.14)$$

where

$$C_{KLMN}=E_{KLMN}-\frac{E_{KLPQ}\frac{\partial f}{\partial S_{PQ}}\cdot\frac{\partial f}{\partial S_{RS}}E_{RSMN}}{H'+\frac{\partial f}{\partial S_{PQ}}E_{PQRS}\frac{\partial f}{\partial S_{RS}}}$$

$$D_{KL}=\beta\delta_{KL}-\frac{E_{KLPQ}\frac{\partial f}{\partial S_{PQ}}\cdot\frac{\partial f}{\partial S_{RS}}\beta\delta_{RS}}{H'+\frac{\partial f}{\partial S_{PQ}}E_{PQRS}\frac{\partial f}{\partial S_{RS}}}$$

$$\frac{\partial f}{\partial S_{KL}}=\frac{\sqrt{3}}{2}\frac{S_{KL}'}{\sqrt{J_2'}}$$

in which H' is strain hardening coefficient given by the experimental ways. Equation (5.14) corresponds to the one obtained by Yamada, Yoshimura and Sakurai[1] in the form of the relation between Kirchhoff's stress tensor and Green's strain tensor.

6. FINITE ELEMENT ANALYSIS

It is supposed that the continuous medium is divided into the several small media called finite elements. The displacement of α th node in the K direction is denoted by $U_{\alpha K}$ on each finite element. With the aid of shape function Φ_α, the displacement inside the finite element U_K is approximated to be,

$$U_K=\Phi_\alpha U_{\alpha K} \cdots\cdots (6.1)$$

Substituting this into equation (2.7), e_{KL} can be expressed as follows.

$$2e_{KL}=\Phi_{\alpha,K}U_{\alpha L}+\Phi_{\alpha,L}U_{\alpha K}+\Phi_{\alpha,K}\Phi_{\beta,L}U_{\alpha M}U_{\beta M} \cdots\cdots (6.2)$$

Strain rate $\dot{e}_{KL}$ can be described as in equation (6.3), using equation (6.1) and equation (2.11).

$$2\dot{e}_{KL}=\Phi_{\alpha,K}\dot{U}_{\alpha L}+\Phi_{\alpha,L}\dot{U}_{\alpha K}+\Phi_{\alpha,K}\Phi_{\beta,L}\dot{U}_{\alpha M}U_{\beta M}+\Phi_{\alpha,K}\Phi_{\beta,L}U_{\alpha M}\dot{U}_{\beta M} \cdots\cdots (6.3)$$

Substituting equation (6.1) and equation (6.3) into equation (4.13) and using the relations,

$$\dot{S}_{KL}\dot{e}_{KL}=\dot{S}_{KL}(\delta_{LM}+\Phi_{\beta,L}U_{\beta M})\Phi_{\alpha,K}\dot{U}_{\alpha M} \cdots\cdots (6.4)$$

$$S_{KL}(0)\dot{U}_{M,K}\dot{U}_{M,L}=S_{KL}(0)\Phi_{\alpha,K}\Phi_{\beta,L}\dot{U}_{\alpha M}\dot{U}_{\beta M} \cdots\cdots (6.5)$$

equation (6.6) can be obtained.

$$\dot{U}_{\beta I}\dot{T}_{\beta I}=\dot{U}_{\beta I}\dot{\Omega}_{\beta I} \cdots\cdots (6.6)$$

where

$$\dot{T}_{\beta I}=\int_{V_0}\{\dot{S}_{KL}(\delta_{LI}+\Phi_{\alpha,L}U_{\alpha I})\cdot\Phi_{\beta,K}+S_{KL}(0)\Phi_{\alpha,K}\Phi_{\beta,L}\dot{U}_{\alpha I}\}dV_0 \cdots\cdots (6.7)$$

$$\dot{\Omega}_{\beta I}=\int_{V_0}\rho_0\dot{F}_I\Phi_\beta dV_0+\int_{A_0}\dot{P}_I\Phi_\beta dA \cdots\cdots (6.8)$$

in which $\dot{P}_I$ is related to $\dot{S}_{KL}$ as shown in equation (6.9).

$$\dot{P}_I=\{\dot{S}_{KL}(\delta_{IK}+\Phi_{\alpha,K}U_{\alpha I})+S_{KL}(0)\Phi_{\alpha,K}\dot{U}_{\alpha I}\}N_L \cdots\cdots (6.9)$$

Since $\dot{U}_{\beta I}$ have to be arbitrary in equation (6.6), the following simultaneous equations can be drived.

$$\dot{T}_{\beta I}=\dot{\Omega}_{\beta I} \cdots\cdots (6.10)$$

The stress-strain equations described in equation (5.10) can be transformed into equation (6.11) with the use of equation (6.3).

$$\dot{S}_{KL}=C_{KLMN}(\delta_{MJ}+\Phi_{\gamma,M}U_{\gamma J})\Phi_{\alpha,N}\dot{U}_{\alpha J}+D_{KL}\dot{T} \cdots\cdots (6.11)$$

Combining equation (6.10) and equation (6.11), and rearranging dummy indices, the governing equations for the finite element method can be obtained in the following incremental form.

$$K_{\beta I\alpha J}\cdot\dot{U}_{\alpha J}=\dot{\Omega}_{\beta I}-\dot{\Omega}_{\beta I}^{*} \cdots\cdots (6.12)$$

where

$$K_{\beta I\alpha J}=\int_{V_0}[\Phi_{\beta,K}(\delta_{LI}+\Phi_{\delta,L}U_{\delta I})\cdot C_{KLMN}(\delta_{MJ}+\Phi_{\gamma,M}U_{\gamma J})\Phi_{\alpha,N}+\delta_{IJ}S_{KL}(0)\Phi_{\alpha,K}\Phi_{\beta,L}]dV_0 \cdots\cdots (6.13)$$

$$\dot{\Omega}_{\beta I}^{*}=\int_{V_0}D_{KL}\dot{T}(\delta_{LI}+\Phi_{\alpha,L}U_{\alpha I})\Phi_{\beta,K}dV_0 \cdots\cdots (6.14)$$

$\dot{\Omega}_{\beta I}$ in equation (6.12) corresponds to the equivalent nodal force caused by the increments of body force $\dot{F}_I$ and surface force $\dot{P}_I$, $\dot{\Omega}_{\beta I}^{*}$ is the equivalent nodal force resulted from the incremets of temperature $\dot{T}$. The right hand side of equation (6.12) is all known terms, and the coefficients of the left hand side consist of $S_{KL}(0)$ and $U_{\alpha K}$ in the reference configuration. In the state that the external load increments are applied, on the assumption that the influence of the changings of $S_{KL}(0)$ and $U_{\alpha K}$ to $K_{\alpha J\beta I}$ can be regarded to be negligibly small, $K_{\beta I\alpha J}$ can be considered as a known and constant array.

In the same manner as of the conventional finite element method, the simultaneous equations on the whole continuum are easily constructed by using the equilibrium equations of the nodal force and the continuity equations of the nodal displacement. The equations can be derived as the relations between the increments of nodal displacement and external load. If $K_{\beta I\alpha J}$ can be considered to be constant, the equations become linear simultaneous ones. The increments of displacement can be obtained by these equations applying the increments of external force. The increments of stress can be derived by substituting these into equation (6.3) and equation (5.10). Adding the increments of stress to the stress in the reference configuration, and referring to the conditions described in equation (5.2)~(5.5), the new coefficients C_{KLMN} in equation (5.10) can be derived. With the use of the new C_{KLMN}, the next cycle of the increments of applied external force is to be followed.

7. ALTERNATIVE APPROACH TO FINITE ELEMENT ANALYSIS

It is noted that by the method described in the proceeding paragraph the different increments of external load sometimes yield the different increments of displacement. Therefore, some iteration computations are necessary. In this paragraph the governing equations are transformed into the convenient forms for the application of the Newton-Raphson iterative method.

To start with, the stress-strain equations is transformed into equation (7.1) under the assumption that increments of external load are small.

$$S_{KL}-S_{KL}(0)=C_{KLMN}(e_{MN}-e_{MN}(0))+D_{KL}\dot{T} \quad \cdots\cdots(7.1)$$

where $S_{KL}(0)$ and $e_{MN}(0)$ denote the stress and strain respectively, in the reference configuration. S_{KL} and e_{MN} are the stress and strain, respectively, after the increments of external load are applied. C_{KLMN} is considered to be constant in each load increment.

Equation (7.1) can be rewritten as,

$$S_{KL}=C_{KLMN}e_{MN}+\bar{S}_{KL}(0) \quad \cdots\cdots(7.2)$$

where

$$\bar{S}_{KL}(0)=S_{KL}(0)-C_{KLMN}e_{MN}(0)+D_{KL}\dot{T} \quad \cdots\cdots(7.3)$$

It follows from equation (4.3) and equation (7.2) that

$$\int_{V_0}C_{KLMN}e_{MN}\dot{e}_{KL}dV_0=\Omega-\int_{V_0}\bar{S}_{KL}(0)\dot{e}_{KL}dV_0 \quad \cdots\cdots(7.4)$$

Introducing equations (6.1), (6.2) and (6.3) into (7.4) and noting that $\dot{U}_{\beta I}$ is arbitrary, equation (7.5) can be obtained.

$$\bar{K}_{\beta I\alpha J}\cdot U_{\alpha J}=\Omega_{\beta I}-\bar{\Omega}_{\beta I} \quad \cdots\cdots(7.5)$$

where

$$\bar{K}_{\beta I\alpha J}=\int_{V_0}\Big[\Phi_{\beta,K}(\delta_{LI}+\Phi_{\delta,L}U_{\delta I})C_{KLMN}\cdot\left(\delta_{MJ}+\frac{1}{2}\Phi_{\gamma,M}U_{\gamma J}\right)\Phi_{\alpha,N}\Big]dV_0$$

$$\Omega_{\beta I}=\int_{A_0}P_I\Phi_\beta dA_0+\int_{V_0}\rho_0F_I\Phi_\beta dV_0$$

$$\bar{\Omega}_{\beta I}=\int_{V_0}[\bar{S}_{KL}(0)(\delta_{LI}+\Phi_{\gamma,L}U_{\gamma I})\Phi_{\beta,K}]dV_0$$

Equation (7.5) is formal relations between displacement $U_{\beta I}$ and equivalent nodal force $\Omega_{\beta I}-\bar{\Omega}_{\beta I}$. However, since the term $\bar{\Omega}_{\beta I}$ includes the stress $S_{KL}(0)$ and strain $e_{MN}(0)$ in the reference configuration, equation (7.5) can be regarded as the modified incremental equations. The subsequent procedures of solution are the same as those of equation (6.12). It is convenient to apply the Newton-Raphson method to equation (7.5), because the equation is the third order simultaneous equations in $U_{\alpha J}$, making it simple to calculate the first and second order derivatives.

Correspondence between equation (6.12) and equation (7.5) is stated in the followings. Equation (7.4) can be rewritten as in equation (7.6).

$$\int_{V_0}[\{C_{KLMN}(e_{MN}-e_{MN}(0))\dot{e}_{KL}+S_{KL}(0)\ddot{e}_{KL}\}+D_{KL}\dot{T}]dV_0=\int_{A_0}P_N\dot{U}_NdA_0+\int_{V_0}\rho_0F_N\dot{U}_NdV_0-\int_{V_0}S_{KL}(0)\dot{e}_{KL}(0)dV_0 \quad \cdots\cdots(7.6)$$

Substituting equation (4.2) into the third term of the right hand side of equation (7.6) and rearranging it lead to,

$$\int_{V_0}(\dot{S}_{KL}\dot{e}_{KL}+S_{KL}(0)\ddot{e}_{KL})dV_0=\int_{A_0}(P_N-P_N(0))\dot{U}_NdA_0+\int_{V_0}\rho_0(F_N-F_N(0))\dot{U}_NdV_0 \quad \cdots\cdots(7.7)$$

Equation (7.7) can be rewritten as,

$$\int_{V_0}(\dot{S}_{KL}\dot{e}_{KL}+S_{KL}(0)\ddot{e}_{KL})dV_0=\dot{\Omega} \quad \cdots\cdots(7.8)$$

where

$$\dot{\Omega}=\int_{A_0}\dot{P}_N\dot{U}_NdA_0+\int_{V_0}\rho_0\dot{F}_N\dot{U}_NdV_0 \quad \cdots\cdots(7.9)$$

in which the following relations are used.

$$\dot{P}_N=P_N-P_N(0) \quad \cdots\cdots(7.10)$$

$$\dot{F}_N=F_N-F_N(0) \quad \cdots\cdots(7.11)$$

Equation (7.8) is the same equation as equation (4.14). Equation (7.5) is the governing equation based on equation (7.8), and equation (6.12) is derived from equation (4.14). Therefore, equation (7.5) can be regarded as the transformed relation of equation (6.12), which is convenient for the iterative computation as stated in the preceedings.

8. NEWTON-RAPHSON METHOD

In order to solve nonlinear algebraic simultaneous equations, Newton-Raphson method is generally introduced.

Nonlinear equations are denoted as,

$$^N\varphi(y_j)=0 \quad \cdots\cdots(8.1)$$

where $y_j(j=1, 2,\cdots M)$ is the unknowns, and M is their total numbers. N shows that the equation is the Nth equation of the whole system. $(N=1, 2, \cdots M)$. Expanding equation (8.1) with the use of power series around arbitrary initial values, it follows that

$$^N\varphi(y_j)={}^N\varphi(0)+{}^NJ_k(0)\cdot(y_k-y_k(0))+\frac{1}{2}{}^NJ_{kl}(0)\cdot(y_k-y_k(0))\cdot(y_l-y_l(0))+\frac{1}{6}{}^NJ_{klm}(0)\cdot(y_k-y_k(0))\cdot(y_l-y_l(0))\cdot(y_m-y_m(0)) \quad \cdots\cdots(8.2)$$

where

$$^NJ_k(0)=\left[\frac{\partial^N\varphi}{\partial y_k}(y_j(0))\right]$$

$$^NJ_{kl}(0)=\left[\frac{\partial^{2N}\varphi}{\partial y_k\partial y_l}(y_j(0))\right]$$

$$ {}^NJ_{klm}(0)=\left[\frac{\partial^3\,{}^N\varphi}{\partial y_k\partial y_l\partial y_m}(y_j(0))\right] $$

Neglecting terms of or higher than second order, equation (8.1) can be denoted as follows :

$$ {}^N\varphi(y_j)={}^N\varphi(0)+{}^NJ_k(0)\cdot(y_k-y_k(0))\approx 0 \quad\cdots\cdots(8.3) $$

y_k is calculated by equation (8.3), as shown in equation (8.4).

$$ y_k=y_k(0)-{}^NJ_k^{-1}(0)\cdot{}^N\varphi_k(0)\quad\cdots\cdots(8.4) $$

Thus, the following algorithm is obtained referring to equation (8.4).

$$ y_k^{(n+1)}=y_k^{(n)}-{}^NJ_k^{-1}(y_j^{(n)})\cdot{}^N\varphi_k(y_j^{(n)})\cdots(8.5) $$

where (n) is the number of iteration cycle.

Equation (8.5) is the first order convergence algorithm of Newton-Raphson method.

In this paper, second order convergence algorithm is also treated. Neglecting terms of or higher than third order in equation (8.2), equation (8.1) can be expressed as,

$$ {}^N\varphi(y_j)={}^N\varphi(0)+{}^NJ_k(0)\cdot(y_k-y_k(0)) +\frac{1}{2}{}^NJ_{kl}(0)\cdot(y_k-y_k(0))\cdot(y_l-y_l(0))\approx 0 \quad\cdots\cdots(8.6) $$

Letting y_k^* denote the corrected value of y_k, it follows from equation (8.4) that

$$ y_k^*-y_k=y_k^*-y_k(0)+{}^NJ_k^{-1}(0)\cdot{}^N\varphi_k(0)\cdots(8.7) $$

Substituting equation (8.7) into equation (8.6) and rearranging it, with the approximation of equation (8.8),

$$ y_k-y_k(0)\approx y_k^*-y_k-{}^NJ_k^{-1}(0){}^N\varphi_k(0)\cdots\cdots(8.8) $$

the following equation (8.9) is drived.

$$ {}^N\varphi_k(0)-{}^NJ_k^{-1}(0)\cdot{}^NJ_k(0)\cdot{}^N\varphi_k(0) +{}^NJ_k(0)(y_k^*-y_k)+{}^N\psi^*(0)=0\cdots\cdots(8.9) $$

where

$$ {}^N\psi^*(0)=\frac{1}{2}{}^NJ_{kl}(0)\cdot(y_k-y_k(0))\cdot(y_l-y_l(0)) \quad\cdots\cdots(8.10) $$

Equation (8.9) can be transformed into equation (8.11).

$$ y_k^*=y_k-{}^NJ_k^{-1}(0)\cdot{}^N\psi^*(0)\cdots\cdots(8.11) $$

Referring to equation (8.11), the following algorithm of the second order Newton-Raphson method is obtained.

$$ y_k^{*(n+1)}=y_k^{(n+1)}-{}^NJ_k^{-1}(y_j^{(n+1)})\cdot{}^N\varphi^*(y_j^{(n+1)}) \quad\cdots\cdots(8.12) $$

where

$$ {}^N\psi^*(y_j^{(n+1)})=\frac{1}{2}{}^NJ_{kl}(y_j^{(n+1)}) \cdot(y_k^{(n+1)}-y_k^{(n)})\cdot(y_l^{(n+1)}-y_l^{(n)})\cdots(8.13) $$

Starting from the appropriate initial values of $y_k(0)$, and using equation (8.5), (8.12) and (8.13), the iteration computation is proceeded.

The second order Newton-Raphson method mentioned above requires less iteration procedures than the first order Newton-Raphson method does.

In practical computation, ${}^NJ_{kl}$ needs the three dimensional array, large core storage and computational time, in spite of the easier formulation of ${}^NJ_{kl}$ with the use of equation (7.5) as the governing equation.

9. NUMERICAL EXAMPLES

In order to compare the incremental loading method (I.L.M.) with the direct iteration method (D.I.M.), simple numerical example is illustrated in Figure (2). The stress-strain equation is regarded to be expressed by that of Hooke's law. Thermal terms are all neglected. As the shape function, the constnt strain type function using area coordinates of a triangular finite element is introduced. Good agreement between I.L.M. and D.I.M. were found in the tension examples. However, in the case of compressive external loads, the different displacements were obtained corresponding to the different increments of external load. It is noted that the displacements of D.I.M. are larger than that of I.L.M. Figure (3) illustrates displacements calculated by D.I.M. both for tension and compression loads.

As to the elastic-plastic numerical example, the simple structure shown in figure (4) is dealt with. The material of the structure is considered to be

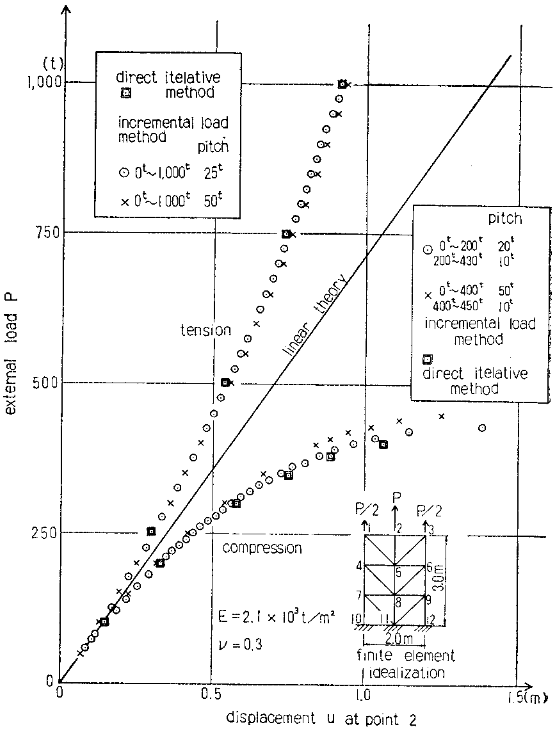

Fig. 2 Comparison of incremental load method with direct itelative method under elastic-stress strain relation.

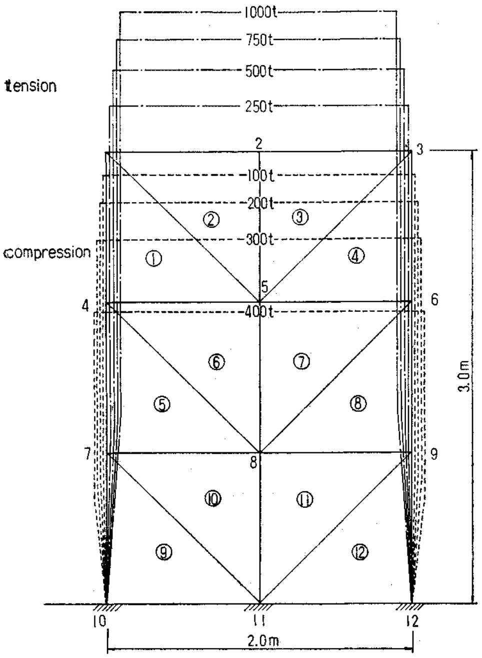

Fig. 3 Displacement

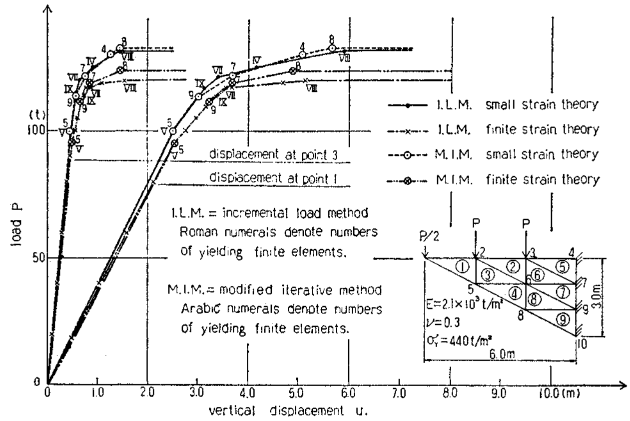

Fig. 4 Load displacement diagram

expressed by equation (5.14) in which $H'=0$ and $\dot{T}=0$. Figure (4) shows the load-displacement diagram. The numerical results derived from equation (6.12) and from equation (7.5) are denoted by I.L.M. and M.I.M., respectively. Numerals in the figure are the numbers of the yielded elements, the ordinates of which indicate the applied load at the yielding points. The figure also shows the comparison between the solutions under the small deformation assumption and the solutions on the basis of finite strain theory. On the whole, the results of M.I.M. give larger displacements than that of I.L.M.

In the computational procedure of M.I.M., Newton-Raphson method is available both for

Table 1 Comparison of iteration cycles

Newton Raphson method \ external force	tension		compression	
	1,000t	2,000t	400t	200t
first order convergence	4cycles	5cycles	5cycles	3cycles
second order convergence	3	3	4	2

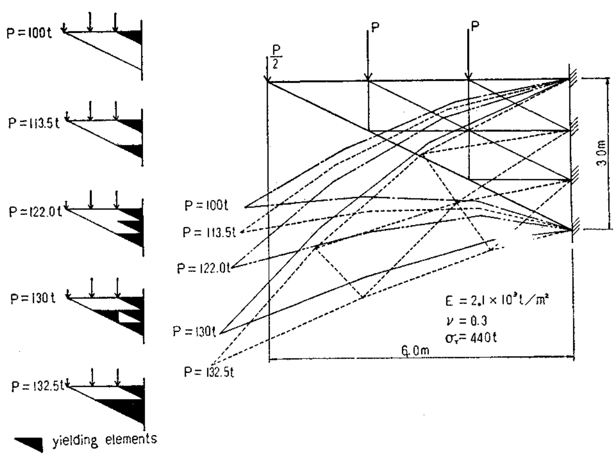

Fig. 5 Deformation of cantilever beam (small strain) and yielding elements.

equation (8.5) and equation (8.12). The comparison of the number of iteration cycles between the first and second order Newton-Raphson method is listed in Table (1). Both methods shows good agreement in the final results. In the procedure using equation (8.12), the time needed for the calculations of array ${}^{N}J_{kl}$ were so long that each algorithm did not make big differences in total times. The first order Newton-Raphson method would be convenient to calculate during the state of rather small external load. Figure (5) and Figure (6) show deformations of cantilever beam and yielding elements under small deformation assumption and on the

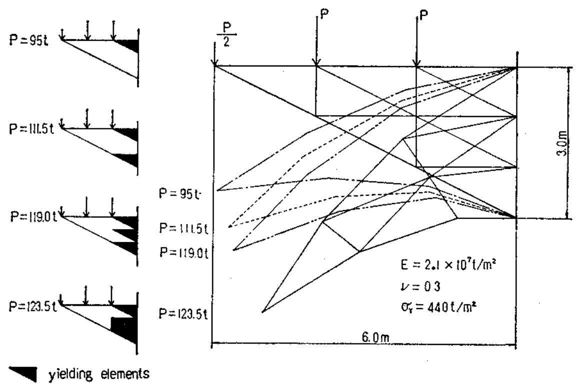

Fig. 6 Deformation of cantilever beam (finite strain) and yielding

basis of finite strain theory, respectively.

10. CONCLUDING REMARKS

The elastic-plastic numerical analysis in this paper is summarized as follows :

To start with, the Green's strain tensor is assumed to be decomposed into the elastic and plastic strain tensor.

The elastic strain is derived from the Helmholtz free energy, and the flow rule is introduced for the plastic strain.

The constitutive equations of the elastic-plastic bodies using Kirchhoff's stress tensor and Green's strain tensor are transformed into the incremental relations between stress rates and strain rates. These equations correspond to the generalized relations proposed by Yamada, Yoshimura and Sakurai[1], Marcal[2)~5)] and others.

Using the constitutive equations, the compatibility equations between strain and displacement and the equations of virtual work, which are derived from the equilibrium, are obtained as the incremental equations between external forces and displacements.

The alternative method is presented on the basis that the stress-strain equations given as the incremental form are transformed into the total strain relations between the stress and strain in the state after deformation.

Using these constitutive equations, the finite element equations of solution are obtained as the total formulation between external forces and displacements. The equations is formally given as the total formulation. However, since these include the stress and strain in the reference configuration, the relations are the incremental equations actually.

In the practical computation, it should be noted that the choice of the increments of the external forces is the main and important techniques. In order to calculate the incremental equations, it is important to know how to choose the external load increment, namely, to select the load step considering how each finite element may yield. The equations described in this paper is the alternative forms of the incremental equations for the purpose of making the Newton-Raphson method applicable. This method is convenient especially in case of applying the second order Newton-Raphson method.

On the assumption that the constitutive equation is given as the relation between Kirchhoff's stress tensor and Green's strain tensor, the generalized constitutive equation can be obtained as the form of equation (5.10) which is based on the free energy equation (3.13). Equation (5.10) can be applied to the every form of free energy assumption. Moreover, the final equations of the finite element method can be led to the convenient form for the practical computation.

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