Proc. of JSCE,
No. 188, April, 1971

RESPONSE OF A SINGLE DEGREE OF FREEDOM SYSTEM WITH PROBABILISTIC PARAMETERS

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I. INTRODUCTION

This study attempts to investigate the dynamic responses of a single degree of freedom system with probabilistic parameters. The structural model we consider herein is a mass, spring and desh pot system in which coefficients of spring and viscous damping are assumed to be random variables (Fig. 1). The assumption of these probabilistic parameters is justified by the fact that owing to the randomness in material properties and etc., the actual values of the parameters cannot be determined exactly.

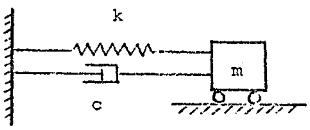

Fig. 1 One Degree of Freedom System

A perturbation method was used in the past for the research of free vibration of elastic beams with stochastic parameters by W.E. Boyce[1]. The author extended Boyce's work to free and forced vibrations of beam-columns and rectangular plates[2),3)]. The recent paper by the author[4] investigated a single degree of freedom system with probabilistic parameters by the same method.

A question has arisen on the errors in the series approximation of response parameters by the perturbation method. For instance, the errors in the probability distributions of output parameters such as natural circular frequency may increase with the increment of the dispersion of spring coefficient. The errors in the displacement parameters of the system may also increase with time. In this paper, the theoretical expressions for the exact output parameters such as natural circular frequency and displacement are developed by a direct approach and a Monte Carlo simulation. The errors by the perturbation method are examined through this study.

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II. FREE VIBRATION

Let us first discuss the response quantities of natural circular frequency, amplitude and phase angle of undamped free vibration of the system given in Fig. 1. The governing equation of the undamped free vibration is given by

$$m\ddot{x}+kx=0 \qquad (1)$$

and the solution is

$$x(t)=x_0\cos\omega t+\frac{\dot{x}_0}{\omega}\sin\omega t \qquad (2)$$

or

$$x(t)=a\cos(\omega t-\phi) \qquad (3)$$

where

m = mass
k = spring coefficient
ω = natural circular frequency and $\omega^2=k/m$
x_0 = initial displacement of mass
$\dot{x}_0$ = initial velocity of mass
a = amplitude $=(x_0^2+\dot{x}_0^2/\omega^2)^{1/2}$

and

ϕ = phase angle $=\tan^{-1}(\dot{x}_0/\omega x_0)$

If the probability density function of k is given, the probability density function of ω is derived from the following formula[5].

$$f_\Omega{}^D(\omega)=f_K(k=m\omega^2)\left|\frac{dk}{d\omega}\right| \qquad (4)$$

where $f_\Omega{}^D(\omega)$ and $f_K(k)$ are probability density functions of ω and k respectively. Super index D indicates the probability density function by the direct method, which is to be differentiated from the corresponding function by the perturbation method.

By introducing $|dk/d\omega|=2m\omega$ in eq. (4), we have

$$f_\Omega{}^D(\omega)=2m\omega f_K(m\omega^2) \qquad (5)$$

Since $k>0$, ω is also a positive random variable.

Amplitude, a is given by

$$a=(x_0^2+\dot{x}_0^2/\omega^2)^{1/2}$$

Differentiating this with respect to ω, we obtain

$$\frac{da}{d\omega}=-\frac{\dot{x}_0^2}{a}\left(\frac{\dot{x}_0^2}{a^2-x_0^2}\right)^{-3/2}$$

Hence, the probability density function of a by the direct method is

$$f_A{}^D(a)=f_\Omega{}^D(\omega)\left|\frac{d\omega}{da}\right|=f_\Omega{}^D\left(\sqrt{\frac{\dot{x}_0{}^2}{a^2-x_0{}^2}}\right)\frac{a}{\dot{x}_0{}^2}\times\left(\frac{\dot{x}_0{}^2}{a^2-x_0{}^2}\right)^{3/2}\cdots\cdots(6)$$

With eq. (5), this becomes

$$f_A{}^D(a)=\frac{2ma\dot{x}_0{}^2}{(a^2-x_0{}^2)^2}f_K\left(\frac{m\dot{x}_0{}^2}{a^2-x_0{}^2}\right)\cdots\cdots(7)$$

where $a\gtrapprox x_0$.

The probability density function of phase angle, ϕ is now obtained from $\phi=\tan^{-1}(\dot{x}_0/\omega x_0)$ and eq. (5) as follows.

$$f_\Phi{}^D(\phi)=f_\Omega{}^D(\omega)\left|\frac{d\omega}{d\phi}\right|=2m\left(\frac{\dot{x}_0}{x_0}\right)^2\frac{\cos\phi}{\sin^3\phi}f_K\left(\frac{m\dot{x}_0{}^2}{x_0{}^2\tan^2\phi}\right)\cdots\cdots(8)$$

On the other hand, the corresponding probability density functions of ω, a and ϕ by the perturbation method are given in reference[4] for a given normal density function of k as follows.

$$f_\Omega{}^P(\omega)\cong N\left(\sqrt{\frac{k_0}{m}},\ \frac{\sigma_k}{2\sqrt{k_0m}}\right)\cdots\cdots(9)$$

$$f_A{}^P(a)\cong N\left(\sqrt{x_0{}^2+\frac{\dot{x}_0{}^2m}{k_0}},\ \frac{\dot{x}_0{}^2m}{2k_0{}^2\sqrt{x_0{}^2+\frac{\dot{x}_0{}^2m}{k_0}}}\sigma_k\right)\cdots\cdots(10)$$

and

$$f_\Phi{}^P(\phi)=\frac{\dot{x}_0\sqrt{2k_0m}}{\sqrt{\pi}\,x_0\sigma_k\sin^2\phi}\exp\left\{-\frac{2k_0m}{\sigma_k{}^2}\left(\frac{\dot{x}_0}{x_0\tan\phi}-\sqrt{\frac{k_0}{m}}\right)^2\right\}\cdots\cdots(11)$$

where we assumed $k=k_0+\varepsilon_k$ in which k_0 is a deterministic constant value and ε_k is a random variable with zero mean and standard deviation σ_k. ε_k is small such that there is almost zero possibility that ε_k exceeds k_0. Symbol $N(,)$ indicates a normal density function with first and second arguments for mean and standard deviation respectively. We should note that if $\varepsilon_k=N(0,\sigma_k)$, then $k=N(k_0,\sigma_k)$.

Since the normal distributions are assumed for the perturbation method, we also assume normal distributions for the direct method. Letting

$$f_K(k)=\frac{1}{\sqrt{2\pi}\,\sigma_k}\exp\left\{-\frac{(k-k_0)^2}{2\sigma_k{}^2}\right\}\cdots\cdots(12)$$

and putting eq. (12) in eqs. (5), (7) and (8) we make the following ratio functions for the two methods.

$$R_\Omega=\frac{f_\Omega{}^D(\omega)}{f_\Omega{}^P(\omega)}=\frac{\omega}{\omega_0}\exp\left\{-\frac{k_0{}^2}{2\sigma_k{}^2}\left(\frac{\omega}{\omega_0}-1\right)^3\left(\frac{\omega}{\omega_0}+3\right)\right\}\cdots\cdots(13)$$

$$R_A=\frac{f_A{}^D(a)}{f^P{}_A(a)}=\left(\frac{\omega}{\omega_0}\right)^4\left(\frac{a}{a_0}\right)\exp\left\{-\frac{k_0{}^2}{2\sigma_k{}^2}\left[\left(\frac{\omega^2}{\omega_0{}^2}-1\right)^2-\frac{4\left(\frac{a}{a_0}-1\right)^2}{\left(\frac{x_0{}^2}{a_0{}^2}-1\right)^2}\right]\right\}\cdots\cdots(14)$$

and

$$R_\Phi=\frac{f_\Phi{}^D(\phi)}{f_\Phi{}^P(\phi)}=R_\Omega\cdots\cdots(15)$$

where

$$\omega_0{}^2=k_0/m\cdots\cdots(16)$$

and

$$a_0=\left\{x_0{}^2+\frac{\dot{x}_0{}^2}{\omega_0{}^2}\right\}^{1/2}\cdots\cdots(17)$$

It is obvious that when $\omega=\omega_0$ and consequently $a=a_0$, the ratio functions (13), (14) and (15) take unity value and differ from unity for any other ω. Thus, these ratio functions give the degree of difference between two distributions of output parameters by the two different methods.

Fig. 2 and 3 show the curves of these ratio functions for the different σ_k values. We note that ω_0

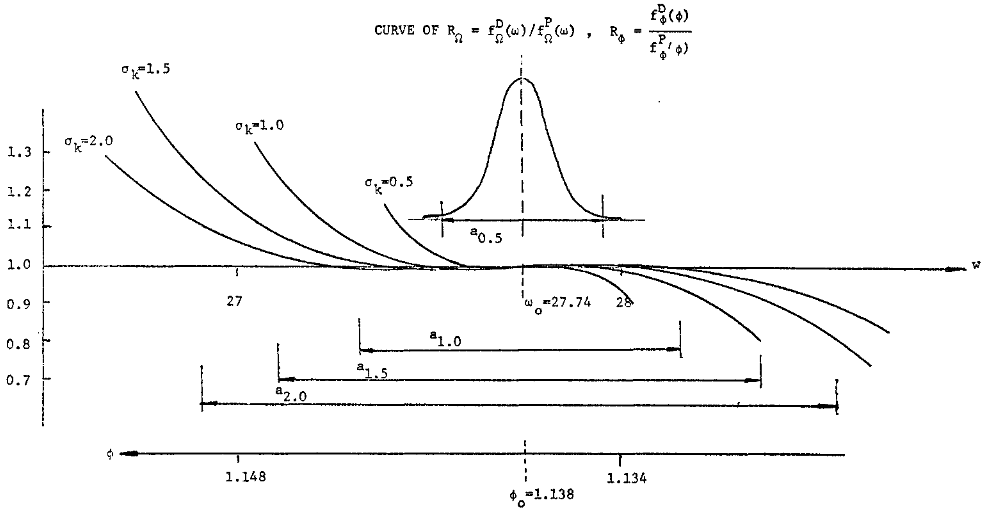

Fig. 2 Ratio Functions R_Ω and R_Φ

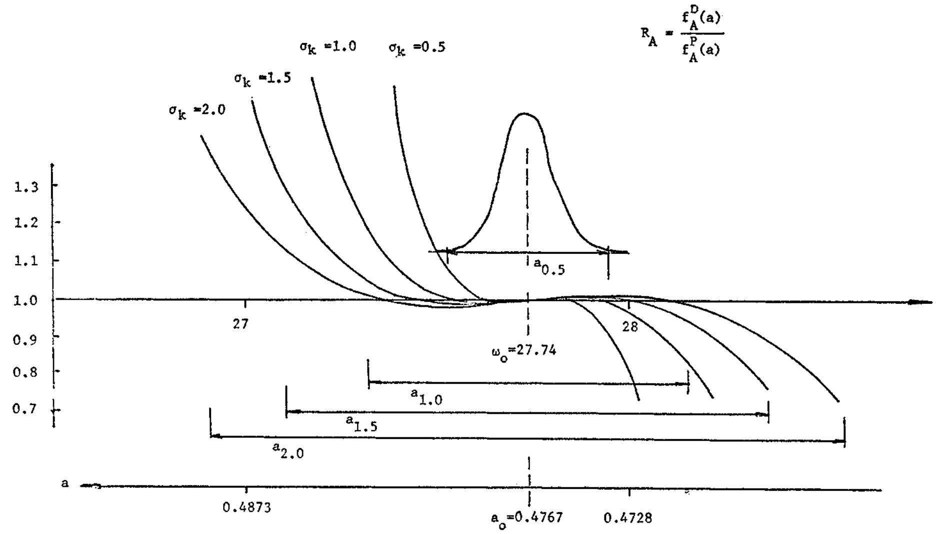

Fig. 3 Ratio Functions R_A

is the mean of natural circular frequency by the perturbation method[4] although the exact mean of ω by the direct method slightly differs from ω_0. The ranges $a_{0.5} \sim a_{2.0}$ are shown along each curve such that probability of $|\omega-\omega_0| \leq 3\sigma_\omega$ is 0.997. It is concluded that

(1) The value of ratio functions increases with the value ω which is away from ω_0. Clearly, it is dependent on the value σ_k.

(2) However, these significant errors occur only for the outside of the ranges $a_{0.5} \sim a_{2.0}$ where we have the tails of the probability density functions of output parameters. Thus, the total errors by the perturbation method may be negligible.

(3) The exact probability density functions of outputs are skewed normal since the values of the ratio functions are positive for $\omega < \omega_0$ and negative for $\omega > \omega_0$.

III. DISPLACEMENT OF FREE VIBRATION

(1) Mean of $x(t)$

For the initial conditions, x_0 and $\dot{x}_0$, the displacement, $x(t)$ of free vibration without damping is given by eq. (2).

$$x(t) = x_0 \cos \omega t + \frac{\dot{x}_0}{\omega} \sin \omega t \quad \cdots\cdots (2)$$

Taking the expectation of eq. (2), we have for the mean of $x(t)$

$$E^D[x(t)] = x_0 E[\cos \omega t] + \dot{x}_0 E\left[\frac{1}{\omega} \sin \omega t\right] \quad \cdots\cdots (18)$$

where $E^D[\ ]$ means the expectation by the direct method. With eq. (5), eq. (18) becomes

$$E^D[x(t)] = x_0 \int_0^\infty \cos \omega t f_\Omega{}^D(\omega) d\omega + \dot{x}_0 \int_0^\infty \frac{1}{\omega} \sin \omega t f_\Omega{}^D(\omega) d\omega \quad \cdots\cdots (19)$$

$$E^D[x(t)] = 2mx_0 \int_0^\infty \omega \cos \omega t f_K(m\omega^2) d\omega + 2m\dot{x}_0 \int_0^\infty \sin \omega t f_K(m\omega^2) d\omega \quad \cdots\cdots (20)$$

Clearly, $\lim_{t \to 0} E^D[x(t)] = x_0$. We also note that if σ_k approaches to zero, the probability density function, $f_K(k)$ becomes the Dirac delta function. In mathematics,

$\lim_{\sigma_k \to 0} f_K(k) = \delta(k-\mu_k)$ in which μ_k=mean of k.
In this case, we also have $\lim_{\sigma_k \to 0} f_\Omega{}^D(\omega) = \delta(\omega-\mu_\omega)$ where $\mu_\omega = \sqrt{\frac{\mu_k}{m}}$. This μ_ω is not necessarily mean of ω. Applying the above relations to eq. (19), we have

$$\lim_{\sigma_k \to 0} E^D[x(t)] = x_0 \int_0^\infty \cos \omega t\, \delta(\omega-\mu_\omega) d\omega + \dot{x}_0 \int_0^\infty \frac{1}{\omega} \sin \omega t\, \delta(\omega-\mu_\omega) d\omega = x_0 \cos \mu_\omega t + \frac{\dot{x}_0}{\mu_\omega} \sin \mu_\omega t \quad \cdots\cdots (21)$$

In words, eq. (21) indicates that for small standard deviation of k, the mean of $x(t)$ is approximately given by the right-hand side of eq. (21). We have also from eq. (19).

$$|E^D[x(t)]| \leq x_0 \int_0^\infty |\cos \omega t f_\Omega{}^D(\omega)| d\omega + \dot{x}_0 \int_0^\infty \left|\frac{1}{\omega} \sin \omega t f_\Omega{}^D(\omega)\right| d\omega$$

$$\leq x_0 \int_0^\infty f_\Omega{}^D(\omega) d\omega + \dot{x}_0 \int_0^\infty \frac{1}{\omega} f_\Omega{}^D(\omega) d\omega$$

Thus,

$$|E^D[x(t)]| \leq x_0 + \dot{x}_0 \int_0^\infty \frac{1}{\omega} f_\Omega{}^D(\omega) d\omega \quad \cdots\cdots (22)$$

(2) Variance of $x(t)$

Squaring eq. (2) and taking the expectation of it, we have for the mean square of $x(t)$

$$E^D[x^2(t)]=x_0^2\int_0^\infty \cos^2\omega t f_\Omega{}^D(\omega)d\omega + x_0\dot{x}_0\int_0^\infty \frac{1}{\omega}\sin 2\omega t f_\Omega{}^D(\omega)d\omega + \dot{x}_0^2\int_0^\infty \frac{1}{\omega^2}\sin^2\omega t f_\Omega{}^D(\omega)d\omega \quad\cdots(23)$$

$$E^D[x^2(t)]=2mx_0^2\int_0^\infty \omega\cos^2\omega t f_K(m\omega^2)d\omega + 2mx_0\dot{x}_0\int_0^\infty \sin 2\omega t f_K(m\omega^2)d\omega + 2m\dot{x}_0^2\int_0^\infty \frac{1}{\omega}\sin^2\omega t f_K(m\omega^2)d\omega \quad\cdots(24)$$

Clearly we have the following relationships;

$$\lim_{t\to 0} E^D[x^2(t)]=x_0^2 \quad\cdots(25)$$

$$\lim_{\sigma_k\to 0} E^D[x^2(t)]=x_0^2\cos^2\mu_\omega t+\frac{x_0\dot{x}_0}{\mu_\omega}\sin 2\mu_\omega t + \frac{\dot{x}_0^2}{\mu_\omega^2}\sin^2\mu_\omega t \quad\cdots(26)$$

The variance of $x(t)$ is given from eqs. (20) and (24) as follows

$$\mathrm{var}^D[x(t)]=E^D[x^2(t)]-E^D[x(t)]^2 \quad\cdots(27)$$

where

$$\lim_{t\to 0}\mathrm{var}^D[x(t)]=\lim_{\sigma_k\to 0}\mathrm{var}^D[x(t)]=0 \quad\cdots(28)$$

(3) Errors by Perturbation Method

Assuming a series approximation of $x(t)$ in the form of

$$x(t)=\sum_{i=0}^\infty x_i(t)\varepsilon_k{}^i \quad\cdots(29)$$

and sbustituting this into eq. (1), we obtain a sequence of differential equations for $x_i(t)$. Then putting the solutions $x_i(t)$ of these equations in eq. (29), we take the expectation of eq. (29) and of the square of eq. (29). Taking terms up to linear one, we obtain[4]

$$E^P[x(t)]\cong x_0(t) \quad\cdots(30)$$

where $E^P[\ ]$ means the expectation by the perturbation method.

and

$$E^P[x^2(t)]\cong x_0^2(t)+x_1^2(t)\sigma_k^2 \quad\cdots(31)$$

where

$$x_0(t)=x_0\cos\omega_0 t+\frac{\dot{x}_0}{\omega_0}\sin\omega_0 t \quad\cdots(32)$$

and

$$x_1(t)=\left(-\frac{\dot{x}_0}{2m\omega_0^3}-\frac{x_0}{2m\omega_0}t\right)\sin\omega_0 t + \frac{\dot{x}_0}{2m\omega_0^2}t\cos\omega_0 t \quad\cdots(33)$$

Thus the variance of $x(t)$ is given by

$$\mathrm{var}^P[x(t)]\cong x_1^2(t)\sigma_k^2 \quad\cdots(34)$$

By using eqs. (20) and (30), we can evaluate the error function for the mean of $x(t)$ as follows.

$$q_x(t)=E^D[x(t)]-E^P[x(t)] \quad\cdots(35)$$

Similarly for the error function of variance,

$$q_{\mathrm{var}}(t)=\mathrm{var}^D[x(t)]-\mathrm{var}^P[x(t)] \quad\cdots(36)$$

Thus, so far we discussed the linear approximation by the perturbation method. The expressions of error functions (35) and (36) can be obtained by considering the higher terms of eq. (29) in the following discussions.

The sequence of differential equations for $x_i(t)$ of eq. (29) are given by

$$m\ddot{x}_i+k_0x_i=-x_{i-1} \text{ for } i\geqq 1 \quad\cdots(37)$$

and the initial conditions are

$$x_i(0)=\dot{x}_i(0)=0 \quad\cdots(38)$$

Thus, the solution $x_i(t)$ in terms of $x_{i-1}(t)$ is given by the following convolution.

$$x_i(t)=-\int_0^t g(t-\tau)x_{i-1}(\tau)d\tau \text{ for } i\geq 1 \quad\cdots(39)$$

where

$$g(t)=\frac{1}{m\omega_0}\sin\omega_0 t \quad\cdots(40)$$

For example, for $i=1$, we can evaluate $x_1(t)$ with eq. (32) as given by eq. (33). In general, $x_i(t)$ is sequentially obtained by knowing $x_{i-1}(t)$ for $i\geq 1$.

Since the sum of the neglected higher terms of eq. (29) is $\sum_{i=2}^\infty x_i(t)\varepsilon_k{}^i$, the error function $q_x(t)$ for mean is given by

$$q_x(t)=\sum_{i=2}^\infty x_i(t)E[\varepsilon_k{}^i] \quad\cdots(41)$$

With eq. (39), eq. (41) becomes

$$q_x(t)=-\int_0^t g(t-\tau)\sum_{i=2}^\infty x_{i-1}(\tau)E[\varepsilon_k{}^i]d\tau \quad\cdots(42)$$

Squaring eq. (29), we have

$$x^2(t)=\sum_{i=0}^\infty x_i^2(t)\varepsilon_k{}^{2i}+\sum_{\substack{i=0\\(i\neq j)}}^\infty\sum_{j=0}^\infty x_i(t)x_j(t)\varepsilon_k{}^{i+j} \quad\cdots(43)$$

Thus, the error function of the mean square value, eq. (30) is

$$q_{x^2}(t)=\sum_{i=2}^\infty x_i^2(t)E[\varepsilon_k{}^{2i}]+\sum_{\substack{i=0\\(i\neq j)}}^\infty\sum_{j=0}^\infty x_i(t)x_j(t)E[\varepsilon_k{}^{i+j}] \quad\cdots(44)$$

Therefore, the error function of variance, eq. (36) is given by

$$\begin{aligned}q_{\mathrm{var}}(t)&=\mathrm{var}^D[x(t)]-\mathrm{var}^P[x(t)]\\&=E^D[x^2(t)]-E^D[x(t)]^2-E^P[x^2(t)]+E^P[x(t)]^2\\&=E^P[x^2(t)]+q_{x^2}(t)-\{E^P[x(t)]+q_x(t)\}^2-E^P[x^2(t)]+E^P[x(t)]^2\\&=q_{x^2}(t)-2E^P[x(t)]q_x(t)-q_x{}^2(t)\end{aligned} \quad\cdots(45)$$

IV. FORCED VIBRATION

The method of the direct approach for the displacement, $x(t)$ due to excitation, $f(t)$ is briefly given herein. The governing equation of the system given in Fig. 1 is

$$m\ddot{x}+c\dot{x}+kx=f(t) \quad \cdots\cdots (46)$$

where c is a viscous damping coefficient.

Letting $\alpha=c/2m$ and $\omega^2=k/m$ and limiting the problem to the case of low damping, we have for $x(t)$

$$x(t)=\frac{1}{m}\int_0^t \frac{1}{\lambda}e^{-\alpha\tau}\sin\lambda\tau f(t-\tau)d\tau \quad \cdots\cdots (47)$$

Squaring $x(t)$ in eq. (47), we have

$$x^2(t)=\frac{1}{m^2}\int_0^t\int_0^t \frac{1}{\lambda^2}e^{-\alpha(\tau_1+\tau_2)}\sin\lambda\tau_1 \times\sin\lambda\tau_2 f(t-\tau_1)f(t-\tau_2)d\tau_1 d\tau_2 \quad \cdots\cdots (48)$$

where $\lambda=\sqrt{\omega^2-\alpha^2}$

Assuming that $f(t)$ is a random process, independent of k and c, we have the expectations of eqs. (47) and (48) as follows.

$$E^D[x(t)]=\frac{1}{m}\int_0^t E\left[\frac{1}{\lambda}e^{-\alpha\tau}\sin\lambda\tau\right] E[f(t-\tau)]d\tau \quad \cdots\cdots (49)$$

and

$$E^D[x^2(t)]=\frac{1}{m^2}\int_0^t\int_0^t E\left[\frac{1}{\lambda}{}^2 e^{-\alpha(\tau_1+\tau_2)}\sin\lambda\tau_1\sin\lambda\tau_2\right] \times E[f(t-\tau_1)f(t-\tau_2)]^{d\tau_1 d\tau_2} \quad \cdots\cdots (50)$$

Thus, the variance of $x(t)$ is given by

$$\mathrm{var}^D[x(t)]=E^D[x^2(t)]-E^D[x(t)]^2 \quad \cdots\cdots (51)$$

To evaluate the integrations of eqs. (49) and (50), we need to have the joint density function of λ and α. Since k and c are to be assumed as random variables, λ and α are also random variables.

Starting with the marginal density functions $f_K(k)$ and $f_C(c)$, we can construct the joint density function of k and c as follows[6].

$$f_{KC}(k,c)=f_K(k)f_C(c)\{1+\rho[2F_K(k)-1][2F_C(c)-1]\} \quad \cdots\cdots (52)$$

where ρ is an input constant, associated with the correlation between k and c and $|\rho|\leq 1.0$. $f_K(k)$ and $f_C(c)$ are respectively the marginal density functions of k and c and, $F_K(k)$ and $F_C(c)$ are the corresponding distribution functions.

Since $\alpha=\frac{c}{2m}$ and $\lambda=\sqrt{\omega^2-\alpha^2}=\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}$, let us define an event, E such that

$$E=\left(\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}\leq\lambda\right)\cap\left(\frac{c}{2m}\leq\alpha\right) \quad \cdots (53)$$

where $\cap$ is intersection.

Now we can evaluate the joint distribution function of λ and α by integrating eq. (52) over the region which is the event, E in $c-k$ plane.

$$F_{\lambda\alpha}(\lambda,\alpha)=\int_E\int f_{KC}(k,c)dkdc \quad \cdots\cdots (54)$$

where the region E is shown in Fig. 4.

The joint density function of λ and α is

$$f_{\lambda\alpha}(\lambda,\alpha)=\frac{\partial^2}{\partial\lambda\partial\alpha}F_{\lambda\alpha}(\lambda,\alpha) \quad \cdots\cdots (55)$$

The evaluation of mean, $E^D[x(t)]$ and variance $\mathrm{var}^D[x(t)]$ can be done by integrating eqs. (49) and (50) with eq. (55).

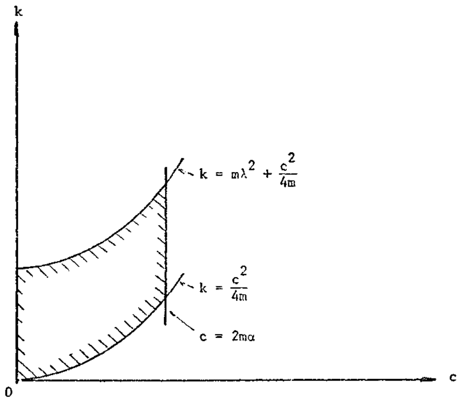

Fig. 4 Range for k and c

The corresponding mean and variance of $x(t)$ by the perturbation method are given in reference[4]. Therefore, the comparative study on the errors by the perturbation method is also possible in forced vibration.

V. MONTE CARLO SIMULATION

In the previous two sections, the direct method was developed and the errors by the perturbation method were theoretically discussed for the mean and variance of displacement. However, since the numerical evaluation of these errors by the direct method requires the higher moment of input parameters as is seen in eqs. (42) and (44) and are rather complicated, a Monte Carlo simulation is carried out in order to examine the errors. Although the Monte Carlo simulation is an approximation method for finite number, N of generated random variable sets (spring coefficient, k_i and damping coefficient, c_i), the results approach to the exact solutions for large N. Thus, in that case, the discussion on the errors by the perturbation method is possible.

The Monte Carlo simulation procedure is summarized as follows :

Step 1 Generate a random variable k_i from a given joint probability density function of k and c. This generation can be ma de by using IBM system/360 Scientific Subroutine Package, Version III. For example, for given Gaussian distribution, use subroutine GAUSS.

Step 2 Given a random variable k_i from Step 1, generate a random variable c_i from the conditional density function, $f(c/k_i)$.

Step 3 Repeating Steps 1 and 2 from $i=1$ to N, N sets of generated random variables, k and c will be obtained.

Step 4 For each set (k_i and c_i), a realization of random processes $x_i(t)$ from eq. (2) for free vibration, and $x_i(t)$ from eq. (47) for forced vibration is obtained for given time t.

Step 5 With estimators

$$\bar{x}(t)=\frac{1}{N}\sum_{i=1}^{N} x_i(t) \text{ for mean} \cdots\cdots\cdots\cdots\cdots (56)$$

and

$$S(t)=\sqrt{\frac{\sum_{i=1}^{N} x_i^2(t)-N\bar{x}(t)^2}{N-1}} \cdots\cdots\cdots\cdots\cdots (57)$$

for standard deviation

the mean and standard deviation of $x(t)$ at given time t are obtained respectively.

In the following example, we assume that k and c are jointly Gaussianly distributed with means k_0 and c_0, and standard deviations σ_k and σ_c respecti-

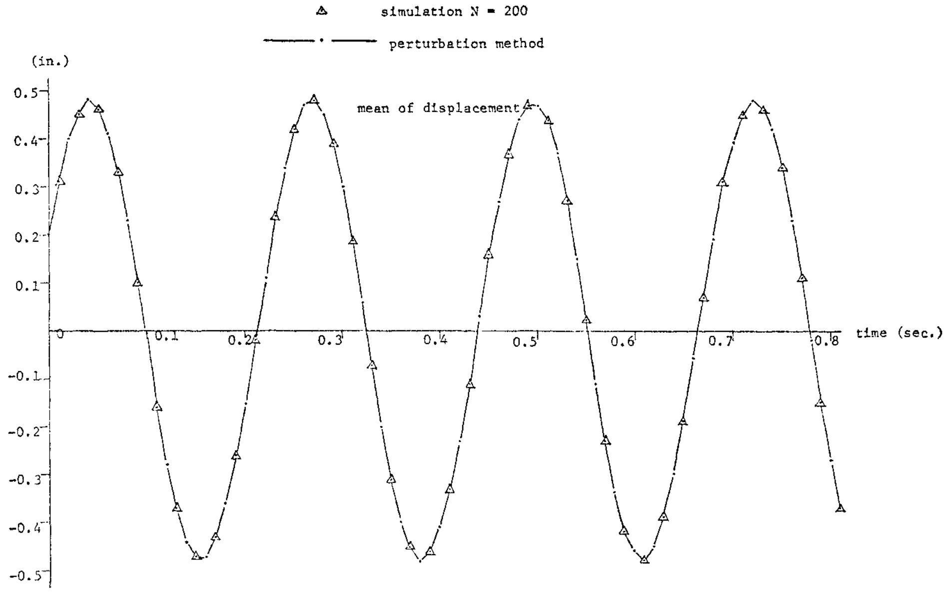

Fig. 5 Mean of Displacement for Free Vibration

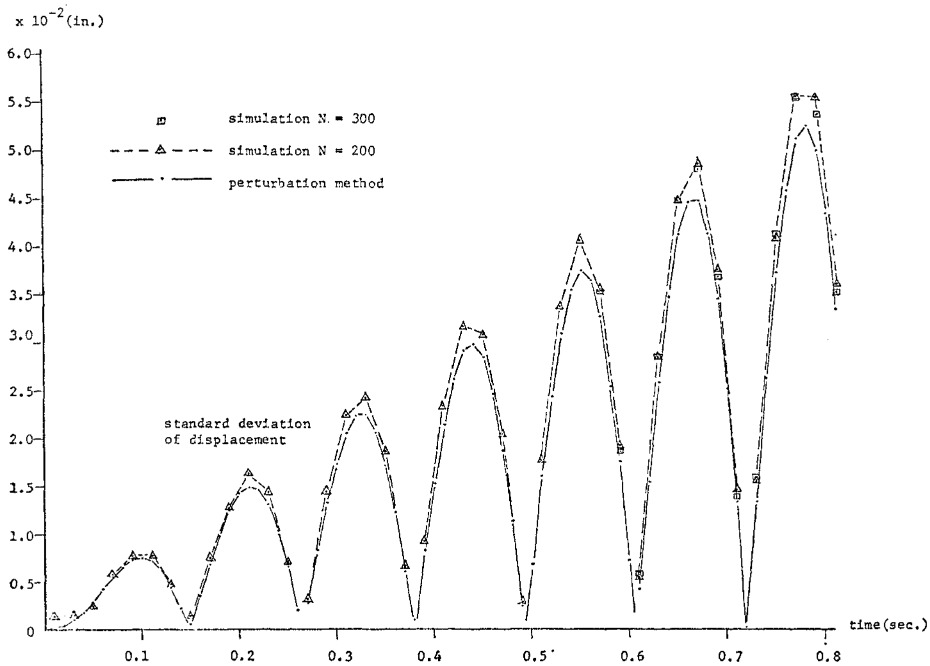

Fig. 6 Standard Deviation of Displacement for Free Vibration

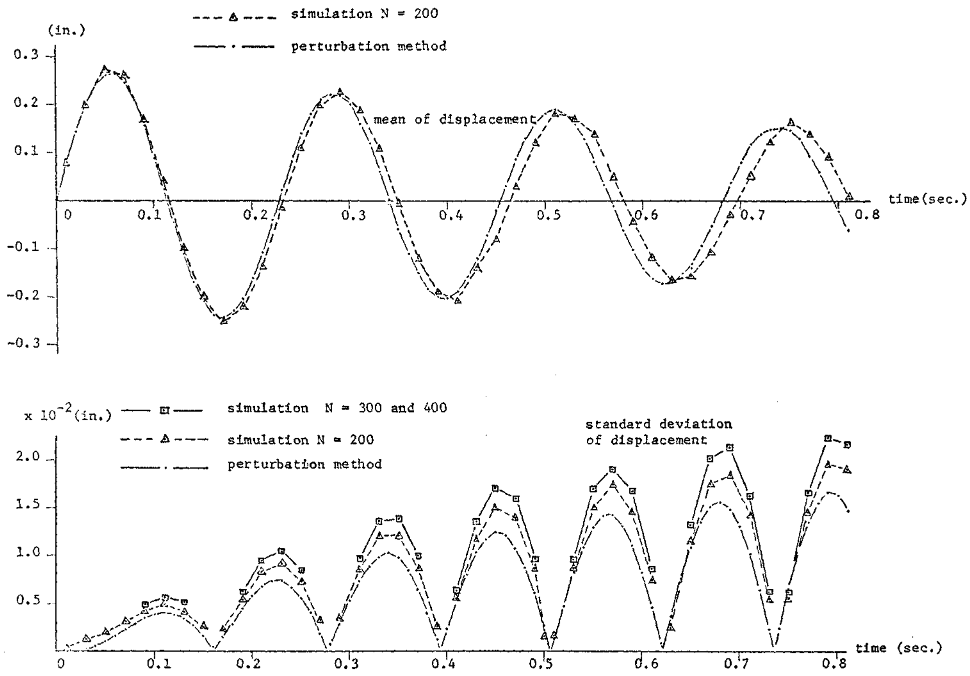

Fig. 7 Displacement Time History for Impulsive Excitation

vely. The mean and standard deviation of $x(t)$ are evaluated both by the prerturbation method and the Monte Carlo simulation. We suppose that the data on the mass, spring, and damping are given as follows :

$m=0.13$ lb·sec²/in (weight=50 lbs)

$k_0=100$ lb/in

$c_0=0.2$ lb·sec/in

$\sigma_k=1$ lb/in

$\sigma_c=0.002$ lb·sec/in

$\rho_{kc}=0.8$(coefficient of correlation of k and c)

$$\left.\begin{array}{l} x_0=0.2\text{ in} \\ \dot{x}_0=12\text{ in/sec} \end{array}\right\} \text{for free vibration}$$

The mean and standard deviation of $x(t)$ are computed with time interval 0.01 sec by the perturbation method and with 0.02 sec by the Monte Carlo simulation.

The mean of $x(t)$ in free vibration are shown in Fig. 5. The time history of the mean shows the very good agreement by the two methods. In this case, we generated $N=200$ random variables, k. Fig. 6 gives the corresponding standard deviation. Since the Monte Carlo simulations with $N=200$ and 300 have the almost same result, it seems that the curve by the Monte Carlo simulation is converged to the exact solution. Then, it is seen that the errors gradually increase with the time. However, for the initial part of time, the errors are small in the magnitude.

For the forced vibration, the dirac delta type of impulsive excitation, $f(t)=\delta(t)$ is imposed on the system at $t=0$. The results are illustrated in Fig. 7. The Monte Carlo simulation with $N=300$ gives the exact solution. The errors by the perturbation method are indicated by the difference between curves by the two approaches. For the engineering purpose, the errors may be small both for the mean and variance again for the initial time duration.

VI. CONCLUSIONS

From the theoretical development of the exact solutions of the responses of a mass, spring, and dash pot system with single degree of freedom, the following conclusions are drawn :

1. The errors by the perturbation method are small for the response parameters such as natural circular frequency if the standard deviation of input parameter, k is small. The perturbation method can be used for the evaluation of natural circular frequency, amplitude, and phase angle for the approximation.

2. The errors of the mean of $x(t)$ by the perturbation method are also small at least for the initial part of time duration. If we are interested in the response for the short time duration, the perturbation method is suggested for use, since this method is rather simple and does not require the knowledge of actual probability distribution.

3. The errors of the standard deviation of $x(t)$ by the perturbation method increase with time, although the order of errors are small. For the

aculate analysis, the solution of the direct method shuold be numerically evaluated, or in a rather simple manner, Monte Carlo simulation should be employed to obtain the variance of $x(t)$. In our example, the total number of data sets, $N=300$ gives almost convergence to the exact solution.

VII. ACKNOWLEDGEMENT

The author's acknowledgement is forwarded to the Engineering Mechanics Department, Virginia Polytechnic Institute and State University.

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(Received Sept. 10, 1970)