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ON THE STRESS CONCENTRATION AT A CORNER OF STRUCTURAL MEMBERS

By Torao TAYA

SYNOPSIS

This paper is a brief abstract of authors's investigation on the stress concentrations in *haunches* or *angle corner parts* of structures through many experiments (about 2,000 pieces of *angle parts* testing models whose shapes are different from each other) with photoelastic and interferometric (Mach-Zehnder isopachic) *moiré* methods.

NOMENCLATURE

b : depth of beam part of the corner model (mm)
D : flexural rigidity of a plate
E : Young's modulus
K (in p-K network) : $=E\phi\beta t\times 10$
M : bending moment (kg-mm)
$|m|$: degree of a *singular point*
m' : reciprocal value of the degree of a *pole*
p : (sum of principal stresses in fringe order unit) $\times 10$
t : thickness of the model (mm)
(r, θ) : polar coordinate in (mm) and (radian), respectively
α : photoelastic stress sensitivity coefficient
β : interferometric isopachic stress sensitivity coefficient
δ : half value of inside corner angle (not part of elastic body)
ϕ : rigid body rotation angle (radian)
ν : Poisson's ratio
σ_{cr} : critical stress
$\sigma_r, \sigma_\theta, \tau_{r\theta}$: stress components in polar coordinates (kg/mm^2)
σ_1, σ_2 : two orthogonal principal stresses (kg/mm^2)
η : stress concentration factor

* Professor of Civil Eng., Tôhoku University.

1. INTRODUCTION

The most critical part in a structural member is not the part of maximum bending moment (*ordinary point*) as generally caculated by elementary skeleton mechanics which is based upon Bernoulli-Euler's theory, and it is generally the *pole* appearing as a sort of *singular point* at the inside angle corner apex.

The stress intensity at the *pole* can never be clarified only by the traditional method of pure analytical elasticity or only by the experimental method such as an electric wire strain gauge; because at the *pole* itself all the formulae in the ordinary elasticity do not hold, and on the other hand on account of an infinite variation of the absolute values and directions of principal stresses in mathematical meaning at the neighbourhood of this point, the observed value such as obtained by electric wire strain gauge of finite dimension has no true meaning actually at this point. Up to this time such defect as above including stress measurement in vibration has often appeared in many papers on stress analysis.

Considering above these facts, many experiments were excecuted about the stress concentration at a *singular point* appearing in *haunches* or *angle corner parts* of structures, which are not subjected to external forces directly, using photoelastic and interferometric (Mach-Zehnder isopachic) *moiré* methods as the means of investigation of two-dimensional stress distribution.

2. GENERAL CONCEPT OF SINGULAR POINT

2.1 "Singular Point"

First of all, the author must establish the definition of "Singular Point" in general case, since he has used this terminology in somewhat different meaning from that defined by M.M. Frocht[1].

If we consider Airy's Stress Function F about a point inside or on free boundary of an elastic body in two-dimensional stress condition, and assume the point as an original point of the polar coordinates (r, θ), then in the neighbourhood of that point we can generally express $F(r, \theta)$ as

$$F(r, \theta)=F(0, \theta)+\frac{r}{1!}\frac{\partial}{\partial r}F(0, \theta)+\frac{r^2}{2!}\frac{\partial^2}{\partial r^2}F(0, \theta)+\frac{r^3}{3!}\frac{\partial^3}{\partial r^3}F(0, \theta)+\cdots$$
$$=\phi_0(\theta)+r\,\phi_1(\theta)+\frac{r^2}{2}\phi_2(\theta)+\frac{r^3}{6}\phi_3(\theta)+\cdots=\sum_{n=0}^{\infty}r^n\Phi_n(\theta)$$

where

$$\Phi_n(\theta)=\frac{1}{n!}\frac{\partial^n}{\partial r^n}F(0, \theta), \text{ and } n\geqq 0.$$

Now among the values of n_s in the above infinite series, let us put the minimum value as

$$n=m+2, \quad (\text{then, } m\geqq -2)$$

and also

$$\Phi_n(\theta)=\Phi_{m+2}(\theta)=f(\theta).$$

Since in the neighbourhood of that origin the terms of higher degree can be neglected as compared with the the term of lowest degree, then we get generally

$$F(r, \theta)=r^{m+2}f(\theta) \quad \cdots\cdots (1)$$

On the other hand,

$$\left.\begin{aligned}\sigma_r&=\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}+\frac{1}{r}\frac{\partial F}{\partial r}\\ \sigma_\theta&=\frac{\partial^2 F}{\partial r^2}\\ \tau_{r\theta}&=\frac{1}{r^2}\frac{\partial F}{\partial\theta}-\frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta}\end{aligned}\right\}\cdots\cdots (2)$$

Substituting eq. (1) into eq. (2), we get

$$\left.\begin{aligned}\sigma_r&=r^m\{f''+(m+2)f\}\\ \sigma_\theta&=r^m(m+2)(m+1)f\\ \tau_{r\theta}&=-r^m(m+1)f'\end{aligned}\right\}\cdots\cdots (3)$$

where we denote the notation $(\partial/\partial\theta)$ by $(')$.

Further, the function F must satisfy the following equation as a double harmonic function;

$$\nabla^4 F=\left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right)^2 F=0 \quad \cdots (4)$$

Now if we substitute eq. (1) into eq. (4), then we get

$$f''''+\{(m+2)^2+m^2\}f''+m^2(m+2)^2f=0 \quad \cdots\cdots (5)$$

Since the above eq. (5) is a linear differential equation of f, solving this equation we get

$$f(\theta)=a_0\sin(m\,\theta+\lambda_1)+b_0\sin(\sqrt{m+2}\cdot\theta+\lambda_2) \quad \cdots\cdots (6)$$

where a_0, b_0, λ_1 and λ_2, are integration constants which are determined by boundary conditions.

a) The case when $m>0$.

In this case, it follows from eq. (3) that $\sigma_r=\sigma_\theta=\tau_{r\theta}=0$ at the origin. We call this point as "*zero-point*". This is a special case of so called *princippal stress isotropic point* ($\sigma_1-\sigma_2=0$ or $\sigma_1=\sigma_2$), viz. a special case where $\sigma_1=\sigma_2=0$. (M.M. Frocht calls only this special case as "Singular Point"[1])

If the point is inside of an elastic body, it must be generally $f(\theta)=f(\theta+2\pi)$ at the neighbourhood of that point, then from eq. (6) it follows that m must be an integer. Further we can see that, if m is an odd integer, $f(\theta+\pi)=-(f(\theta)$ from eq. (6), and the principal stresses at two opposite points along any straight line through the origin are absolutely equal and opposite in sign to each other; and if m is an even integer, similarly from eq. (6) $f(\theta+\pi)=f(\theta)$, and the principal stresses at two opposite points as above are perfectly equal to each other. If the point is on a free boundary, it is not necessary that $f(\theta)=f(\theta+2\pi)$ is satisfied, and then m is not necessarily an integer.

b) The case when $m=0$.

In this case, the eq. (5) becomes as follows;

$$f''''+4f''=0 \quad \cdots\cdots (7)$$

Integrating the above equation, we get

$$f=a_1\sin(2\theta+\lambda)+a_2\theta+a_3 \quad \cdots\cdots (8)$$

where a_1, λ, a_2 and a_3 are the integration constants.

From the above eq. (8) and eqs. (1), (2), we get as stress intensities

$$\left.\begin{aligned}\sigma_r&=-2a_1\sin(2\theta+\lambda)+2a_2\theta+2a_3\\ \sigma_\theta&=2a_1\sin(2\theta+\lambda)+2a_2\theta+2a_3\\ \tau_{r\theta}&=-2a_1\cos(2\theta+\lambda)-a_2\end{aligned}\right\}\cdots\cdots (9)$$ [2),3)]

Since the equation

$$f(\theta)=f(\theta+2\pi)$$

must be satisfied when the origin of coordinates is in an elastic body, it is necessary in the above eq. (9) that $a_2=0$; however it is not necessary that the equation, $f(\theta)=f(\theta+2\pi)$, is satisfied when the origin of coordinate is on the free boundary or coincides with an angle corner apex.

In the above eq. (9), if $a_2=0$ and $a_1\neq 0$, then we call the point as "*ordinary point*"; and if $a_2\neq 0$, then we call the point as "*point of finite discontinuity*"; and further if it is the special case where $a_2=0$ and $a_1=0$, then we call the point as "*principal stress isotropic point*".

In the case when $a_2\neq 0$, or when the point is *a point of finite discontinuity*, the stress variation at the neighbourhood of that point is continuous about the variable θ, but on the other hand it varies suddenly or discontinuously at that point, when the point in question moves along any radial direction crossing that point.

In the case when $a_2=0$ and $a_1=0$, or when the point is a *principal stress isotropic point*, it follows from eq. (9) that $\sigma_r=\sigma_\theta=2a_3$ and $\tau_{r\theta}=0$. Then in this case the two orthogonal principal stresses σ_1 and σ_2 at this point are equal to each other, and at this point all the radial directions through this point are the directions of principal stresses. The case of "*zero-point*" in the preceding case **a**) is a special case of this "*principal stress isotropic point*" where $a_3=0$ ($\sigma_r=\sigma_\theta=0$, $\tau_{r\theta}=0$).

c) The case when $m<0$.

In this case, from the eq. (3) the stress intensity at the origin becomes infinitely large (∞) in the limit $r\to 0$, and we call this point as "*pole*". In this case too as in the case **a**), if the point is inside of an elastic body, the necessary condition exists that m must be an integer.

We generically call, (1) the *principal stress isotropic point* including *zero-point*, (2) the *point of finite discontinuity* and (3) the *pole*, as "*singular point*" against "*ordinary point*". Among these *singular points*, it is the *pole* that is especially important from a view point of strength of structural members; and it appears generally at a straight edge type inside angle corner apex of structures.

2.2 "Degree" and "Order" of a Singular Point

When $m\neq 0$ in eq. (3), we call the absolute value of m as the "*degree*" of a *zero-point* or a *pole*.

On the other hand, supposing a point moving counterclockwisely along a roop around a *singular point*, if it is the case where the principal stress direction of the point in question varies i times ($i=$ positive or negative integer including *zero*) of 180°, then we call the number i as the "*order*" of the *singular point*.

3. PROPERTIES OF A SINGULAR POINT

Since any point in the neighbourhood of a *singular point* is generally an *ordinary point*, any formula in the analytical elasticity is effective in this region except the *singular point*. However the stress at a *singular point* itself as a limiting value obtained by coverging a point to the *singular point* infinitely, is not necessarily the same value depending on the route through which a point converges to the limit, and we get generally some different limiting values as in pure mathematics.

On the other hand, it is generally common at the *singular point* that not only two orthogonal but many directions of principal stresses exist.

Further, if we consider a minute element in an elastic body including a *singular point* in it, the stress distribution on each side of this minute element does not converge to a uniform stress distribution as generally assumed in the analytical elasticity, no matter how small the minute element may become: and on the contrary it always varies on each side of the minute element without limit in the process in which the element reduces to a very minute one.

From all the above matters, we can see that formulae and general ideas in the analytical elasticity generally are effective at a point in the nieghbourhood of a *singular point*, however at the *singular point* itself as its limit never hold.

4. STRESS CONFIGURATION AT AN ANGLE CORNER APEX OF STRAIGHT EDGE TYPE

In the following, let us consider about the special case of the *singular point* which lies at an intersecting point of two straight free edges.

a) The case when $m\neq 0$.

When $m\neq 0$, the origin is either a *zero-point* or a *pole* as mentioned in the preceding article 2.

Putting $a=ma_0$, $b=(m+2)b_0$, in eq. (6), we get

$$f(\theta)=\left\{\frac{a}{m}\sin(m\theta+\lambda_1)+\frac{b}{m+2}\sin(\overline{m+2}\cdot\theta+\lambda_2)\right\} \qquad\cdots\cdots(10)$$

and

$$\left.\begin{aligned}\sigma_r&=-(m+1)r^m\left\{\frac{(m-2)}{m}a\sin(m\theta+\lambda_1)+b\sin(\overline{m+2}\cdot\theta+\lambda_2)\right\}\\ \sigma_\theta&=(m+1)r^m\left\{\frac{(m+2)}{m}a\sin(m\theta+\lambda_1)+b\sin(\overline{m+2}\cdot\theta+\lambda_2)\right\}\\ \tau_{r\theta}&=-(m+1)r^m\{a\sin(m\theta+\lambda_1)+b\sin(\overline{m+2}\cdot\theta+\lambda_2)\}\end{aligned}\right\} \qquad\cdots\cdots(11)^{2),3)}$$

Now if we assume the two directions of free edges as $\theta=\pm\theta_0$ for the original line of coordinate, then there must be a condition $\sigma_\theta=\tau_{r\theta}=0$ along these boundaries. The condition, $\sigma_\theta=\tau_{r\theta}=0$, in the case when $\theta=\theta_0$, is

$$\left.\begin{aligned}&(m+2)\,a\sin(m\theta_0+\lambda_1)+mb\sin(\overline{m+2}\cdot\theta_0+\lambda_2)=0\\ &a\cos(m\theta_0+\lambda_1)+b\cos(\overline{m+2}\cdot\theta_0+\lambda_2)=0\end{aligned}\right\}$$

If we put in the above equations $-\theta_0$ instead of θ_0, and combine them with the above equations, then as the conditions to be $\sigma_\theta=\tau_{r\theta}=0$ at $\theta=\pm\theta_0$ we get the following simultaneous equation.

$$\left.\begin{aligned}&(m+2)a\sin\lambda_1\cos m\theta_0+mb\sin\lambda_2\cos(m+2)\theta_0=0\\ &(m+2)a\cos\lambda_1\sin m\theta_0+mb\cos\lambda_2\sin(m+2)\theta_0=0\\ &a\cos\lambda_1\cos m\theta_0+b\cos\lambda_2\cos(m+2)\theta_0=0\\ &a\sin\lambda_1\sin m\theta_0+b\sin\lambda_2\sin(m+2)\theta_0=0\end{aligned}\right\} \qquad\cdots\cdots(12)$$

The above eq. (12) is the very condition to determine the values of m and the integration constants a/b, λ_1 and λ_2.

Now if we assume that one of the values of $a\cdot\cos\lambda_1$ or $b\cdot\cos\lambda_2$ is zero in the 2 nd. and 3 rd. equations of eq. (12), then another value out of them must also be zero. Then let us consider the case other than $a\cdot\cos\lambda_1=b\cdot\cos\lambda_2=0$. We get from the 2 nd. and 3 rd. equations of eq. (12)

$$\frac{\sin 2(m+1)\theta_0}{\sin 2\theta_0}=m+1=\frac{2(m+1)\theta_0}{2\theta_0}\quad\cdots\cdots(13)^{2),3)}$$

Similarly as above, from the 1 st. and 4 th. equations of eq. (12), we get excepting the case $a\cdot\sin\lambda_1=b\cdot\sin\lambda_2=0$,

$$\frac{\sin 2(m+1)\theta_0}{\sin 2\theta_0}=\frac{-2(m+1)\theta_0}{2\theta_0}\quad\cdots\cdots\cdots(14)^{2),3)}$$

However, since the eqs. (13) and (14) can not be satisfied at the same time, then if eq. (13) is satisfied, the eq. (14) can not be satisfied and the condition $a\cdot\sin\lambda_1=b\cdot\sin\lambda_2=0$ must be satisfied. Therefore we can determine the degree m as the value which satisfy either the eq. (13) or (14).

b) The case when $m=0$.

In this case the origin is at once either an *ordinary point* or a *point of finite discontinuity* as already mentioned in the preceding article 2. (However, the case when the angle corner apex is an *isotropic point* or when $a_1=0$, $a_2=0$ and $a_3\neq 0$, does not exist in this case; because at that time it becomes $\sigma_r=\sigma_\theta=2a_3$, $\tau_{r\theta}=0$ i.e. $\sigma_1=\sigma_2=2a_3$ and it contradicts to the actual stress condition, $\sigma_\theta=0$ at $\theta=\pm\theta_0$.), and the eq. (9) holds.

If we assume that $\theta=\pm\theta_0$ on two side edges at the corner and the region of elastic body lies in the range $\theta_0>\theta>-\theta_0$, then from eq. (9) and the condition $\sigma_\theta=\tau_{r\theta}=0$ at $\theta=\pm\theta_0$, we get, the following conditions;

$$\left.\begin{aligned}&a_1\sin\lambda\cos 2\theta_0+a_3=0\\&a_1\cos\lambda\sin 2\theta_0+a_2\theta_0=0\\&2a_1\cos\lambda\cos 2\theta_0+a_2=0\\&2a_1\sin\lambda\sin 2\theta_0=0\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots(15)$$

From the 4 th equation in eq. (15) we get

$$a_1=0 \text{ or } \lambda=0.$$

If we assume $a_1=0$, then we get $a_3=0$ and $a_2=0$ from the 1 st., 2 nd. and 3 rd. equations in eq. (15). Then this case is included in the case **a**) in this article, and the origine is a *zero point* ($m>0$). If we assume $a_1\neq 0$, then it must be $\lambda=0$ and from the 1 st. equation in eq. (15) we get $a_3=0$, and from the 2 nd. or 3 rd. equation in eq. (15) we get $a_2\neq 0$.

Therefore we can see that in this case the origin is a *point of finite discontinuity*.

Eliminating a_1 and a_2 from the 2 nd. and 3 rd. equations in eq. (15), we get

$$\tan 2\theta_0=2\theta_0\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(16)^{2),3)}$$

The above eq. (16) is the condition for the value of θ_0 which is necessary for the angle corner apex to be a *point of finite discontinuity*.

Solving the eq. (16) graphically, we obtain

$$2\theta_0=257°.0.$$

Thus the case when the angle corner apex is an *ordinary point* also does not exist in this case.

Putting all together the above matter and the case **a**) in this article, we can see there is no case where the angle corner apex is either an *ordinary point* or an *ordinary principal stress isotropic point*; i.e., the angle corner apex is always a *singular point* except *ordinary principal stress isotropic point* (Namely, the apex is either a *pole*, or a *zero-point*, or a *point of finite discontinuity*).

Such a result as above can also be presumed more simply from the reason as follows : —Both the principal stresses perpendicular to the two straight edges forming the angle corner must be zero, then if we consider this condition to converge infinitely to the angle corner apex, we necessarily reach to the conclusion that both the two principal stresses at the apex must be zero and the apex must be a *singular point*. (From only one condition that two principal stresses are both zero, it does not necessarily follow the conclusion that the apex is always a *zero-point*, and it follows only a result that the apex is a *singular point*, which includes *zero-point* excepting *ordinary principal stress isotropic point*. Refer, for instance, the stress configuration of a point under a concentrated load on free boundary of semi infinite plate).

Solving the eq. (13) or (14) in the case **a**) in this article graphically, we get the following tables for the value of m.

In the case when the corner angle $2\theta_0$ is larger than 180 (i.e., when the angle corner apex is inside one) and futhter when the apex is a *pole*, the solu-

Table 1 Solutions of Eq. (13)

$2\theta_0$	$2(m+1)\theta_0$	m	Angle Corner Apex
159.1°	441.7°	+1.78	Zero-Point
167.9°	805.1°	+3.80	〃
180.0°	180.0°	0	Ordinary Point
180.0°	360.0°	+1	Zero-Point
180.0°	540.0°	+2	〃
180.0°	720.0°	+3	〃
180.0°	900.0°	+4	〃
198.7°	625.1°	+2.51	〃
208.3°	314.4°	+0.509	〃
257.0°	257.0°	0	Point of Finite Discontinuity
⋮	⋮	⋮	⋮
328.3°	625.1°	+0.904	Zero-Point
360.0°	540.0°	+0.500	〃
360.0°	180.0°	−0.500	Pole
328.3°	198.7°	−0.395	〃
314.4°	208.3°	−0.337	〃
270.0°	246.4°	−0.0874	〃
257.0°	257.0°	0	Point of Finite Discontinuity

Table 2 Solutions of Eq. (14)

$2\theta_0$	$2(m+1)\theta_0$	m	Angle Corner Apex
146.3°	257.0°	+0.757	Zero-Point
164.7°	198.9°	+0.185	〃
164.7°	328.7°	+0.996	〃
164.7°	624.2°	+2.79	〃
180.0°	180.0°	0	Ordinary Point
180.0°	360.0°	+1	Zero-Point
180.0°	540.0°	+2	〃
194.2°	805.1°	+3.15	〃
208.3°	442.7°	+1.13	〃
⋮	⋮	⋮	⋮
314.4°	442.7°	+0.408	〃
335.5°	805.1°	+1.40	〃
360.0°	360.0°	0	Ordinary Point
360.0°	540.0°	+0.500	Zero-Point
360.0°	180.0°	−0.500	Pole
270.0°	146.8°	−0.456	〃
257.0°	146.3°	−0.431	〃
225.0°	151.5°	−0.327	〃
202.5°	162.0°	−0.200	〃
191.25°	169.5°	−0.114	〃
180.0°	180.0°	0	Ordinary Point

tions for the value of m are always obtained as the value ($0 > m \geqq -1/2$) as shown in the following Table 1 and 2 ; but these facts are only true when the angle corner apex is not subjected directly to external force, and generally in two dimensional statical case we can say only $m \geqq -2$.* In three dimensional general case, the value of m lies in the range $m \geqq -3$.* On the other hand as a peculiar case, there is a *singular point* formed by the combination of a *zero-point* and a *pole*.*

5. SOME EXAMPLES OF STRESS CONFIGURATION AT A STRAIGHT EDGE TYPE ANGLE CORNER APEX OF INSIDE CORNER AND ITS EXPERIMENTAL FORMULAE

From many examples of straight edge type angle corner apexes, the author will cite the following experimental results.

Figs. 1～6 show *isoclinics, principal stress trajectories, isochromatics, p-K network(p=isopachics,* K=line of epui rigid body rotation) and deflection curves of straight edge type angle corner models under compressive loadings. (Only Fig. 5 shows the case of curved type Angle Corner model).

Figs. 7 and 8 show stress distributions on some cross-sections of angle corner model of straight edge type in portal rigid frame under a compressive loading.

The author has obtained from the shapes of curves

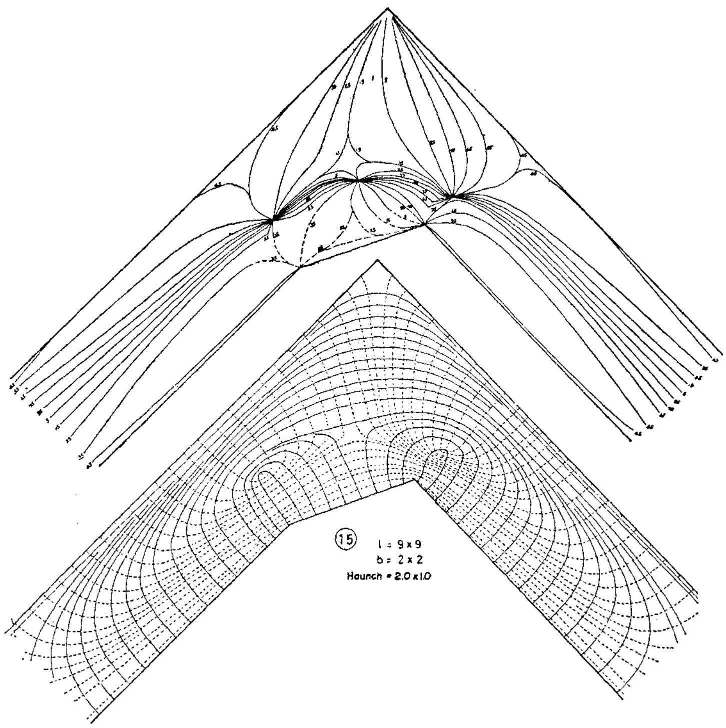


Fig. 1 Isoclinics and Principal Stress Trojectories of Rectangular Angle Corner Part with Haunch 20×10

* Regarding the proof of these above facts, see bibliography 3), Vol. 33, No. 2, pp. 294～300 and Vol. 34, No. 1, pp. 76～101.

of isochromatic fringe pattern by photoelasticity and isopachic moiré pattern by Mach-Zehnder interferometer the following stress-formula for the neighbourhood of angle corner apexes at inside corner subjected intermediately to pure bending moment, assuming the angle corner apex as the original point and one of the inside free edges as the original line of polar coordinates and expressing their curves as simultaneous equations and then solving them*,

$$\left.\begin{aligned}\sigma_1&=\frac{K''M}{2b^2t}\left(\frac{b}{r}\right)^{1/m'}\cdot\{0.843^{1/m'}+\cos^{1/m'}(\theta+32°30')\}\\ \sigma_2&=\frac{K''M}{2b^2t}\left(\frac{b}{r}\right)^{1/m'}\cdot\{0.843^{1/m'}-\cos^{1/m'}(\theta+32°30')\}\end{aligned}\right\}\quad(17)^{3)}$$

where K'' is the coefficient shown in the Table 3.

Table 3 Value of K''

δ	11.25°	22.5°	33.75°	45°	56.25°	67.5°	78.75°	90°
K''	2.55	2.60	2.70	3.00	3.40	4.00	4.85	6.00

The above equation shows that the stress intensity at the apex of inside corner becomes infinitely large (*pole*) when $r\to 0$. But in actual case, the neighbourhood around the Apex becomes plastic region where the stress intensity is constant. If we put the restriction for actual safety after Neumann's theory for notches[6]) that the radius r of this plastic region should be smaller than 1% of the beam depth b, then we get the following formula for practical designing.

$$\left.\begin{aligned}\sigma_1&=\frac{K''M}{2b^2t}(100)^{1/m'}\cdot\{0.843^{1/m'}+\cos^{1/m'}(\theta+32°30')\}\\ \sigma_2&=\frac{K''M}{2b^2t}(100)^{1/m'}\cdot\{0.843^{1/m'}-\cos^{1/m'}(\theta+32°30')\}\end{aligned}\right\}\cdots(18)^{3)}$$

The stress σ_1 in eq. (18) becomes maximum when the point is on a free boundary ($\theta=0$), and thus the stress concentration factor η of an angle corner apex in the case when we assume as a base stress intensity the Bernoulli-Euler's maximum fiber stress, becomes generally as follows : —

$$\eta=\left\{\frac{K''M}{b^2t}(84.3)^{1/m'}\Big/\left(\frac{6M}{b^2t}\right)\right\}=\frac{K''}{6}(84.3)^{1/m'}\quad\cdots\cdots\cdots\cdots\cdots(19)$$

In the case of same model under same loading, the stress in the neighbourhood of an inside angle corner apex is generally expressed from eq. (17)

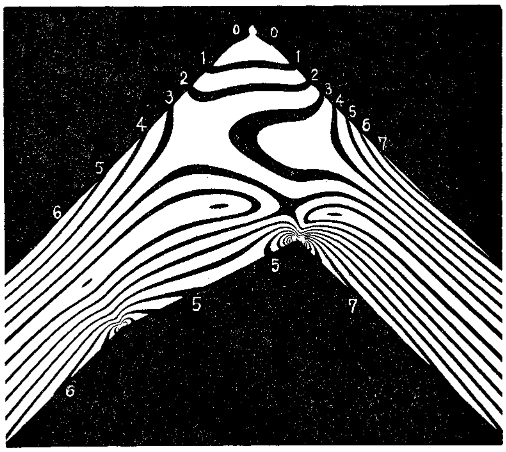


Fig. 2 Isochromatics of Rectangular Angle Corner Part with Haunch 30×10 (Photograph)

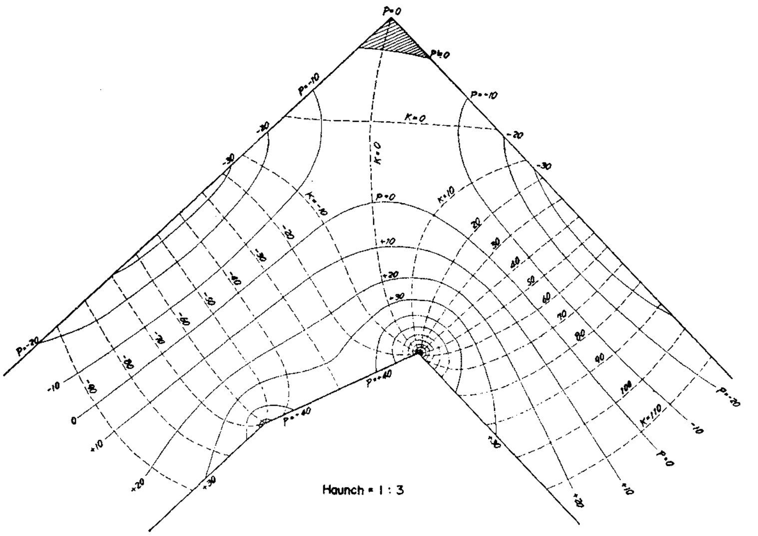
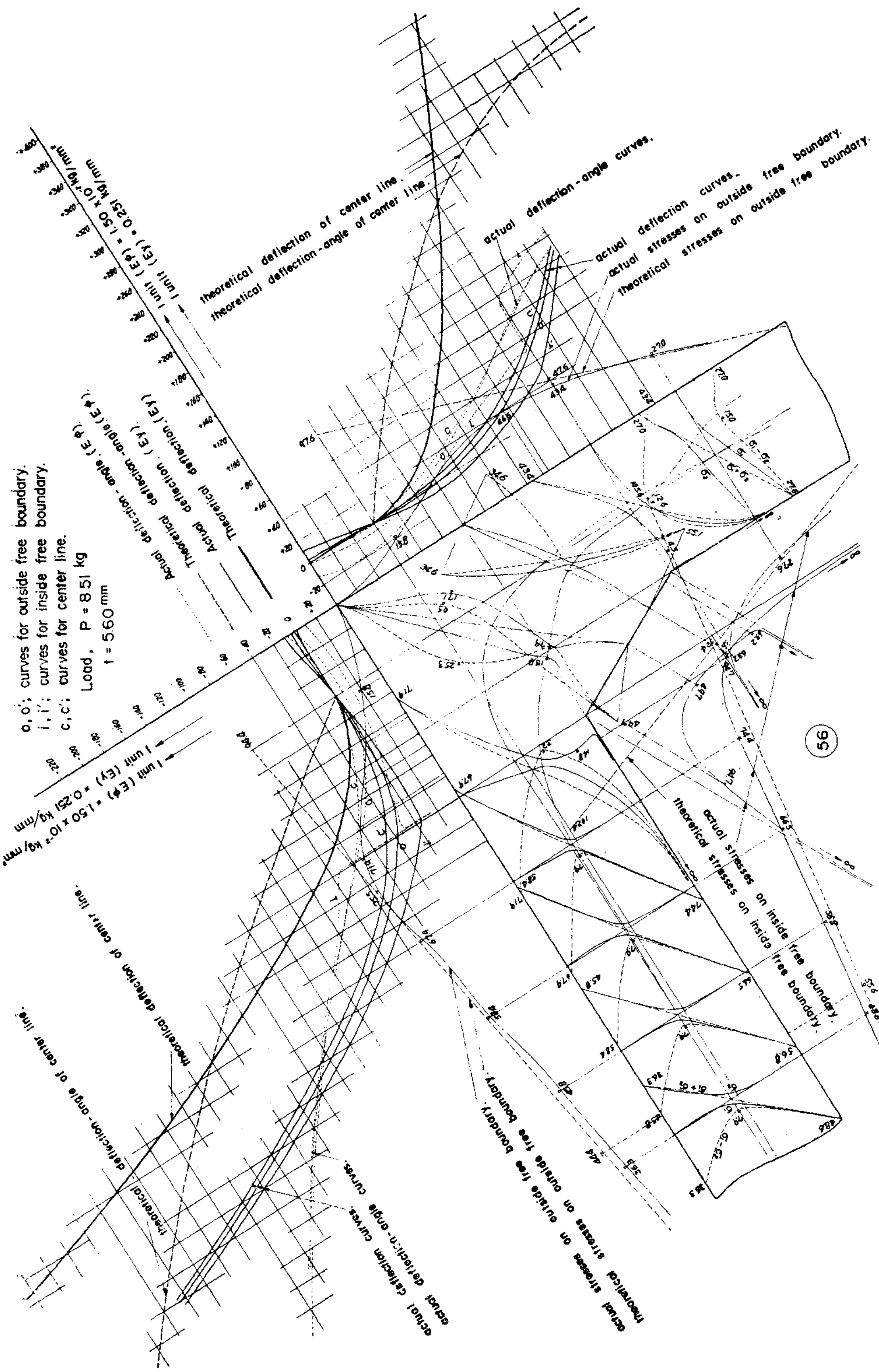


Fig. 3 p-K Network in Rectangular Angle Corner Part with Haunch 30×10

* Bibliography 3), Vol. 33, No. 1, pp. 54~82.

** It is not necessarily $m<0$ (*pole*) when the external force is not only pure bending moment. See Tables 1 and 2 in this paper.



Comparison of actual principal stresses in various sections and actual deflection curves with their theoretical values. (l=14cm x 6cm, b = 2cm x 2cm, Haunch = 1cm x 2cm)
Stress values are all in (%), provided that theoretical value $(M_s/I)\cdot y = (6Pl\sin\theta/tb^2)$ is 100 %.

Fig. 56

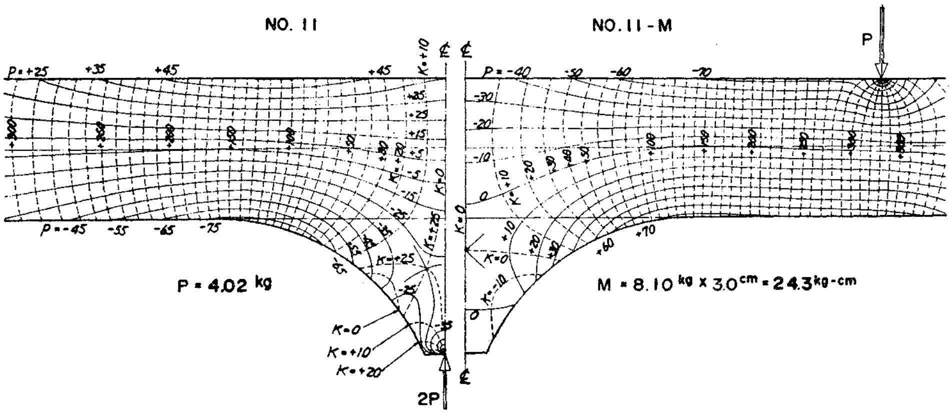


Fig. 5 *p-K* Networks in Supporting Parts of Girder Bridge with Haunch $r=3.20$ cm

simply as

$$\sigma_1 = K \cdot r^m,\ (m<0)^{**} \cdots\cdots\cdots (20)$$

In the case of a different similar model under same loading in the same dimensional problem, the stress σ_1 at a comparatively large distance r from an apex of inside corner (when it is a *pole*) is almost equal to the other (see Fig. 9) regardless of the value of m, but the stress σ_1 in the neighbourhood of an apex becomes larger when $|m|$ is larger, and it is more dangerous when $|m|$ is larger; i.e., the stress concentration factor η becomes larger, when m' is smaller. On the other hand, $|m|$ is larger when the corner angle $2\delta=360°-2\theta_0$ is smaller, as shown in the lower part of the Table 2.

As mentioned above, we generally use the terminology "stress concentration factor η" in the case when we take the ratio between the stress at a certain small distance from a *pole* and the definite value of stress σ_1 at the point comparatively large distance r from the *pole* (we call this stress as ultimate stress), but on the other hand there is a similar terminology called "stress diffusion velocity v".

The terminology "stress diffusion velocity" means the ratio between the stress at a certain distance from the point and the definite maximum stress (In the case when it is a *pole*, we take the constant stress in the plastic region around the *pole*); i.e., it expresses the decreasing velocity of stress when we assume a definite maximum stress as a basic stress.

Therefore both the values of η and v generally become larger when the degree $|m|$ of a *pole* is larger[3] but it is to be noticed that in the two cases above the stress assumed as a definite basic stress are different from each other.

In the case of comparison between a model in two dimensions and the similar one in three dimensions under equal loading, we can conclude that the degree $|m|$ of a *pole* in three dimensions is generally larger by 1 than that in two dimensions.*

In the case of an angle corner part of rigid frame of prestressed concrete, the value of $|m|$ due to external load is generally larger than that due to initial prestress at the corner; and therefore, if the external load acting intermediately to this part is the one so as to cause tensile stress at the apex, then the resultant stress at a certain small region around that apex becomes already tensile at the time when the extreme fiber stress at the beem part due to loading becomes just to extinguish the compressive prestress at the same beam part of the prestressed concrete structure.[4]

6. RELATION BETWEEN THE HAUNCH SLOPE AND THE STRENGTH OF CORNER PART

Up to the present, it has been used to design an angle corner part of reinforced concrete structure by the standard specification[7] that, if the extreme fiber stress calculated by applying the Bernoulli-Euler's beam theory to an imaginary angle corner part, which has smaller beam depth than actual cross sections and gentle haunch slope not more than 1 : 3, is smaller than the allowable stress of the material, then this part of angle corner is in safe.

However, if the actual beam depth is larger than the imaginary one used in calculating the stress in designing, this part of actual corner becomes proportionally stronger comparing with other parts of structural members, and on the other hand the upper corner angle 2δ of this actual hanch part becomes generally smaller and the value of $|m|$ of this corner apex becomes larger than that of imaginary one used in designing, and thus the actual stress becomes to be more concentrated to the proportionally weak

* Regarding the reason, see bibliography 3), Vol. 34, No. 1, p. 102.

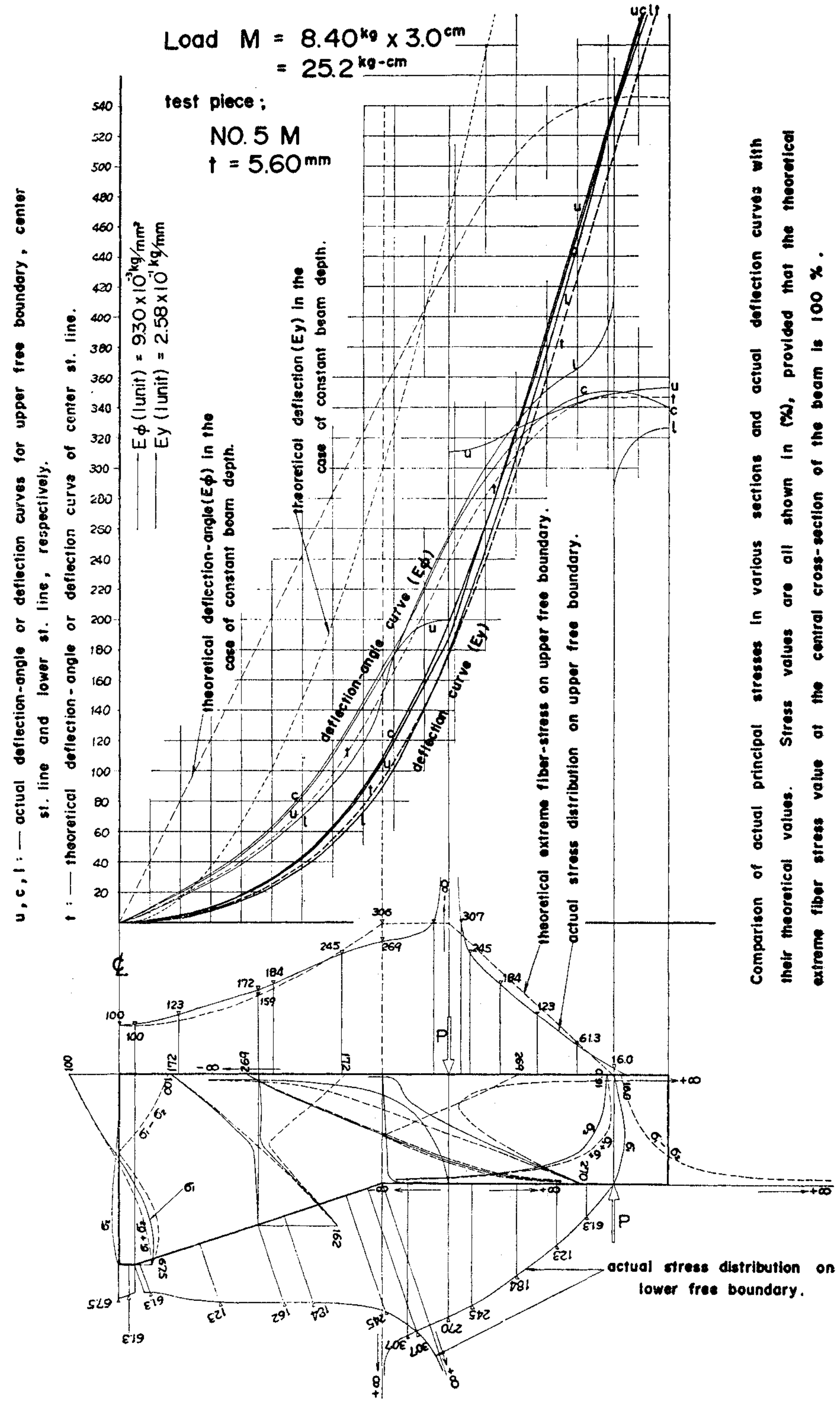


Fig. 6 Deflection curves and stress distributions in various cross section of a model of Girder Bridge (Haunch 45×15) under pure bending moment.

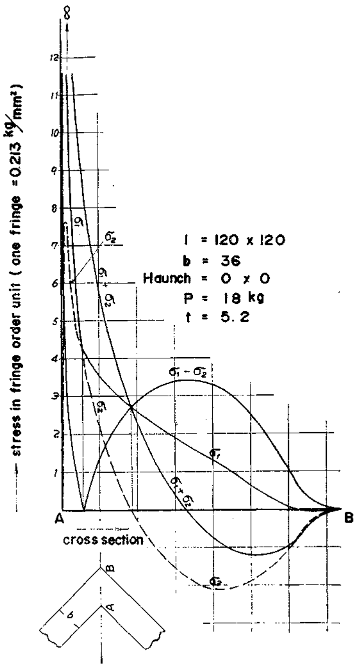


Fig. 7 Principal Stress Distribution in a Cross-Section A-B of a Rectangular Angle Corner Part with Haunch 0×0.

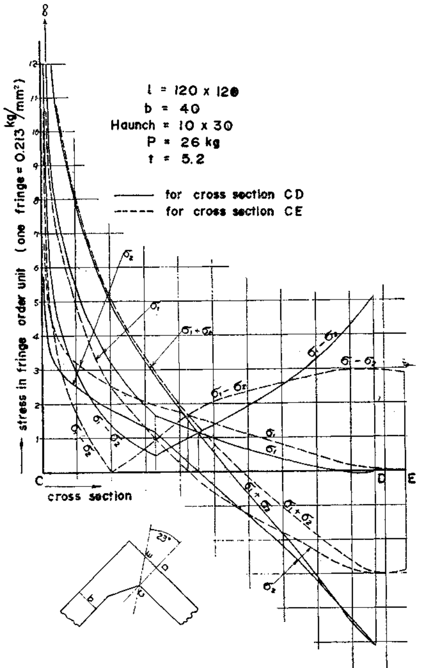


Fig. 8 Principal Stress Distribution in Cross-Sections, C-D, C-E, of a Rectangular Angle Corner Part with Haunch 30×10.

point, namely to the corner apex, than in the imaginary haunch part assumed in designing. Therefore we cannot necessarily say that the haunch part having larger beam depth is safer than the one having smaller beam depth.

7. COMPARISON OF MAXIMUM PRINCIPAL STRESSES IN STRAIGHT EDGE TYPE ANGLE CORNER PARTS AND INSCRIBED CIRCULAR ARC TYPE ANGLE CORNER PARTS

Fig. 9 shows examples of actual principal stress variation (full lines) on a free edge in the neighbourhood of an inside angle corner apex when a tensile load $P=6.27$ kg acts intermediately. In this figure, the abscissa expresses the distance from angle corner apex in mm. when it is a straight edge type, but in a special case when it is an inscribed circular arc type ($R=20$ mm.), the abscissa expresses the distance from the symmetry axis of angle corner in cm.

Fig. 10 shows in the case of circular arc type angle corner the relation between the radius r of inscribed circular arc and the value, $(\sigma_{max}/\sigma')\%$, where σ_{max} is actual maximum stress at the angle corner obtained by experiment, and σ' is extreme fiber stress calculated by the elementary Bernoulli-Euler's beam-theory, namely $\frac{6Pl\sin\theta}{b^2 t}+\frac{P\cos\theta}{bt}$.

In this case of inscribed circular arc type, as shown in Fig. 10 the stress σ_{max} does not appear generally on the axis of symmetry of corner but appears in the neighbourhood of contact point of inscribed circular arc. (generally in the beam-side in which the extreme fiber stress by the external force is larger than that in another beamside).

We can see that the value σ_{max}/σ' is mainly dependent on the value of r/b and further, this value increases rapidly from the neighbourhood smaller than $r/b=1/2$ and at last when $r=0$ (This condition means it is the case where the angle corner is a straight edge type), the value σ_{max}/σ' becomes infinitely large (∞) and thus the angle corner apex becomes a *pole*.

And further we can say from Fig. 10 that, in order to get haunch strength having the stress concentration factor $\eta=1$, we must adopt in the corner part an inscribed circular arc with radius r at least equal to the beam-depth.

8. LOCAL ELASTIC INSTABILITY OF A CORNER

When corner part includes a *pole* under compressive stress and constituted by a thin material such as steel plate, the fracture of this part may be caused not only by the maximum stress or by the

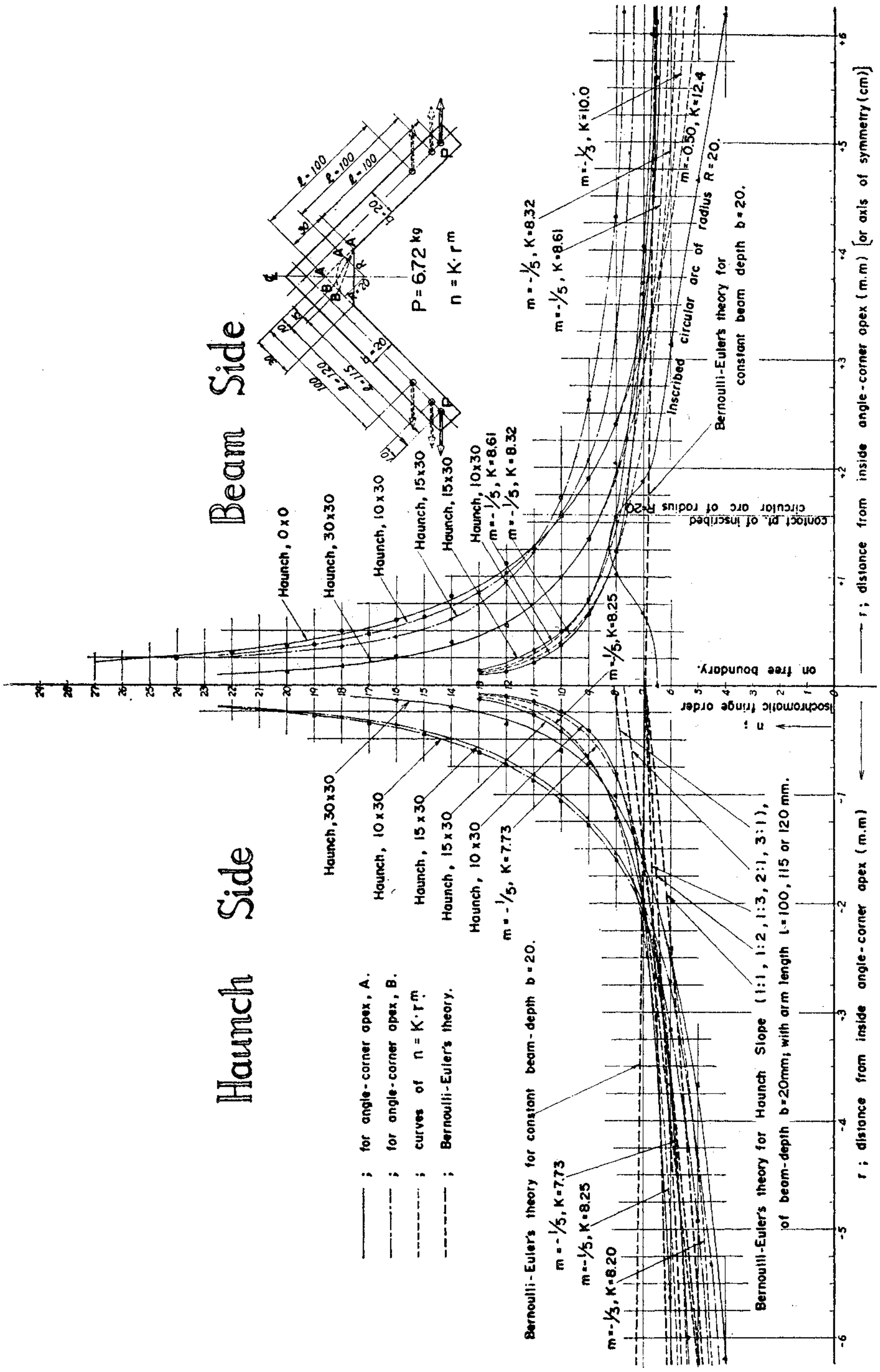


Fig. 9 The Stress Variation in the Neighbourhood of an Inside Angle Corner Apex. (Pole))

breadth of plastic region but also by the local elastic buckling of this part in the direction perpendicular to the surface.

In this case, if we apply to this part of structural member the buckling formula of a circular plate uniformly compressed in radial directions and simply supported at the circumference, the critical stress is given as follows.

$$\left.\begin{aligned}\sigma_{cr}&=\frac{4.20\,D}{r^2t}\\ D&=\frac{Et^3}{12(1-\nu^2)}\end{aligned}\right\}\qquad\cdots\cdots\cdots\cdots\cdots\cdots(21)^{5)}$$

where r is the radius of a certain region subjected to buckling.

The above eq. (21) is regarded to be a formula showing the relation between the critical stress σ_{cr} and the radius r of a region subjected to buckling in the neighbourhood of an angle corner apex.

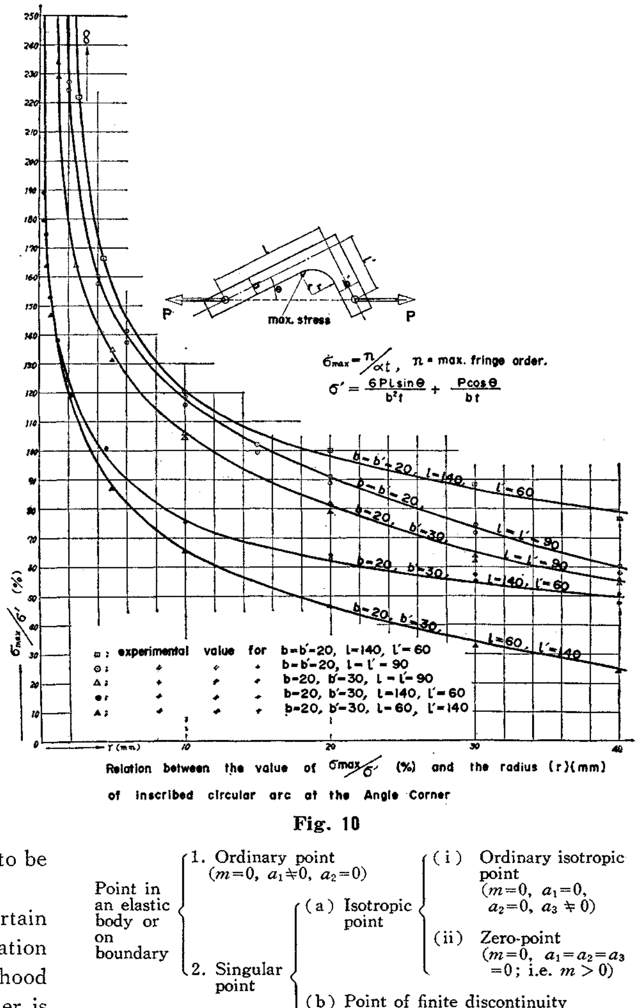
On the other hand, the working stress σ in the neighbourhood of an inside angle corner apex (when it is a *pole*) varies almost as

$\sigma=Kr^m$, $(m<0)$.$\cdots$(20) (inserted again)

As already mentioned in the end of the article 4, the value of m is generally $0>m\geqq -1/2$, when the corner apex is not subjected directly to external force and, in addition, the corner apex is a *pole*; on the other hand σ_{cr} grows up with r^{-2} as r reduces to 0. Therefore if the part of angle corner is designed so as to be

$\sigma_{cr}>\sigma$

for any one value of $r=a$, where a is a certain finite distance, then it is sure that the above relation holds at any point of $r\leqq a$ in the neighbourhood of the corner apex and the part of angle corner is in a condition of elastic stability.



Relation between the value of σ_{max}/σ' (%) and the radius (r)(mm) of inscribed circular arc at the Angle Corner

Fig. 10

9. CONCLUSIONS

Summarizing the contents of this paper, the essential points are as follows.

1. The stresses at a *singular points*, σ_r, σ_θ and $\tau_{r\theta}$ are generally expressed as a form of Kr^m (the term of lowest degree of r in an expansion of stresses in infinite series.). And further in precisely, they are expressed as follows.

$(m=0)$:

$$\left.\begin{aligned}\sigma_r&=-2\,a_1\sin(2\,\theta+\lambda)+2\,a_2\theta+2\,a_3.\\ \sigma_\theta&=2\,a_1\sin(2\,\theta+\lambda)+2\,a_2\theta+2\,a_3.\\ \tau_{r\theta}&=-2\,a_1\sin(2\,\theta+\lambda)-a_2.\end{aligned}\right\}\cdots(9)$$

$(m\neq0)$:

$$\left.\begin{aligned}\sigma_r&=r^m\{f''+(m+2)f\}.\\ \sigma_\theta&=r^m(m+2)(m+1)f.\\ \tau_{r\theta}&=-r^m(m+1)f'.\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

In the above equations, the kinds of *singular points* are classified as follows.

Point in an elastic body or on boundary
- 1. Ordinary point ($m=0$, $a_1\neq0$, $a_2=0$)
- 2. Singular point
  - (a) Isotropic point
    - (i) Ordinary isotropic point ($m=0$, $a_1=0$, $a_2=0$, $a_3\neq0$)
    - (ii) Zero-point ($m=0$, $a_1=a_2=a_3=0$; i.e. $m>0$)
  - (b) Point of finite discontinuity ($m=0$, $a_2\neq0$)
  - (c) Pole ($m<0$)

2. The theoretical absolute value of stress at a *pole* is infinitely large as a limiting value when $r=0$, and the larger is the value of $|m|$, the more rapidly the stress increases as r reduces to zero.

3. At an inside corner apex of straight edge type in the range of $2\,\theta_0=360°\sim180°$ (namely, $2\,\delta=0°\sim180°$), which is not subjected to external forces directly, there appears generally a *pole*; and the value of m of that *pole* lies in the range $-0.500\sim0$ corresponding to $2\,\delta=0°\sim180°$.

However, at an inside angle corner of inscribed circular arc type, there does not appear any *pole*, and the stress concentration factor η of inscribed circular arc type becomes almost smaller than 1 when $r/b=1$.

4. The stresses at an inside angle corner apex which is subjected intermediately only to pure bending moment, are expressed as follows.

$$\sigma_1=\frac{K''M}{2\,b^2t}\left(\frac{b}{r}\right)^{1/m'}\{0.843^{1/m'}+\cos^{1/m'}(\theta+32°30')\}$$

$$\sigma_2=\frac{K''M}{2b^2t}\left(\frac{b}{r}\right)^{1/m'}\{0.843^{1/m'}-\cos^{1/m'}(\theta+32°30')\} \quad \cdots\cdots\cdots\cdots\cdots(17)$$

where K'' is a coefficient shown in the Table 3.

5. The haunch part having larger beam depth is not always safer than the one having smaller beam depth.

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