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A STUDY ON THE BEHAVIOUR OF BASIN WATER BY MEANS OF THE VARIATIONAL TECHNIQUE

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I. INTRODUCTION

Runoff phenomena in a river basin occur according to the variation of water states in a basin. That is, runoff shows only one phase of basin water behaviour.

Several years ago, the author discussed[1),2)] the recession characteristics and rising states of ground water runoff by means of runoff models. Through the research works, the physical significances of recession characteristics have been made clear and it is pointed out that the runoff component from the unconfined aquifer plays an important role in the ground water runoff during a prolonged period. The research work has been carried out successfully especially for the basins with catchment area 100～300 km^2. The method, moreover, has been extended to the simulation technique[3)] of stream discharge for such basins.

For rather large basins, however, it is difficult to clarify the several characteristics because of the scattering and variation of several parameters in various cases. The variation is due to the change of transformations in the runoff process; for example the interactions among stream water, ground water and hydrological quantities with different distributions in time and space. In other words, the different ways of interactions in each case have large effects on ground water runoff when it appears in our sights. Therefore, it may be hoped that the problems on ground water runoff are discussed in connection with the behaviours of all water components in basins.

Runoff process, in general, is of uni-directional system, that means the several phenomena in downstream reaches are affected by those in upstream ones. Through these processes, the runoff water changes and variates its kinematical characteristics. However, phenomenological consideration in details indicates that there are situated in a watershed such regions that each of them acts as a kinematical system. In the region, besides giving considerable effects on water state in downstreams, the phenomena in upstreams are controled by the states in downstreams. For this example, we may take the plain regions and cupshaped regions. Therefore, field of runoff in actual basins may be understood as the links of several kinematical systems.

Generally, natural phenomena in a kinematical system occur in pursuit of the most stable and equilibrium state through the averaging and uniformalizing processes. The process towards the equilibrium state itself is understood as to be dependent upon a certain kinematical balance. Therefore, it may be expected that the behaviour of whole basin water also follows a balance in a system in the same meaning. As the results and one phase of the behaviour, we observe the runoff phenomena and the variations of stream discharge.

There are various kinds of components in the basin water; stream water, ground water, soil moisture and others. Regarding the respect mentioned above, it may be considered that these components also behave following to their own balances under such situation that the basin water, as a whole, behaves in chase of the stable and equilibrium state.

The movement of each component is expressed by the fundamental equation. It is, however, still unknown what quantity and law govern the balance and behaviour of total basin water in a system. Such quantity and law, if could be found, may contribute to the understanding of the runoff phenomena in connection with all the components of basin water.

In many cases of water resources ploblems, we plan and construct several hydraulic structures at the boundaries of the kinematical systems, for example dams at narrow valleys which separate the downstreams from upstream region, and the pipe lines and watercourses from the head of alluvial zones. Due to the artificial water controls, we will face changes of the states of basin water in the system. In these problems, we should handle the several water components in a system together.

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Realizing the point, the author commenced a study of the problem whether we can formulate any law which governs the basin water behaviour all at once. As the first step of the approach, the author has focused his attention to ground water, stream water, and their interactions in a runoff process by means of mathematical model. The purpose of the present paper is to introduce a variational formulation for the basin water behavior, especially with respect to the process of ground water runoff. Applying the method, moreover, the variation of the recession characteristics in the runoff process are also discussed.

II. VARIATIONAL FORMULATION FOR BASIN WATER BEHAVIOUR

II. 1 Variational Principle for Energy Dissipative System

There are many variational expressions for kinematical systems, including Lagrangian and Hamiltonian. In general, the methods treat the energy conservative systems, but not the energy dissipative systems. Prigogine and his colleagues carried out studies[4] on the problems whether there are any functions which play such roles in the dissipative system as Lagrangian function in a conservative system. Thereafter, the variational principles have been formulated for the linear system at first and extended for the nonlinear system. The variational principle so introduced is now applied as a new tool to the macroscopic and statistical treatment of various phenomena. The fundamental assumption is so called "principle of local equilibrium", that is, although the system as a whole will not be equilibrium, there exists at every point a state of local equilibrium with local entropy defined by the classical Gibbs formura.

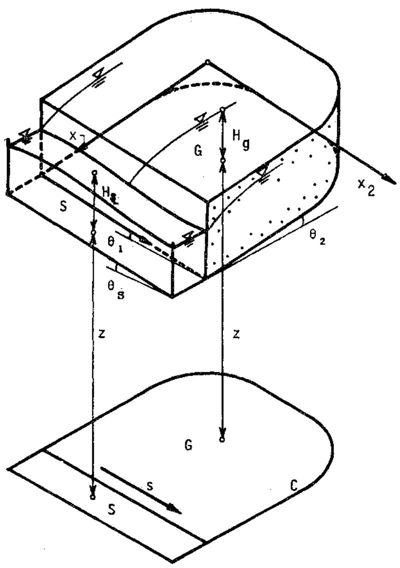

Fig. 1 Runoff model

Recently, the method has been applied in the various fields of hydrodynamics by W.H. Reid[5], P. Glansdorff and D.F. Hays[6], R.S. Schechter and D. M. Himmelblau[7] and others. The author considered the method proposed by Prigogine may give us an useful tool to understand the hydrological problems described in the last chapter.

In this chapter, the variational formulation is attempted for the basin water behaviour. The expression partially differs from that after Prigogine because of the particularity of runoff system. The method discussed here is of course not perfect as the numerous problems are yet unsolved, but it will become an initiator for the better understanding of runoff phenomena.

II. 2 Variational Formulation for Ground Water Behaviour

Let us consider the movement of ground water which occurs in the ground water region G as shown in Fig. 1. The fundamental equation of the motion is as well known:

$$\gamma \frac{\partial H_g}{\partial t} = \frac{\partial}{\partial x_i}\left\{k \cdot H_g \cdot \frac{\partial H_g}{\partial x_i} - f_i H_g\right\} + r. \quad \cdots\cdots (1)$$

The symbols in the figure and the equation denote:

G : ground water region,

γ : porosity,

k : permeability coefficient,

H_g : water depth of ground water,

r : recharge intensity per unit area of the ground water region,

x_i : the rectangular coordinates downward positive, $i=1, 2$,

θ_i : inclinations of the impermeable bed,

f_i : $=k \sin \theta_i = -k \dfrac{\partial z}{\partial x_i}$,

z : elevation of the impermeable bed from the reference horizon.

The water depth H_g considered here may be expressed as of the macroscopic water depth or mean water depth $H_g{}^*$ plus the small arbitrary variations δH_g around the macroscopic water depth. Both of them are functions of space coordinates as well as time, that is,

$$H_g(x_i, t) = H_g{}^*(x_i, t) + \delta H_g(x_i, t), \cdots\cdots (2)$$

moreover, we assume

$$|H_g{}^*(x_i, t)| \gg |\delta H_g(x_i, t)|. \cdots\cdots (3)$$

Provided the gradient of ground water surface and bed are very small, Dupuit-Forchheimer's assumption leads that the potential energy per unit area of ground water region is proportional to

$$H_g(x_i, t)+z. \qquad (4)$$

Thereafter, the variation of potential energy due to the small variation of water depth δH_g, is given by

$$\delta(H_g+z),$$

in which z does not change its value due to the variation of water depth, that is

$$\delta z=0. \qquad (5)$$

Multiplying $-\delta(H_g+z)$ to the both hand sides of Eq. (1) and replacing H_g by $H_g^*+\delta H_g$, we obtain the following relationship:

$$-\gamma\delta(H_g+z)\frac{\partial \delta H_g}{\partial t}=\gamma\frac{\partial H_g^*}{\partial t}\delta(H_g+z)$$
$$+\frac{1}{2}\Sigma\, kH_g^*\cdot\delta\left(\frac{\partial(H_g+z)}{\partial x_i}\right)^2-r\,\delta(H_g+z)$$
$$-\Sigma\frac{\partial}{\partial x_i}\left\{\left(kH_g\frac{\partial H_g}{\partial x_i}-f_iH_g\right)\delta(H_g+z)\right\}$$

in which the higher order infinitesimals are ignored. Integrating this equation over the region G and any time interval, we obtain :

$$-\frac{1}{2}\int_t\int_G\gamma\frac{\partial}{\partial t}(\delta(H_g+z))^2dx_idt$$
$$=-\frac{1}{2}\int_G\gamma(\delta(H_g+z))^2dx_i$$
$$=\delta\int_t\int_G\left[\gamma\frac{\partial H_g^*}{\partial t}(H_g+z)+\frac{1}{2}\Sigma\, kH_g^*\cdot\left(\frac{\partial(H_g+z)}{\partial x_i}\right)^2-r(H_g+z)\right]dx_idt$$
$$+\int_t\int_C\left[\left\{\left(kH_g^*\frac{\partial H_g^*}{\partial x_1}-f_1H_g^*\right)\frac{dx_2}{ds}-\left(kH_g^*\frac{\partial H_g^*}{\partial x_2}-f_2H_g^*\right)\frac{dx_1}{ds}\right\}\delta(H_g+z)\right]$$
$$\cdot dsdt\leq 0. \qquad (6)$$

In this equation, $\int_G dx_i$ means double integral over the ground water region, ie. $\iint dx_1dx_2$, and $\int_C ds$ line integral along the boundary of ground water region. The first term of the right hand side in the equation may also be written as;

$$\delta\iint\left[\gamma\frac{\partial H_g^*}{\partial t}(H_g+z)+\sum_i\left(kH_g^*\frac{\partial H_g^*}{\partial x_i}-f_iH_g^*\right)\left(\frac{\partial H_g}{\partial x_i}+\frac{\partial z}{\partial x}\right)-r(H_g+z)\right]dx_idt. \qquad (7)$$

In the case of

$$H_g(x_i, t)=H_g^*(x_i, t) \qquad (8)$$

all over the region G, the equality in Eq. (6) should be satisfied. Moreover, the last term in the right hand side of Eq. (6) vanishes provided boundary conditions are given on the boundary C or no flow flux crosses the boundary. In such case, since only the first term remains, we may write the equation of motion as,

$$\delta\iint\left[\gamma\frac{\partial H_g^*}{\partial t}(H_g+z)+\frac{1}{2}\Sigma\, kH_g^*\left(\frac{\partial(H_g+z)}{\partial x_i}\right)^2-r(H_g+z)\right]dtdx_i=0, \qquad (9)$$

or approximately

$$\delta\iint\left[\gamma\frac{\partial H_g^*}{\partial t}(H_g+z)+\Sigma\left(kH_g^*\frac{\partial H_g^*}{\partial x_i}-f_iH_g^*\right)\frac{\partial(H_g+z)}{\partial x_i}-r(H_g+z)\right]dtdx_i=0. \qquad (9)'$$

In the equation, the variation should be taken with respect to only the quantity H_g, keeping the macroscopic water depth H_g^* fixed according to the principle of local equilibrium. As well known, these equations have the equation of motion for the macroscopic distributions as their Euler-Lagrangian equation. These presentations of the problem need the subsidiary condition Eq. (8).

II. 3 Variational Formulation for Stream Water Flow

For the stream water flow as shown in Fig. 1, the one-dimensional equation of continuity is

$$D\frac{\partial H_s}{\partial t}+\frac{\partial Q}{\partial s}=0. \qquad (10)$$

Manning's formula is

$$V=\frac{1}{n}H_s^{2/3}\left\{-\frac{\partial H_s}{\partial s}+\sin\theta_s\right\}^{1/2}, \qquad (11)$$

in which the symbols denote as follows;

H_s : water depth in the stream,
D : width of the stream,
Q : discharge,
V : mean velocity,
s : distance along the stream, downward positive,
θ_s : bed slope of the stream, $\sin\theta_s=-\frac{\partial z}{\partial s}$,
n : Manning's roughness coefficient.

We write the water depth $H_s(s, t)$ in the stream as the summation of macroscopic depth distribution $H_s^*(s, t)$ and the small arbitrary deviations $\delta H_s(s, t)$ around the macroscopic distribution,

$$H_s(s, t)=H_s^*(s, t)+\delta H_s(s, t). \qquad (12)$$

The following inequality is also assumed,

$$|H_s^*(s, t)|\gg|\delta H_s(s, t)|. \qquad (13)$$

For the brief treatment, let us consider the case of stream with very gentle slope, that is,

$$\cos\theta_s\simeq 1,$$

then the variation of the potential energy due to the variation of water depth δH_s may be proportional to

$$\delta(H_s+z),$$

in which δz is equal to zero. Then the following expression is obtained in the same way as described in the last article.

$$-\frac{1}{2}D\int(\delta(H_s+z))^2ds$$
$$=\delta\iint D\frac{\partial H_s^*}{\partial t}(H_s+z)+\frac{2}{3}\frac{D}{n}H_s^{*5/3}\cdot\left(-\frac{\partial H_s}{\partial s}-\frac{\partial z}{\partial s}\right)^{3/2}dsdt\cdot\iint\left[\frac{D}{n}H_s^{5/3}\right.$$

$$\cdot\left(-\frac{\partial H_s}{\partial s}-\frac{\partial z}{\partial s}\right)^{1/2}\delta(H_s+z)\Big]_{s=0}^{s=l}dt\leq 0. \quad \cdots\cdots(14)$$

Thus it may be concluded that the equality in this equation holds in the case the water depth equals to the macroscopic water depth all over the region S,

$$H_s(s,t)=H_s{}^*(s,t). \quad \cdots\cdots(15)$$

Moreover, the equation leads the following variational formulation provided either the boundary conditions are given or no flow flux crosses the boundary.

$$\delta\iint\left[D\frac{\partial H_s{}^*}{\partial t}(H_s+z)+\frac{2}{3}\frac{D}{n}H_s{}^{*5/3}\cdot\left(-\frac{\partial H_s}{\partial s}-\frac{\partial z}{\partial s}\right)^{3/2}\right]dsdt=0. \quad \cdots\cdots(16)$$

This equation is also written approximately in the form of

$$\delta\iint\left[D\frac{\partial H_s{}^*}{\partial t}(H_s+z)+Q^*\left(-\frac{\partial H_s}{\partial s}-\frac{\partial z}{\partial s}\right)\right]\cdot dsdt=0. \quad \cdots\cdots(16)'$$

If we treat Chézy flow, we gain the same type of equation. These equations have the equation of motion for the macroscopic behaviour of stream flow as their Euler-Lagrangian equation with the help of the subsidiary condition Eq. (15). In this case too, the variation should be taken with respect to only the quantity H_s, keeping $H_s{}^*$ fixed.

II. 4 Variational Formulation for Basin Water Behaviour

In this article, we will introduce the variational formulation for the behaviour of the basin water in a system as a whole, on the basis of the expressions obtained in the last two articles.

Now we consider the water movement containing the interaction of ground water and stream water in the mathematical model as shown in Fig. 1. We treat the following problem;

$$\delta\left\{\iiint\mathcal{L}_g dx_i dt+\iint\mathcal{L}_s dsdt\right\}=0, \quad \cdots\cdots(17)$$

in which,

$$\mathcal{L}_g=\begin{cases} r\dfrac{\partial H_g{}^*}{\partial t}(H_g+z)+\dfrac{1}{2}\Sigma\, kH_g{}^*\cdot\left(\dfrac{\partial H_g}{\partial x_i}+\dfrac{\partial z}{\partial x_i}\right)^2-r(H_g+z), \quad \cdots(18) \\ \text{or approximately,} \\ r\dfrac{\partial H_g{}^*}{\partial t}(H_g+z)+\Sigma\, kH_g{}^*\cdot\left(\dfrac{\partial H_g}{\partial x_i}-f_iH_g{}^*\right)\dfrac{\partial(H_g+z)}{\partial x_i}-r(H_g+z), \quad \cdots\cdots(18)' \end{cases}$$

$$\mathcal{L}_s=\begin{cases} D\dfrac{\partial H_s{}^*}{\partial t}(H_s+z)+\dfrac{2}{3}\dfrac{D}{n}H_s{}^{*5/3}\cdot\left(-\dfrac{\partial H_s}{\partial s}-\dfrac{\partial z}{\partial s}\right)^{3/2}, \quad \cdots\cdots(19) \\ \text{or approximately,} \\ D\dfrac{\partial H_s{}^*}{\partial t}(H_s+z)+Q^*\left(-\dfrac{\partial H_s}{\partial s}-\dfrac{\partial z}{\partial s}\right) \quad \cdots\cdots(19)' \end{cases}$$

In Eq. (17), variation is taken with respect to only H_g and H_s, keeping $H_g{}^*$ and $H_s{}^*$ fixed. In this meaning, the quantities $\mathcal{L}_g$ and $\mathcal{L}_s$ are called "local potential". Now we have the following equations as the Euler-Lagrangian equation for the variational problem Eq. (17) and as the natural boundary condition along the boundary between ground water region and the stream, together with subsidiary conditions Eqs. (8) and (15).

Euler-Lagrangian equation:

$$r\frac{\partial H_g{}^*}{\partial t}-\sum_i\frac{\partial}{\partial x_i}\left\{kH_g{}^*\left(\frac{\partial H_g{}^*}{\partial x_i}+\frac{\partial z}{\partial x_i}\right)\right\}-r=0, \quad \cdots\cdots(20)$$

Natural boundary condition along the stream:

$$D\frac{\partial H_s{}^*}{\partial t}+\frac{\partial Q^*}{\partial s}+\left\{\left(kH_g{}^*\frac{\partial H_g{}^*}{\partial x_1}-f_1H_g{}^*\right)\frac{dx_2}{ds}-\left(kH_g{}^*\frac{\partial H_g{}^*}{\partial x_2}-f_2H_g{}^*\right)\frac{dx_1}{ds}\right\}=0. \quad \cdots\cdots(21)$$

Take $H_g{}^*=H_s{}^*$ at the boundary between ground water and stream, then Euler-Lagrangian equation leads to the fundamental equation of ground water movement and the natural boundary condition to the equation of motion of stream water, containing the interaction with ground water. The discharge in Eq. (21) should be expressed in terms of $H_s{}^*$ and its derivatives with respect to space coordinate. Consequently, we may write the basin water behaviour in terms of a simple variational form Eq. (17). In other words, it may be understood that the water movement in stream and ground water region occurs as the result of the behaviour of whole basin water which follows the variational principle.

Moreover, since $\mathcal{L}_g$ and $\mathcal{L}_s$ in Eqs. (18), (18)′, (19) and (19)′ do not contain $\partial H_g/\partial t$ and $\partial H_s/\partial t$ explicitly, time t is only a parameter in the variational calculation in Eq. (17). Therefore, we may write the variational formulation in the following form too,

$$\delta\left\{\frac{\partial}{\partial t}\int_t\int_G\mathcal{L}_g dx_i dt+\frac{\partial}{\partial t}\int_t\int_S\mathcal{L}_s dsdt\right\}=0 \quad \cdots\cdots(22)$$

or,

$$\delta\left\{\int_G\mathcal{L}_g dx_i+\int_S\mathcal{L}_s ds\right\}=0. \quad \cdots\cdots(22)'$$

II. 5 Physical Significances of Local Potential and Variational Principle

It is apparent that each term of the local potential shown in Eqs. (18), (18)′, (19) and (19)′, has dimension of flux of potential energy in terms of water head. In order to understand the physical significances of the local potentials, we will take them defined by the prime-system, that is, by Eqs.

(18)′ and (19)′. If we multiply H_g+z and H_s+z to Eqs. (20) and (21), which should be satisfied in the case $H_g=H_g^*$ and $H_s=H_s^*$, respectively, and thereafter subtract them from the local potentials Eqs. (18)′ and (19)′, we have another expressions for $\mathcal{L}_g$ and $\mathcal{L}_s$ as follows:

$$\mathcal{L}_g=\Sigma-\frac{\partial}{\partial x_i}\left\{-(H_g+z)\cdot\left(kH_g^*\frac{\partial H_g^*}{\partial x_i}-f_iH_g^*\right)\right\}, \quad\cdots\cdots\cdots\cdots(23)$$

and

$$\mathcal{L}_s=-\frac{\partial}{\partial s}(Q^*(H_s+z))+(H_s+z)\cdot\left\{\left(kH_g^*\frac{\partial H_g^*}{\partial x_1}-f_1H_g^*\right)\frac{dx_2}{ds}-\left(kH_g^*\frac{\partial H_g^*}{\partial x_2}-f_2H_g^*\right)\frac{dx_1}{ds}\right\}. \quad\cdots\cdot(24)$$

These expressions show that the local potentials $\mathcal{L}_g$ and $\mathcal{L}_s$ are of the following physical significances:

In the case we change the water depth around the macroscopic ones, keeping the flow fluxes fixed, $\mathcal{L}_g$ corresponds to the summation of the inflow flux of potential energy per unit area due to the flow flux and that due to the work done by the pressure, and $\mathcal{L}_s$ corresponds to the summation of the inflow flux of potential energy per unit length of stream due to the flow flux and that due to the work done by the pressure.

These significances are rather loosely defined but it is apparent that the local potentials $\mathcal{L}_g^*$ and $\mathcal{L}_s^*$ when $H_g=H_g^*$ and $H_s=H_s^*$, correspond to the actual values of inflow fluxes of potential energy demonstrated above. In the discussion here, it should be noted that we assume very gentle slopes of stream bed and ground water surface. Though there are certain restrictions, the physical significances of each term in local potential Eqs. (18) and (19) may also be summarized as Table 1. Thus

Table 1 Physical significances of each term of local potential

	Physical Significances
$D\frac{\partial H_s^*}{\partial t}H_s^*$, $\gamma\frac{\partial H_g^*}{\partial t}H_g^*$,	Change of the potential energy stored in the region per unit time and unit area.
$-Q^*\frac{\partial H_s^*}{\partial s}$, $\sum_i\left\{kH_g^*\frac{\partial H_g^*}{\partial x_i}-f_iH_g^*\right\}\frac{\partial H_g^*}{\partial x_i}$,	Potential energy carried into the region of unit area due to flow flux per unit time.
$H_g^*\left(kH_g^*\frac{\partial H_g^*}{\partial x_1}-f_1H_g^*\right)\frac{dx_2}{ds}-H_g^*\left(kH_g^*\frac{\partial H_g^*}{\partial x_2}-f_2H_g^*\right)\frac{dx_1}{ds}$, rH_g^*,	Potential energy carried into the region of unit area per unit time.
$-H_s^*\frac{\partial Q^*}{\partial s}$, $\Sigma H_g^*\frac{\partial}{\partial x_i}\left\{kH_g^*\frac{\partial H_g^*}{\partial x_i}-f_iH_g^*\right\}$,	Potential energy carried into the region of unit area due to the works caused by the pressure per unit time.

the local potential is closely related to the potential energy of water. For the generalization of the local potential, if the kinetic energy which is not considered here because of the mathematical difficulty is introduced, we may also obtain the similar expressions of the local potential but in a rather complex form.

As mentioned above, it may be concluded that the behaviour of whole basin water takes place within the basin so that the local potential closely related to the potential energy has stationary value for the variation of water depth. For the approximate local potential in the prime system, although the variational problem degenerates, the equation (17) itself remains theoretically correct and may be available for the brief treatment of the practical problems. Moreover, it may be easily understood that the type of variation Eq. (17) holds for the evolution of the system during arbitrary time interval, in addition the latter form Eq. (22) holds for the whole system at any moment.

Although the physical significances of the local potential and the variational priciple are rather loosely defined, it is interesting that they are closely related to the potential energy of basin water.

III. ON THE RECESSION CHARACTERISTICS OF GROUND WATER RUNOFF

The variational technique formulated here, may be applicable to the analyses of many hydraulic and hydrological problems. In this paper, the author wishes to discuss the variation of the recession characteristics of ground water runoff discharge in the runoff process as an example of the applications.

In advance of the arguments on the application of the variational technique, several basic points obtained through the author's precedent research[1),2)], are outlined in this chapter.

The author has clarified that the ground water runoff is characterized by runoff components from the unconfined and confined aquifers. But we treat here only the unconfined component, since it plays a dominant role in the runoff process during a prolonged period. Figure 2 shows the runoff model for this component. The symbols indicate,

H_u : water depth,
x : distance along the unconfined aquifer,
t : time,
k_u : permeability coefficient,
γ_u : porosity,
L_u : length of the unconfined aquifer,

The fundamental equation for this component is written as:

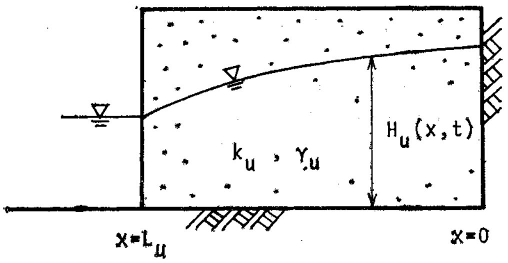

Fig. 2 Runoff model for the unconfined component

$$\frac{\partial H_u}{\partial t}=\beta H_u\frac{\partial^2 H_u}{\partial x^2}+\beta\left(\frac{\partial H_u}{\partial x}\right)^2,\quad \beta=\frac{k_u}{\gamma_u}, \quad\cdots\cdots(25)$$

for the recession state. If the gradient of water surface is very small, the equation may de rewritten simply as follows:

$$\frac{\partial H_u}{\partial t}=\beta H_u\frac{\partial^2 H_u}{\partial x^2}. \quad\cdots\cdots(26)$$

Under certain conditions, this equation is easily solved and we have

$$H_u(x,t)=\frac{1}{(at+1)}\left\{-\frac{a}{2\beta}x^2+\frac{h_{u0}-H_{u0}}{L_u}x+\frac{a}{2\beta}L_u x+H_{u0}\right\}, \quad\cdots\cdots(27)$$

in which

$h_{u0}=H_u(L_u, 0)$,
$H_{u0}=H_u(0, 0)$.

Write the runoff discharge from the unconfined aquifer in the basin as Q_u and its initial value Q_{u0}, then the recession state of the runoff discharge is given by

$$Q_u(t)=\frac{Q_{u0}}{(at+1)^2}. \quad\cdots\cdots(28)$$

The quantity a in Eqs. (27) and (28) is defined as;

$$a=\frac{2\beta}{L_u^2}(H_{u0}-h_{u0}), \quad\cdots\cdots(29)$$

which is a function of the geological factors of the basin as well as the initial state. In the case the discharge decreases along the same one curve (so-called Normal recession curve) regardless the initial discharge, the following relationship between the quantity a and the initial discharge Q_{u0} for the recession limb is theoretically derived.

$$a=K\sqrt{Q_{u0}}. \quad\cdots\cdots(30)$$

New quantity K defined above, is a constant for a basin, because it consists of geological and geographical factors.

The detailed results obtained by the field data are omitted here, since they were discussed in the precedent papers. Here we summarize below the results closely related to the discussions in the following chapter.

i) In the small mountainous basin with basin area 100~300 km², K-values are almost constant for each basin, respectively. Therefore, the normal recession curves are defined obviously.

ii) In the case of a rather large basin with plain or cupshaped region, considerable scattering appears in the analyzed K-values, although they are defined for each recession state. Hence it is rather difficult to hold the unique normal recession curve even for a basin.

iii) However, we can define unique K-value for certain basivs of which catchment area are very large.

It is, of course, difficult to distinguish these groups in terms of basin area alone. Regarding the obtained K-values for several basins, discussions are rather restricted because of the assumption that the runoff process in a basin has been regarded as a lumped system. Therefore, we cannot search for the detail phenomena which transform the recession characteristics in runoff process. It is the variational principle introduced in the precedent chapter, that gives us a clue to discuss these problems.

IV. VARIATION OF RECESSION CHARACTERISTICS OF GROUND WATER RUNOFF IN RUNOFF PROCESS

As described in the opening chapter, the actual river basin consists of the links of regions and each of them acts as a kinematical system. Realizing the facts, Dooge[8] and Nash[9] have developed their runoff models as links of reservoirs. In their studies, the reservoirs are regarded as kinematical lumped systems. The author recognized that the runoff phenomena would variate its characteristics within the region which is considered as a system with various distributions of hydrological quantities in time and space. In this chapter, the discussions are made theoretically on the problems of what kinds of transformations and averagings happen in the system with respect to the recession characteristics of ground water runoff.

IV. 1 Assumptions

The behaviour of water within a system is expressed by the variational principle Eqs. (17) or (22). If we use proper trial functions for H_g, H_g^*, H_s and H_s^*, and define the values of parameters in such way that the variational principle holds, we will obtain the approximate behaviour of basin water as a whole. Strictly speaking, the procedure should be repeated until the obtained values become equal to the assumed ones for H_g^* and H_s^*. However, we treat the method simply by several assumptions as described below.

i) Runoff region: For the brief treatment, the model shown in Fig. 3 is adopted for the region which acts a kinematical system. The region consists

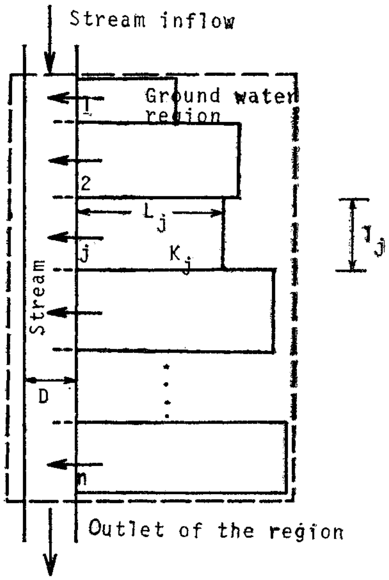

Fig. 3 Model for the watershed as a kinematical system

of n-ground water regions with different characters, and of n-stream reaches. Moreover, both flows through the ground water regions and stream reaches are assumed one dimensional. The new symbols indicate;

l : the width of the ground water region or the length of the each stream reach,
G_j : j-th ground water region ($j=1, 2, \cdots, n$),
S_j : j-th stream reach,
H_u : water depth in the ground water region,
H_s : water depth in the stream reach.

and the subscripts u and s denote the values for the unconfined component of ground water and for the stream water, respectively. Moreover, the subscript j indicates the value for the j-th ground water region and the stream reach.

ii) Ground water: In each ground water regions G_j, the ground water flow is assumed to follow the behaviour of the unconfined flow obtained in the chapter III, if in the region the interactions between hydrological situations do not occur. Then, we may define several quantities or relationships introduced in the precedent chapter.

$$H_{uj}^*(x, t)=\frac{1}{(a_jt+1)}\left\{-\frac{a_j}{2\beta_j}x^2+\frac{h_{u0j}-H_{u0j}}{L_j}x+\frac{a_j}{2\beta_j}L_jx+H_{u0j}\right\}, \cdots\cdots(31)$$

$$a_j=\frac{2\beta_j}{L_j}\{H_{u0j}-h_{u0j}\}, \cdots\cdots(32)$$

$$a_j=K_j\sqrt{Q_{u0j}}\ . \cdots\cdots(33)$$

The equation (31) is used for the approximate trial functions H_g^* in each region G_j.

According to the results obtained through the precedent research, each recession state is expressed considerably well by the equations on the unconfined component even for the rather large basins, although the values for each occasion scatter. In other words, it seems that the mean behaviour of water may follow those of the unconfined component as a lumped system. Therefore, the equation of which parameter is replaced by a for a_j in Eq. (31) is also useful as a trial function H_g, that is

$$H_{uj}(x, t)=\frac{1}{(at+1)}\left\{-\frac{a}{2\beta_j}x^2+\frac{h_{u0j}-H_{u0j}}{L_j}x+\frac{a}{2\beta_j}L_jx+H_{u0j}\right\}, \cdots\cdots(34)$$

in which value a is common to all region $G_j(j=1, 2,\cdots, n)$.

iii) Channel water: We assume that the stream water flowing into the system is characterized by Eq. (28) with recession coefficient a_s. The coefficient a_s in stream discharge is also assumed to keep its value constant, if no interaction would occur in the system. Moreover, assume the water depth distributed linearly within the stream reach S_j from the value H_{sj} at upstream end to that H_{sj+1} at downstream end, then we may use the following trial functions for H_{sj} and H_{sj}^* in the reach S_j;

$$H_{sj}^*=\frac{1}{a_st+1}\left\{\frac{h_{u0j+1}-h_{u0j}}{l_j}s+h_{u0j}\right\}, \cdots\cdots(35)$$

$$H_{sj}=\frac{1}{at+1}\left\{\frac{h_{u0j+1}-h_{u0j}}{l_j}s+h_{u0j}\right\}, \cdots\cdots(36)$$

in which

$$H_{sj}(t)=H_{uj}(L_j, t),$$
$$H_{sj}(0)=h_{u0j}.$$

In addition, we further assume the relationship between the stream discharge and the water depth in the reach S_j as follows;

$$Q_{sj}=C_jH_{sj}^2. \cdots\cdots(37)$$

IV. 2 Theoretical Treatment

We derive the expressions on variation of recession characteristics in the system by means of variational principle in the type of Eq. (22). The equation is written as

$$\delta\left\{\sum_j\iint_{G_j}\mathcal{L}_{gj}dxdy+\sum_j\int_{S_j}\mathcal{L}_sds\right\}=0. \cdots(38)$$

To calculate the integrals in the blacket, we use the expressions of prime-system for local potentials. Although the equation (27), (31) and (34) are the solutions of Eq. (26), if we consider them as the solutions of Eq. (25) approximately, then we have

$$\mathcal{L}_{gj}=r_j\frac{\partial}{\partial x}\left\{\beta_jH_{gj}^*\frac{\partial H_{gj}^*}{\partial x}H_{gj}\right\},$$

for the trial functions H_{gj}^* and H_{gj}. Thereafter, we obtain by the simple calculations

$$\sum_j\iint\mathcal{L}_{gj}dxdy=\sum_j\frac{r_j\beta_jl_j}{(a_jt+1)^2(at+1)}\cdot\left\{\frac{(h_{u0j}^2-H_{u0j}^2)(h_{u0j}-H_{u0j})}{L_i}\right.$$

$$-(h_{u0j}^2+H_{u0j}^2)\frac{a_j}{2\beta_j}L_j\Big\}, \quad \cdots\cdots(39)$$

On the other hand, the integral at the stream region becomes

$$\sum_j\int\mathcal{L}_{sj}ds=\sum_j\frac{1}{(a_st+1)^2(at+1)}\Big\{-\frac{D_ja_sl_j}{3}\cdot(h_{u0j+1}^2+h_{u0j}h_{u0j+1}+h_{u0j}^2)-\frac{C_j}{3}(h_{u0j+1}^3-h_{u0j}^3)\Big\}. \quad \cdots\cdots(40)$$

Therefore, value a which satisfies Eq. (38), is defined by the following equation.

$$\frac{\partial}{\partial a}\Big\{\sum_j\iint\mathcal{L}_{gj}dxdy+\sum_j\int\mathcal{L}_{sj}ds\Big\}=0. \cdots(41)$$

To find the behaviour of system as a whole, we write the recession factors a_j as follows

$$a_j=a+\delta a_j \quad \cdots\cdots(42)$$

and assume

$$|a|\gg|\delta a_j|. \quad \cdots\cdots(43)$$

Then the above equation (41) can be reduced to

$$\sum_j\Big\{\frac{1}{(a+\delta a_j)t+1}\Big\}^2(a_st+1)^2\prod_{p=1}^{n}\{(a+\delta a_p)t+1\}^2\cdot\{\Lambda_{gj}-\Gamma_{gj}(a+\delta a_j)\}+\prod_{p=1}^{n}\{(a+\delta a_p)t+1\}^2\cdot\sum_j\{\Lambda_{sj}-\Gamma_{sj}a_s\}=0 \quad \cdots\cdots(44)$$

in which

$$\left.\begin{aligned}\Lambda_{gj}&=\frac{\gamma_j\beta_jl_j(h_{u0j}^2-H_{u0j}^2)(h_{u0j}-H_{u0j})}{L_j},\\ \Gamma_{gj}&=\frac{\gamma_jl_jL_j}{2}(h_{u0j}^2+H_{u0j}^2),\\ \Lambda_{sj}&=-\frac{C_j}{3}(h_{u0j+1}^3-h_{u0j}^3),\\ \Gamma_{sj}&=\frac{D_jl_j}{3}(h_{u0j+1}^3+h_{u0j}h_{u0j+1}+h_{u0j}^2).\end{aligned}\right\} \quad \cdots\cdots(45)$$

Consequently, the approximate solution of Eq. (44) is obtained as

$$a^*=\frac{\sum_j\Lambda_{gj}+\sum_j(\Lambda_{sj}-\Gamma_{sj}a_s)}{\sum_j\Gamma_{gj}} \quad \cdots\cdots(46)$$

Although the solution is rather approximate one, the value a^* obtained shows the recession coefficient of stream discharge at the downstream end or also the one averaged all over the system.

IV. 3 Variation of Recession Characteristics in the Runoff Process

The equation (46) is a fraction of series. If we divide each term in the series in the numerator by the corresponding term in the denominator, the remainder becomes nearly equal to the recession coefficient a_j for each ground water zone G_j, that is,

$$\frac{\Lambda_{gj}}{\Gamma_{gj}}\simeq\frac{2\beta_j}{L_j}(H_{u0j}-h_{u0j})=a_j \quad \cdots\cdots(47)$$

Write $H_{u0j}/h_{u0j}=m_j$, then m_j value is constant for the basin with constant K_j in Eq. (33) and we gain approximately the following relationship by simple calculation;

$$\Gamma_{gj}=\Big(\frac{1}{\beta_j}\Big)^2L_j^2(m_j+1)Q_{u0j}, \quad \cdots\cdots(48)$$

where Q_{u0j} indicates the initial discharge out of the j-th ground water zone G_j. Therefore, Eq. (33), (47) and (48) lead to the expression

$$\Lambda_{gj}=\Big(\frac{1}{\beta_j}\Big)^2L_j^2(m_j+1)K_jQ_{u0j}^{3/2}. \quad \cdots\cdots(49)$$

On the other hand, if the channel is prismatic and uniform, that is, D_j, C_j are constant through the stream reaches, we may write

$$\sum\Lambda_{sj}=\frac{1}{3\sqrt{C}}(Q_{u0s}^{3/2}-Q_{u0}^{3/2}) \quad \cdots\cdots(50)$$

and

$$\sum\Gamma_{sj}=\frac{Dl}{C}Q_{u0} \quad \cdots\cdots(51)$$

in which Q_{u0} is the initial discharge at the downstream end, and Q_{u0s} is that at the upstream of this region. Therefore, we have following expressions for a^*.

$$a^*=\frac{\sum p_jq_j^{3/2}K_j+\frac{1}{3\sqrt{C}}(q_s^{3/2}-1)-\frac{Dl}{C}q_s^{3/2}K_s}{\sum p_jq_j}\cdot\sqrt{Q_{u0}}, \quad \cdots\cdots(52)$$

in which

$$\left.\begin{aligned}p_j&=\Big(\frac{1}{\beta_j}\Big)^2L_j^2(m_j+1),\\ q_j&=Q_{u0j}/Q_{u0},\\ q_s&=Q_{u0s}/Q_{u0}.\end{aligned}\right\} \quad \cdots\cdots(53)$$

Moreover, if we adopt the value K in Eq. (30) for the recession characteristics of the system as a whole, the value may be written in terms of the recession characteristics in each basin as

$$K^*=\frac{\sum p_jq_j^{3/2}K_j+\frac{1}{3\sqrt{C}}(q_s^{3/2}-1)-\frac{Dl}{C}q_s^{3/2}K_s}{\sum p_jq_j} \quad \cdots\cdots(54)$$

Gathering the water out of each ground water zone and interacting with these characteristics, the stream water of which recession characteristics is K_s when it flows into the region, will flow out from the outlet of the region with the value K^*. Equation (54) indicates that the value K^* is a weighted mean with respect to the value K_j for the individual zone and the value K_s for the stream. In addition, the weights are defined by the geological and geographical factors as well as the initial distributions q_j and q_s within the region.

Though the above discussion is restricted to the case shown by Eq. (43), it may be understood that the averaging process and interactions cause the variation of recession characteristics in the runoff process. In the actual watershed, it is apparent that besides being averaged in a system, the recession characteristics are averaged repeatedly through many systems in the runoff process.

IV. 4 Discussion

In Chapter III, the recession characteristics K values are discussed for each basin as lumped system. The value corresponds to the value K^* which is averaged within the region.

If we discuss the averaging process in an actual basin based on the characteristics in the small individual regions, many hydrologic data should be necessary. Unfortunately, there are very few such detailed data for sufficient discussions. Therefore, we have to handle the actual problems by another way, that is, discussion on the value K for discharge at the gauging stations located along a stream.

In this section, we wish to check the the expression Eq. (54) for the Nagara River for an example and to make several discussions on the averaging process in the runoff process.

The Nagara River is one of the largest rivers in Central Japan, and its fountain head is located at the mountainous area of the Japan Alps and it runs through the center of the Nohbi Plain and flows into Ise Bay. The general remarks of the river and the basins concerned here, are summarized in Table 2 and Fig. 4. Since no artificial operations have been made for flood controls and any water policies, the river basin and runoff phenomena remain at natural state and also ideal for the studies on ground water runoff.

The water which flows through Tsurugi gauging station, reaches at Kamita station together with the runoff discharge out of the basins situated between these two stations (say the Basin 1). The analyzed K values for these stations are also listed in Table 3. In the table only those which are used in the following discussions are listed.

Table 2 General remarks of the watersheds of the River Nagara

	Tsurugi	Kamita
Catchment area	223.0 km^2	713.0 km^2
Length of the main stream	29.6 km	57.0 km

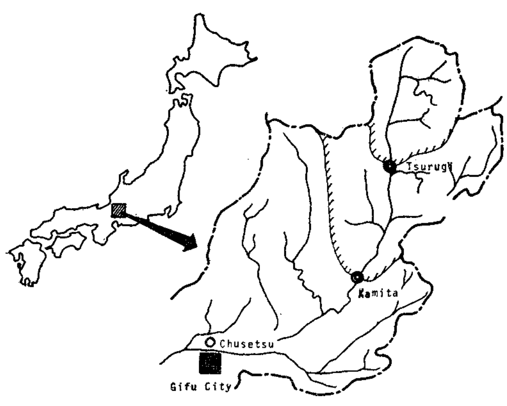

Fig. 4 General Remarks of the Watershed

Table 3 The recession factor K obtained for the watersheds of the River Nagara

Initial day for the recession	Tsurugi		Kamita	
	Initial discharge (m^3/s)	Recession factor K ($\times 10^{-5}$)	Initial discharge (m^3/s)	Recession factor K ($\times 10^{-5}$)
Sept. 21, 1965	12.1	820	28.0	380
July 27, 1965	14.0	820	34.0	380
July 23, 1964	14.0	850	43.0	337
Sept. 2, 1963	11.0	836	30.0	350
Oct. 7, 1954	12.3	720	35.0	350
Oct. 10, 1953	9.0	780	30.0	350

unit K : m$^{-3/2}$ sec$^{1/2}$ day^{-1}

In order that the results of the preceding section may be made useful, we consider the K value at Tsurugi and Kamita to be K_s and K^* in Eq. (54), respectively, and K_1 as the K value for Basin 1. Then, we may discuss the averaging and interaction processes in Basin 1. Since the values K^* and K_s are known quantities, the unknown factors K_1 and the weights for the averagings may be estimated by Eq. (54), which should be satisfied for each recession state. In the practical computation, in this section, the second term in the numerator of Eq. (54) is ignored approximately, and the values q_1 and q_s are calculated for the values listed in Table 3. The distributions of K_1 values and the ratios of the weights estimated are shown in Fig. 5. In the figure, we can see the concentrative distributions around certain values with respect to both K_1 and Dl/p_1C. In other words, these values seem to be almost constant for the region, in spite of various values of q_1 and q_s.

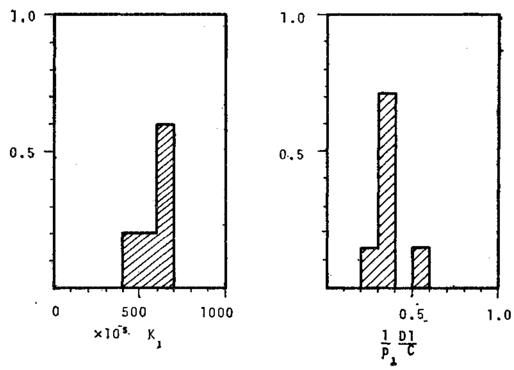

Fig. 5 Normalized Histogram of K_1 and Dl/p_1C (Unit, K_1 : m$^{-3/2}$ sec$^{1/2}$ day^{-1})

As inferred from the facts and also from the discussion made in the last section, value p_1 defined by Eq. (53) consists of the geological and geographical factors in the basin and should be constant for the basin.

Although there are still many problems to be solved, such as the validity of the assumption to adopt Basin 1 as a kinematical system, the fact mentioned above suggests that the averaging of recession states takes place within the Nagara River as expressed by Eq. (54).

Therefore, the factor $q(q_j$ and $q_s)$ is the cause of the variation of recession characteristics for each occassion. The factors q, which indicate the ratios of the discharges out of the small individual regions and of inflow through stream to the total discharge at the outlet, may change their values due to the rainfall distributions case by case. Therefore, the phenomena in the basins for which we can observe the definite K values for any recession states, may correspond to the following cases:

a) q_j and q_s are almost constant for every case,
b) the K^* values are not substantially infiuenced by the q values, regardless the changes of them.

In the small basins, the geological and geographical factors may be considered to distribute uniformly all over the region in the average sense. The small basin in general will be often covered by a rainfall region, and the spatial distributions of rainfall are in nearly the same state for any rainfall. The small basin may be thus considered as of the case a) described above. Moreover, it may be concluded that we can observe the definite K value for the small basin as a whole, as described for the case i) in the last chapter. In other words, it suggests that we may treat such small basin as an unit basin for the studies on ground water runoff.

On the contrary, the spatial and temporal distributions of rainfall will change their states in wide divergences in the very large basin. The runoff in such large basin, moreover, takes place in various regions with different properties. Since the roles of the individual basins are very slight in the whole basin, no matter how the q values variate case by case, K^* value may be kept constant because of the dominant averaging processes in statistical meanings. This case which corresponds to the case b), will explain well the tendency seen in the case iii) in Capter III.

In the basin of which catchment area is neither sufficiently large for the statistical treatment nor small as the unit basins, the runoff phenomena in the various individual regions will have appreciably large effects on the water behaviour in the whole basin. Therefore, it may be expected that the various states of rainfall distributions within the basin result in the variations of recession factor K for each recession state as seen in the case ii) in the precedent chapter.

As discussed above, the tendency of recession characteristics observed in the last chapter may be explained qualitatively by the averaging process expressed by Eq. (54). While the discussion in this chapter has been made on the averaging process within a region as a kinematical system, such averaging will take place repeatedly through many systems in the actual basin.

A detailed conclusion cannot be drawn due to an insufficient number of actual hydrologic data, however, the discussion may indicate the possibility that Eq. (54) will become a new tool to understand the variation of recession characteristics in the runoff process.

V. CONCLUSION

As the first step of studies of which final goal is to clarify the runoff phenomena in connection with the behaviour of basin water within the watershed, the author has made the variational formulation for the movements of ground water, stream water and their interactions as a whole. The analyses indicate that the evolution of water behaviour takes place in a region as a kinematical system so that the local potential takes the stationary value. As the result following to such behaviour, the runoff phenomena and/or the interactions between water components are visually observed.

The physical significances of local potential is loosely discussed because of simplification and several assumptions. It should be noted, however, that the local potential is closely related to the potential energy of the water within the basin. Moreover, the result obtained through the analyses suggests the possibilities that we may understand synthetically the runoff phenomena in connection with the behaviour of water components as a whole. The further generalization of the variational formulation and its physical significances are now in progress.

Regarding the recession characteristics of ground water runoff, the application of the method introduced has been attempted. The expression derived in the present paper seems to explain considerably well in what way the recession characteristics change through the runoff process which involves the various individual regions with different properties. The reason why we may observe K value obviously in the small basin and in the very large basin, has been also explained qualitatively by the expression Eq. (54). Then the problems were reduced in the uniformity of hydrologic quantities for small basins and the repeated statistical processes for the large basins.

The great difficulty encountered when we make the method useful for the practical problem, is what region should be taken as a kinematical system for runoff phenomena. For the actual basin, there are many problems yet unsolved, and further investigation should be made for the detail discussions. However, the method proposed here contributes to the better understandings of runoff and also to relate the mechanism of runoff to the statistical treatments.

We have only noted a few problems of the application, however, many hydraulic and hydrological problems such as changes of behaviour of channel water and/or ground water due to artificial discharge controles may be satisfactorily analyzed by means of variational technique.

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