

FIRST CROSSING PROBABILITY OF TWO RANDOM PROCESSES

*Masaru Hoshiya**

I. INTRODUCTION

We consider herein the probability of crossing of two random stationary Gaussian processes with zero means. When a random excitation is applied to two systems with an initial separation D , they start to oscillate in random fashion. This paper derives the probability of non-crossing (reliability) of the systems for a given time interval $(0, t)$. First, converting the problem into a crossing problem of a random process with level D in the time interval $\{t, t+dt\}$ and then directly applying the available formula by Bendat¹⁾, we set up the reliability of the systems. It is obvious that this probability can be applied to the estimation of the reliability of structural and mechanical systems against a catastrophic failure.

II. CROSSING PROBLEMS OF TWO RANDOM PROCESSES

Let $X(t)$ and $Y(t)$ be a set of sample functions of time, t , from stationary random processes, $\{X(t), Y(t)\}$ with zero means, whose instantaneous amplitudes are associated with a joint probability density function $f_{XY}(x, y)$. Letting $u(t) = X(t) + D$, we assume that

$$P(u(0) = D) = 1 \quad (1)$$

and
$$P(Y(0) = 0) = 1 \quad (2)$$

where D is a deterministic constant. In other words, D indicates the initial distance between $u(0)$ and $Y(0)$. $P(\)$ reads "probability that." A set of sample functions, $u(t)$ and $Y(t)$ are illustrated in Fig. 1.

Let us first consider the probability, $N(D, t)dt$ that $u(t)$ and $Y(t)$ cross each other in the infinitesimal time interval, t and $t+dt$. Define the following events:

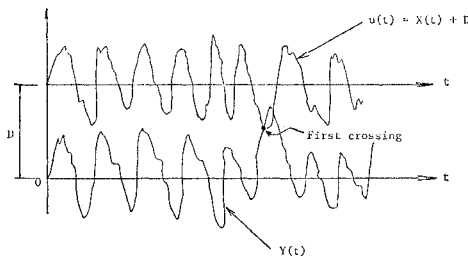


Fig. 1 A set of sample functions, $u(t)$ and $Y(t)$

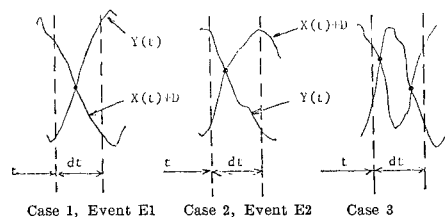


Fig. 2 Events of Crossing

* Assistant Professor of Engineering Mechanics, Virginia Polytechnic Institute and State University.

$$E_1 = X(t) + D > Y(t) \cap Y(t + dt) > X(t + dt) + D \quad (3)$$

and

$$E_2 = X(t) + D < Y(t) \cap Y(t + dt) < X(t + dt) + D \quad (4)$$

where intersection, $A \cap B$ means the simultaneous occurrence of events, A and B .

Then the probability of crossing, $N(D, t)dt$, is given by

$$N(D, t)dt = P(E_1 \cup E_2) \quad (5)$$

where symbol \cup indicates union.

Equation (5) is derived from the consideration that the crossing occurs at most only once during the infinitesimal time interval, t and $t + dt$, and consequently the crossing is either case 1 or case 2, and we never have case 3 in Fig. 2.

Since events E_1 and E_2 are obviously mutually exclusive, Eq. (5) becomes

$$\begin{aligned} N(D, t)dt &= P(E_1) + P(E_2) \\ &= P(X(t) + D > Y(t) \cap Y(t) + Y'(t)dt > X(t) + X'(t)dt + D) \\ &\quad + P(X(t) + D < Y(t) \cap Y(t) + Y'(t)dt < X(t) + X'(t)dt + D) \end{aligned} \quad (6)$$

where we assumed that $X(t)$ and $Y(t)$ are linear for small dt , and we have

$$X(t + dt) = X(t) + X'(t)dt \quad \text{and} \quad Y(t + dt) = Y(t) + Y'(t)dt.$$

Let

$$Z(t) = X(t) - Y(t) \quad (7)$$

and therefore

$$Z'(t) = X'(t) - Y'(t). \quad (8)$$

With Eqs. (7) and (8), Eq. (6) becomes

$$N(D, t)dt = P(-Z'(t)dt > Z(t) + D > 0) + P(-Z'(t)dt < Z(t) + D < 0). \quad (9)$$

Given a joint probability density function, $f_{ZZ'}(\eta, \xi)$, we can calculate Eq. (9) as follows:

$$N(D, t)dt = \int_{-\infty}^0 \int_{-D}^{-\xi dt - D} f_{ZZ'}(\eta, \xi) d\eta d\xi + \int_0^{\infty} \int_{-\xi dt - D}^{-D} f_{ZZ'}(\eta, \xi) d\eta d\xi. \quad (10)$$

For small dt , the η variable is substantially equal to $-D$ in the η integration of Eq. (10), and hence we have

$$N(D, t)dt = \int_0^{-\infty} \xi dt f_{ZZ'}(-D, \xi) d\xi + \int_0^{\infty} \xi dt f_{ZZ'}(-D, \xi) d\xi. \quad (11)$$

The probability of crossing (or the expected number of crossings) per unit time at any given time, t , except for $t=0$ is given by

$$N(D, t) = \int_0^{\infty} \xi f_{ZZ'}(-D, \xi) d\xi + \int_0^{\infty} \xi f_{ZZ'}(-D, \xi) d\xi. \quad (12)$$

We should note that Eq. (12) is time independent because $Z(t)$ and $Z'(t)$, are stationary, and that Eq. (12) is essentially identical to the expected number of crossing at level $(-D)$ per unit time for the process, $Z(t)$, which is given by reference 2).

If we should assume that $X(t)$ and $Y(t)$ are independent of $X'(t)$ and $Y'(t)$

respectively, the probability density function, $f_{ZZ'}(\eta, \xi)$ can be calculated with given probability density functions, $f_{XY}(x, y)$ and $f_{X'Y'}(x', y')$ as follows.

In this case, we have

$$f_{ZZ'}(\eta, \xi) = f_Z(\eta) f_{Z'}(\xi). \tag{13}$$

The probability distribution function of $Z(t)$ is

$$\begin{aligned} F_Z(a) &= P(Z \leq a) = P(X - Y \leq a) \\ &= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\xi+a} f_{XY}(\eta, \xi) d\eta = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^a f_{XY}(\eta + \xi, \xi) d\eta. \end{aligned} \tag{14}$$

Thus, the probability density function, $f_Z(a)$ is given by

$$f_Z(a) = \frac{dF_Z(a)}{da} = \int_{-\infty}^{\infty} f_{XY}(a + \xi, \xi) d\xi. \tag{15}$$

Similarly

$$f_{Z'}(a) = \int_{-\infty}^{\infty} f_{X'Y'}(a + \xi, \xi) d\xi. \tag{16}$$

The assumption that $X(t)$ or $Y(t)$ is independent of $X'(t)$ or $Y'(t)$ is not true in general. However, if $X(t)$ and $Y(t)$ are Gaussian in addition to being stationary with zero means, then $X'(t)$ and $Y'(t)$ are also stationary, Gaussian with zero means and, $X'(t)$ and $Y'(t)$ are independent of $X(t)$ and $Y(t)$ ³⁾.

In the following analysis, assume that $X(t)$ and $Y(t)$ are given by a jointly Gaussian density function with zero means. That is,

$$f_{XY}(\eta, \xi) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \exp\left\{ \frac{-1}{2(1-\rho_{XY}^2)} \left[\frac{\eta^2}{\sigma_X^2} - \frac{2\rho_{XY}\eta\xi}{\sigma_X\sigma_Y} + \frac{\xi^2}{\sigma_Y^2} \right] \right\} \tag{17}$$

where σ_X and σ_Y are standard deviations of X and Y respectively. ρ_{XY} is a correlation coefficient of X and Y .

Introducing Eq. (17) to Eq. (15), and performing the integration, we have

$$f_Z(a) = \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left(-\frac{a^2}{2\sigma_Z^2}\right). \tag{18}$$

Similarly

$$f_{Z'}(a) = \frac{1}{\sqrt{2\pi}\sigma_{Z'}} \exp\left(-\frac{a^2}{2\sigma_{Z'}^2}\right) \tag{19}$$

where

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y\rho_{XY} \tag{20}$$

and

$$\sigma_{Z'}^2 = \sigma_{X'}^2 + \sigma_{Y'}^2 - 2\sigma_{X'}\sigma_{Y'}\rho_{X'Y'} \tag{21}$$

σ_b is the standard deviation of b .

Now back to Eq. (12), together with Eqs. (13), (18), and (19), we have for the probability of crossing per unit time

$$N(D, t) = \frac{1}{\pi} \frac{\sigma_{z'}}{\sigma_z} \exp\left(-\frac{D^2}{2\sigma_{z'}^2}\right) \quad (22)$$

$$\text{or } N(D, t) = \frac{1}{\pi} \sqrt{\frac{\sigma_{x'}^2 + \sigma_{y'}^2 - 2\sigma_{x'}\sigma_{y'}\rho_{x'y'}}{\sigma_{x'}^2 + \sigma_{y'}^2 - 2\sigma_{x'}\sigma_{y'}\rho_{xy}}} \exp\left\{\frac{-D^2}{2(\sigma_{x'}^2 + \sigma_{y'}^2 - 2\sigma_{x'}\sigma_{y'}\rho_{xy})}\right\}. \quad (23)$$

Furthermore, all parameters in Eq. (23) can be determined if the autocorrelation functions, $R_x(\tau), R_y(\tau), \dots$ or the spectral density functions, $S_x(\omega), S_y(\omega), \dots$ of $X(t), Y(t), \dots$ are available as follows⁴⁾

$$\sigma_{x'}^2 = \int_{-\infty}^{\infty} S_x(\omega) d\omega = R_x(0) \quad (24)$$

$$\sigma_{y'}^2 = \int_{-\infty}^{\infty} S_y(\omega) d\omega = R_y(0) \quad (25)$$

$$\sigma_{x'}^2 = \int_{-\infty}^{\infty} \omega^2 S_x(\omega) d\omega = -R_x''(0) \quad (26)$$

$$\text{and } \sigma_{y'}^2 = \int_{-\infty}^{\infty} \omega^2 S_y(\omega) d\omega = -R_y''(0). \quad (27)$$

Regarding the correlation coefficients ρ_{xy} and $\rho_{x'y'}$, we know at least that $0 \leq \rho_{xy}$ or $\rho_{x'y'} \leq 1$. However, the actual values of ρ_{xy} and $\rho_{x'y'}$ may depend upon the value D and the other environmental factors of the real phenomena.

Given $N(D, t)$, the probability of non crossing in a time interval $(0, t)$ can be determined as follows¹⁾

$$P(t) = \exp\{-N(D, t)t\}. \quad (28)$$

Eq. (28) indicates the reliability of two random processes, $U(t)$ and $Y(t)$ against the catastrophic failure defined by the first crossing.

If the evaluation of safe initial spacing, D , is required, the equation for D corresponding to a given level of safety or reliability, $P(t)$, for a time interval $(0, t)$ is obtained from Eqs. (22) and (28) as follows.

$$D = \sqrt{2\sigma_{z'}^2 \ln\left[-\frac{t}{\pi} \frac{\sigma_z}{\sigma_{z'}} \frac{1}{\ln P(t)}\right]}. \quad (29)$$

III. APPLICATION

To show the application of the preceding analyses, a simple mass, spring, and dash pot system is considered (Fig. 3).

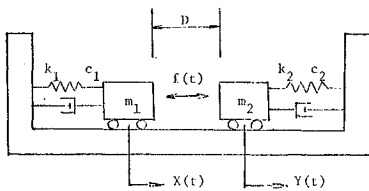


Fig. 3 Structural Model

Determine the allowable safe distance D with the reliability level of 99% for the time duration, t , when the two masses are excited by a random white noise, $f(t)$.

The governing equations for masses m_1 and m_2 are

$$m_1 \ddot{X} + c_1 \dot{X} + k_1 X = f(t) \quad (30)$$

$$\text{and } m_2 \ddot{Y} + c_2 \dot{Y} + k_2 Y = f(t) \quad (31)$$

Frequency response functions for Eqs. (30) and (31) are respectively

$$H_X(\omega) = \frac{1}{m_1} \frac{1}{\omega_{n1}^2 - \omega^2 + 2i\xi_1\omega_{n1}\omega} \quad (32)$$

and

$$H_Y(\omega) = \frac{1}{m_2} \frac{1}{\omega_{n2}^2 - \omega^2 + 2i\xi_2\omega_{n2}\omega} \quad (33)$$

where
$$\xi_i = \frac{c_i}{2\sqrt{k_i m_i}} \quad \text{and} \quad \omega_{ni}^2 = \frac{k_i}{m_i} \quad i=1, 2$$

By applying the well known relationship for spectral density functions of input and output with Eqs. (32) and (33), together with $S_f(\omega) = S_0$ for a white noise, we have

$$\sigma_{X^2} = \int_{-\infty}^{\infty} S_X(\omega) d\omega = \int_{-\infty}^{\infty} |H_X(\omega)|^2 S_f(\omega) d\omega = \frac{S_0}{m_1^2} \frac{\pi}{2\xi_1\omega_{n1}^3} \quad (34)$$

and
$$\sigma_{X'}^2 = \int_{-\infty}^{\infty} \omega^2 S_X(\omega) d\omega = \int_{-\infty}^{\infty} \omega^2 |H_X(\omega)|^2 S_f(\omega) d\omega = \frac{S_0}{m_1^2} \frac{\pi}{2\xi_1\omega_{n1}} \quad (35)$$

Similarly we have

$$\sigma_{Y^2} = \frac{S_0}{m_2^2} \frac{\pi}{2\xi_2\omega_{n2}^3} \quad (36)$$

and
$$\sigma_{Y'}^2 = \frac{S_0}{m_2^2} \frac{\pi}{2\xi_2\omega_{n2}} \quad (37)$$

If $\frac{m_1}{m_2} = \frac{k_1}{k_2} = \frac{c_1}{c_2} = 2$, then we have

$$\frac{\sigma_{Z'}}{\sigma_Z} = \sqrt{\left(\frac{k_1}{m_1}\right) \frac{5-4\rho_{X'Y'}}{5-4\rho_{XY}}} \quad \text{and} \quad \sigma_{Z^2} = \frac{S_0\pi}{c_1 k_1} \quad (39)$$

Introducing Eq. (39), together with $P(t) = 0.99$ into Eq. (29), we have

$$D = \sqrt{\frac{2S_0\pi}{c_1 k_1} \ln \left[-\frac{t}{\pi} \sqrt{\left(\frac{k_1}{m_1}\right) \frac{5-4\rho_{X'Y'}}{5-4\rho_{XY}}} \frac{1}{\ln 0.99} \right]} \quad (40)$$

If moreover $\rho_{XY} = \rho_{X'Y'} = 1$, $\frac{S_0\pi}{c_1 k_1} = 2$ and $\frac{k_1}{m_1} = \pi^2$, then we have the following results, where the units of parameters are not specified.

For 99% Reliability				
Time duration t	(10 ²) 100	(10 ³) 1000	(10 ⁴) 10000	(10 ⁵) 100000
Safe distance D	6.0	6.8	7.4	8.0

The following arguments are valid from this example.

1. Correlation between $X(t)$ and $Y(t)$, and $X'(t)$ and $Y'(t)$, must be strong since the two mass systems are subjected to the identical excitation force. Thus we may assume $\rho_{XY} = \rho_{X'Y'} = 1$ (perfect correlation) in this particular example.
2. In general, however, if $\rho_{XY} = \rho_{X'Y'}$, the safe distance becomes independent of

the correlation.

3. If a higher reliability for a longer time duration is specified, more distance is required for sufficient safety.
4. In the extreme case when the two mass systems are identical ($m_1=m_2$, $k_1=k_2$, and $c_1=c_2$), these two systems never cross and the safe distance will be zero.
5. Although we discussed the simple model, the applications of this probability theory are unlimited. For example, an entanglement probability of two parallel, vertical long cables in ocean due to dynamic turbulence forces can be calculated if we assume the forces are stationary Gaussian with zero means⁵⁾.

IV. CONCLUSION AND ACKNOWLEDGEMENT

The probability of crossing of two random stationary Gaussian processes with zero means was obtained. The possible applications of this probability to actual systems were discussed.

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