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ANALYSIS OF ISOTROPIC AND ORTHOTROPIC RECTANGULAR PLATES WITH TWO OPPOSITE SIDES SUPPORTED BY EDGE-COLUMNS

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I. INTRODUCTION

A plate structure under which columns are directly connected with the plate at the edges, as shown in Fig. 1, is a kind of flat slab, such a structure is often found in the fields of civil engineering, architecture, ship building and machinery. However, the merits of this structure, such as more economic, more rational and more functional compared with the other plate structures with beams and columns, are not utilized enough in their construction, since the practical method of analysis and design of this structure is not settled.

The analysis of the structure with edge-columns as in Fig. 1 is different from that of the flat slab with intermediate-columns in former investigations[1),2),3)]. That is, the redundants of columns in the latter are dealt as a given load on plate, or a plate in the latter is divided into two parts, strip with colums and without, in analysis. On the other hand, the redundants of columns in the structure with edge-columns should be dealt in the boundary conditions of plate. Therefore, it is unable to apply the theories proposed for the latter to the analysis of flat slab in this paper. And, except for such approximate solutions as the difference method and the finite element method, there are only a few investigations for the exact solutions on circular plates supported by edge-columns[4),5),6)], that on a rectangular plate simply supported at conners[4)], that on a isotropic rectangular plate supported at a few points[7)], etc.

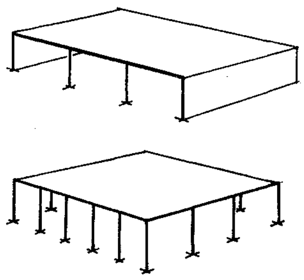

Fig. 1

In this paper, the plate in flat slab is simply supported at two opposite sides. The other sides are supported by the edge-columns arranged arbitrarily. It is provided that the thickness of plate in flat slab is constant and that the deflection of a plate is small in comparison with its thickness.

Since a plate is directly supported by edge-columns, vertical reactions, horizontal reactions in the x and y directions, bending moments in the x and y directions, and twisting moments in the plane of plate are transmitted from the plate to the columns in the structure as in Fig. 2. If the plate is only subjected to lateral loads, then twist-

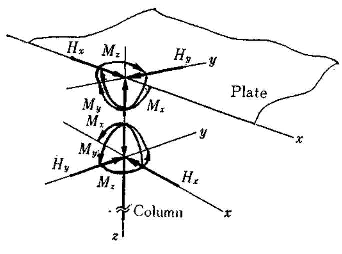

Fig. 2

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ing moments and horizontal reactions at the plate to column connections may be small and neglected. Also, the influence of bending moments, which is generally omitted in the theoretical analysis for a flat slab, can be neglected. Hence, it may be assumed that only vertical reactions are transmitted from the plate to the column.

II. ANALYSIS

(1) Deflection of Plate

To analyze this plate structure, the $x-y$ plane as the middle plane of plate before deformation is introduced. The x-axis is the side AB, the y-axis is the side AC, and the positive direction of z-axis tends downward, as shown in Fig. 3.

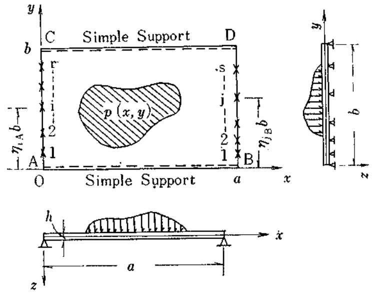

Fig. 3

The known fundamental differential equation for a orthotropic plate under an arbitrary lateral load, $p(x, y)$, can be represented as follows[4];

$$D_x\frac{\partial^4 w}{\partial x^4}+(\nu_y D_x+4C+\nu_x D_y)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_y\frac{\partial^4 w}{\partial y^4}=p(x, y)\ , \tag{1}$$

in which

w: deflection of plate,
h: thickness of plate,
E_x, E_y: Young's moduli in x and y directions of plate,
ν_x, ν_y : Poisson's ratios in x and y directions of plate,
$D_x=E_x h^3/\{12(1-\nu_x\nu_y)\}$, $D_y=E_y h^3/\{12(1-\nu_x\nu_y)\}$, $C=Gh^3/12$,
G: shear modulus of plate.

The general solution of Eq. (1) is given as the sum of the complementary function w_1, and the particular solution w_0.

In this paper, the plate in flat slab is simply supported at two opposite sides, AB and CD, and supported by edge-columns at the other sides. Then, the following equation is assumed as w_1, which is satisfied with the boundary conditions for simple supports at sides AB and CD:

$$w_1=\sum_{n=1}^{\infty} X_n \sin \beta_n y\ , \tag{2}$$

in which

X_n: function of x only,
b : length of sides AC and BD, $\beta_n=n\pi/b$.

Substituting Eq. (2) into the homogeneous equation of Eq. (1), the differential equation for X_n is given as follows:

$$\frac{\partial X_n^4}{\partial x^4}-(\kappa_1^2+\kappa_2^2)\beta_n^2\frac{\partial^2 X_n}{\partial x^2}+(\kappa_1\kappa_2)^2\beta_n^4 X_n=0 \tag{3}$$

where

$$\kappa_1^2=\bar{\kappa}^2+\sqrt{\bar{\kappa}^4-\kappa^2}\ ,\quad \kappa_2^2=\bar{\kappa}^2-\sqrt{\bar{\kappa}^4-\kappa^2}\ ,\quad \kappa^2=D_y/D_x\ ,\quad \bar{\kappa}^2=(\nu_y+\nu_x\kappa^2+4C/D_x)/2\ .$$

Three kinds of general solution of Eq. (3) are found in the following forms depending on the value of $(\bar{\kappa}^4-\kappa^2)$:

① Case I $(\bar{\kappa}^4-\kappa^2>0)$:

$$X_n^{\mathrm{I}}=A_n^{\mathrm{I}}\sinh\kappa_1\beta_nx+B_n^{\mathrm{I}}\cosh\kappa_1\beta_nx+C_n^{\mathrm{I}}\sinh\kappa_2\beta_nx+D_n^{\mathrm{I}}\cosh\kappa_2\beta_nx\ , \tag{4}$$

② Case II $(\bar{\kappa}^4-\kappa^2=0)$:

$$X_n^{\mathrm{II}}=(A_n^{\mathrm{II}}+B_n^{\mathrm{II}}x)\sinh\bar{\kappa}\beta_nx+(C_n^{\mathrm{II}}+D_n^{\mathrm{II}}x)\cosh\bar{\kappa}\beta_nx\ , \tag{5}$$

③ Case III $(\bar{\kappa}^4-\kappa^2<0)$:

$$X_n^{\mathrm{III}}=(A_n^{\mathrm{III}}\sinh\kappa_1'\beta_nx+B_n^{\mathrm{III}}\cosh\kappa_1'\beta_nx)\sin\kappa_2'\beta_nx$$
$$+(C_n^{\mathrm{III}}\sinh\kappa_1'\beta_nx+D_n^{\mathrm{III}}\cosh\kappa_1'\beta_nx)\cos\kappa_2'\beta_nx\ , \tag{6}$$

where

$$\kappa_1'=\sqrt{(\bar{\kappa}^2+\kappa)/2}\ ,\qquad \kappa_2'=\sqrt{(\bar{\kappa}^2-\kappa)/2}$$

in which A_n, B_n, C_n and D_n are constants of integration.

In Case II, the solution of complementary equation for w_1 becomes multiple root, i.e. $\bar{\kappa}^4-\kappa^2=0$, which is the same kind of solution as one of an isotropic plate. Substituting $D_x=D_y=D$ and $\nu_x=\nu_y=\nu$ into Eq. (5), κ and $\bar{\kappa}$ result in unit, which is the value for the solution of isotropic plate.

Since many isotropic plates are used in practice, a complementary function for them will be given as Case IV.

④ Case IV

$$X_n^{\mathrm{IV}}=(A_n^{\mathrm{IV}}+B_n^{\mathrm{IV}}x)\sinh\beta_nx+(C_n^{\mathrm{IV}}+D_n^{\mathrm{IV}}x)\cosh\beta_nx\ . \tag{7}$$

Expanding a given load, $p(x, y)$, by double trigonometric series, we obtain

$$p(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{mn}\sin\alpha_mx\sin\beta_ny\ , \tag{8}$$

where $F_{mn}=\dfrac{4}{ab}\displaystyle\int_0^a\int_0^b p(x,y)\sin\alpha_mx\sin\beta_ny\,dxdy$,

a: length of sides AB and CD, $\alpha_m=m\pi/a$.

And, the particular solution of Eq. (1), w_0, may be assumed by

$$w_0=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}G_{mn}\sin\alpha_mx\sin\beta_ny\ , \tag{9}$$

in which G_{mn} is a constant depending on m and n.

Substituting Eqs. (8) and (9) into Eq. (1), the following equation for G_{mn} in Eq. (9) is found;

$$G_{mn}=F_{mn}/D_x(\alpha_m^4+2\bar{\kappa}^2\alpha_m^2\beta_n^2+\kappa^2\beta_n^4)\ . \tag{10}$$

Substituting Eq. (10) into Eq. (9) and adding to Eqs. (4)~(7), the general solutions of Eq. (1) can be written as follows, depending on the cases;

① Case I:

$$w^{\mathrm{I}}=\sum_{n=1}^{\infty}(A_n^{\mathrm{I}}\sinh\kappa_1\beta_n x+B_n^{\mathrm{I}}\cosh\kappa_1\beta_n x+C_n^{\mathrm{I}}\sinh\kappa_2\beta_n x+D_n^{\mathrm{I}}\cosh\kappa_2\beta_n x)\sin\beta_n y$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{F_{mn}}{\bar{K}_{mn}^{\mathrm{I}}}\sin\alpha_m x\sin\beta_n y\ , \tag{11}$$

in which $\bar{K}_{mn}^{\mathrm{I}}=D_x(\alpha_m^2+\kappa_1^2\beta_n^2)(\alpha_m^2+\kappa_2^2\beta_n^2)$,

② Case II:

$$w^{\mathrm{II}}=\sum_{n=1}^{\infty}\{(A_n^{\mathrm{II}}+B_n^{\mathrm{II}}x)\sinh\bar{\kappa}\beta_n x+(C_n^{\mathrm{II}}+D_n^{\mathrm{II}}x)\cosh\bar{\kappa}\beta_n x\}\sin\beta_n y$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{F_{mn}}{\bar{K}_{mn}^{\mathrm{II}}}\sin\alpha_m x\sin\beta_n y\ , \tag{12}$$

in which $\bar{K}_{mn}^{\mathrm{II}}=D_x(\alpha_m^2+\bar{\kappa}^2\beta_n^2)^2$,

③ Case III:

$$w^{\mathrm{III}}=\sum_{n=1}^{\infty}\{(A_n^{\mathrm{III}}\sinh\kappa_1'\beta_n x+B_n^{\mathrm{III}}\cosh\kappa_1'\beta_n x)\sin\kappa_2'\beta_n x$$
$$+(C_n^{\mathrm{III}}\sinh\kappa_1'\beta_n x+D_n^{\mathrm{III}}\cosh\kappa_1'\beta_n x)\cos\kappa_2'\beta_n x\}\sin\beta_n y$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{F_{mn}}{\bar{K}_{mn}^{\mathrm{III}}}\sin\alpha_m x\sin\beta_n y\ , \tag{13}$$

in which $\bar{K}_{mn}^{\mathrm{III}}=D_x(\alpha_m^4+2\bar{\kappa}^2\alpha_m^2\beta_n^2+\kappa^2\beta_n^4)$,

④ Case IV:

$$w^{\mathrm{IV}}=\sum_{n=1}^{\infty}\{(A_n^{\mathrm{IV}}+B_n^{\mathrm{IV}}x)\sinh\beta_n x+(C_n^{\mathrm{IV}}+D_n^{\mathrm{IV}}x)\cosh\beta_n x\}\sin\beta_n y$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{F_{mn}}{\bar{K}_{mn}^{\mathrm{IV}}}\sin\alpha_m x\sin\beta_n y\ , \tag{14}$$

in which $\bar{K}_{mn}^{\mathrm{IV}}=D(\alpha_m^2+\beta_n^2)^2$.

The integral constants, $A_n^{\mathrm{I\sim IV}}$, $B_n^{\mathrm{I\sim IV}}$, $C_n^{\mathrm{I\sim IV}}$ and $D_n^{\mathrm{I\sim IV}}$, can be determined by the boundary conditions at sides AC and BD. That's, if the vertical reactions, at edge-columns are expressed by trigonometric series, then the boundary conditions are given by

$$\left.\begin{aligned}
&M_x=-D_x\left(\frac{\partial^2 w}{\partial x^2}+\nu_y\frac{\partial^2 w}{\partial y^2}\right)=0\ , \quad V_x=-D_x\left\{\frac{\partial^3 w}{\partial x^3}+\left(\nu_y+\frac{4C}{D_x}\right)\frac{\partial^3 w}{\partial x\partial y^2}\right\}\\
&\qquad=\sum_{n=1}^{\infty}V_{An}\sin\beta_n y \qquad \text{for } x=0\ ,\\
&M_x=-D_x\left(\frac{\partial^2 w}{\partial x^2}+\nu_y\frac{\partial^2 w}{\partial y^2}\right)=0\ , \quad V_x=-D_x\left\{\frac{\partial^3 w}{\partial x^3}+\left(\nu_y+\frac{4C}{D_x}\right)\frac{\partial^3 w}{\partial x\partial y^2}\right\}\\
&\qquad=\sum_{n=1}^{\infty}V_{Bn}\sin\beta_n y \qquad \text{for } x=a\ ,
\end{aligned}\right\} \tag{15}$$

in which V_{An} and V_{Bn} are the coefficients of Fourier series for the vertical reactions V_A and V_B.

Substituting Eqs. (11), (12), (13) and (14) into Eq. (15), the integral constants are obtained as follows;

$$\left.\begin{aligned}
A_n^{\mathrm{I}\sim\mathrm{IV}}&=\frac{a^3}{S_0^{\mathrm{I}\sim\mathrm{IV}}}\left[\left(\sum_{m=1}^{\infty}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{An}}{D_x}\right)U_0^{\mathrm{I}\sim\mathrm{IV}}+\left\{\sum_{m=1}^{\infty}(-1)^m R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{Bn}}{D_x}\right\}T_0^{\mathrm{I}\sim\mathrm{IV}}\right]\\
B_n^{\mathrm{I}\sim\mathrm{IV}}&=\frac{a^2}{S_0^{\mathrm{I}\sim\mathrm{IV}}}\left[\left(\sum_{m=1}^{\infty}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{An}}{D_x}\right)H_0^{\mathrm{I}\sim\mathrm{IV}}+\left\{\sum_{m=1}^{\infty}(-1)^m R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{Bn}}{D_x}\right\}J_0^{\mathrm{I}\sim\mathrm{IV}}\right]\\
C_n^{\mathrm{I}\sim\mathrm{IV}}&=\frac{a^3}{S_0^{\mathrm{I}\sim\mathrm{IV}}}\left[\left(\sum_{m=1}^{\infty}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{An}}{D_x}\right)E_0^{\mathrm{I}\sim\mathrm{IV}}+\left\{\sum_{m=1}^{\infty}(-1)^m R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{Bn}}{D_x}\right\}F_0^{\mathrm{I}\sim\mathrm{IV}}\right]\\
D_n^{\mathrm{I}\sim\mathrm{IV}}&=\frac{a^2}{S_0^{\mathrm{I}\sim\mathrm{IV}}}\left[\left(\sum_{m=1}^{\infty}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{An}}{D_x}\right)P_0^{\mathrm{I}\sim\mathrm{IV}}+\left\{\sum_{m=1}^{\infty}(-1)^m R_{mn}^{\mathrm{I}\sim\mathrm{IV}}-\frac{V_{Bn}}{D_x}\right\}R_0^{\mathrm{I}\sim\mathrm{IV}}\right]
\end{aligned}\right\}\quad(16)$$

in which $\gamma_n=n\pi a/b$, $R_{mn}^{\mathrm{I}\sim\mathrm{IV}}=\{\alpha_m^2+\chi\beta_n^2\}\alpha_m F_{mn}/\bar{K}_{mn}^{\mathrm{I}\sim\mathrm{IV}}$, $\chi=\nu_y+4C/D_x$.

① Case I:

$$S_0^{\mathrm{I}}=\left(\lambda_2\frac{\omega_1^2}{\omega_2}+\omega_2\frac{\lambda_1^2}{\lambda_2}\right)\sinh\kappa_1\gamma_n\sinh\kappa_2\gamma_n-2\lambda_1\omega_1\cosh\kappa_1\gamma_n\cosh\kappa_2\gamma_n+2\lambda_1\omega_1$$

$$U_0^{\mathrm{I}}=\frac{\lambda_1}{\lambda_2}\omega_2\sinh\kappa_1\gamma_n\sinh\kappa_2\gamma_n-\omega_1\cosh\kappa_1\gamma_n\cosh\kappa_2\gamma_n+\omega_1$$

$$T_0^{\mathrm{I}}=\omega_1(\cosh\kappa_1\gamma_n-\cosh\kappa_2\gamma_n)$$

$$H_0^{\mathrm{I}}=\omega_1\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n-\omega_2\frac{\lambda_1}{\lambda_2}\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n$$

$$J_0^{\mathrm{I}}=-\omega_1\sinh\kappa_1\gamma_n+\omega_2\frac{\lambda_1}{\lambda_2}\sinh\kappa_2\gamma_n$$

$$E_0^{\mathrm{I}}=\frac{\omega_1^2}{\omega_2}\sinh\kappa_1\gamma_n\sinh\kappa_2\gamma_n-\omega_1\frac{\lambda_1}{\lambda_2}\cosh\kappa_1\gamma_n\cosh\kappa_2\gamma_n+\omega_1\frac{\lambda_1}{\lambda_2}$$

$$F_0^{\mathrm{I}}=-\omega_1\frac{\lambda_1}{\lambda_2}(\cosh\kappa_1\gamma_n-\cosh\kappa_2\gamma_n)$$

$$P_0^{\mathrm{I}}=-\frac{\omega_1^2}{\omega_2}\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n+\omega_1\frac{\lambda_1}{\lambda_2}\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n$$

$$R_0^{\mathrm{I}}=\frac{\omega_1^2}{\omega_2}\sinh\kappa_1\gamma_n-\omega_1\frac{\lambda_1}{\lambda_2}\sinh\kappa_2\gamma_n$$

$$\lambda_1=\kappa_1(\kappa_1^2-\chi),\qquad\lambda_2=\kappa_2(\kappa_2^2-\chi),\qquad\omega_1=\kappa_1^2-\nu_y,\qquad\omega_2=\kappa_2^2-\nu_y.$$

② Case II:

$$S_0^{\mathrm{II}}=\left\{4\bar{\kappa}(3\bar{\kappa}^2-\chi)-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)^2}{\bar{\kappa}(\bar{\kappa}^2-\chi)}-\frac{4\bar{\kappa}^3(\bar{\kappa}^2-\chi)}{\bar{\kappa}^2-\nu_y}\right\}\sinh^2\bar{\kappa}\gamma_n+\bar{\kappa}(\bar{\kappa}^2-\nu_y)(\kappa_1^2-\chi)\gamma_n^2$$

$$U_0^{\mathrm{II}}=\frac{1}{\gamma_n^3}\left\{\frac{2(3\bar{\kappa}^2-\chi)}{\bar{\kappa}^2-\chi}-\frac{4\bar{\kappa}^2}{\bar{\kappa}^2-\nu_y}\right\}\sinh^2\bar{\kappa}\gamma_n+\frac{1}{\gamma_n}(\bar{\kappa}^2-\nu_y)$$

$$T_0^{\mathrm{II}}=\frac{(3\bar{\kappa}^2-\chi)(\bar{\kappa}^2-\nu_y)}{\bar{\kappa}(\bar{\kappa}^2-\chi)\gamma_n^2}\sinh\bar{\kappa}\gamma_n$$

$$H_0^{\mathrm{II}}=-\frac{1}{\gamma_n^2}\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\cosh\bar{\kappa}\gamma_n-\frac{1}{\gamma_n}(\bar{\kappa}^2-\nu_y)$$

$$J_0^{\mathrm{II}}=\frac{1}{\gamma_n^2}\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n+\frac{1}{\gamma_n}(\bar{\kappa}^2-\nu_y)\cosh\bar{\kappa}\gamma_n$$

$$E_0^{\mathrm{II}}=\frac{1}{\gamma_n^3}\left\{\frac{4\bar{\kappa}^2}{\bar{\kappa}^2-\nu_y}-\frac{2(3\bar{\kappa}^2-\chi)}{\bar{\kappa}^2-\chi}\right\}\sinh\bar{\kappa}\gamma_n\cosh\bar{\kappa}\gamma_n+\frac{2\bar{\kappa}}{\gamma_n^2}$$

$$F_0^{\mathrm{II}}=-\frac{1}{\gamma_n^3}\left\{\frac{4\bar{\kappa}^2}{\bar{\kappa}^2-\nu_y}-\frac{2(3\bar{\kappa}^2-\chi)}{\bar{\kappa}^2-\chi}\right\}\sinh\bar{\kappa}\gamma_n-\frac{2\bar{\kappa}}{\gamma_n^2}\cosh\bar{\kappa}\gamma_n$$

$$P_0^{\mathrm{II}}=\frac{1}{\gamma_n^2}\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh^2\bar{\kappa}\gamma_n$$

$$R_0^{\mathrm{II}}=-\frac{1}{\gamma_n}(\bar{\kappa}^2-\nu_y)\sinh\bar{\kappa}\gamma_n\ .$$

③ Case III:

$$S_0^{\mathrm{III}}=\frac{\gamma_n^3}{\rho_1\tau_2}\{(\rho_1^2+\rho_2^2)(\tau_1^2+\tau_2^2)\sinh^2\kappa_1'\gamma_n\sin^2\kappa_2'\gamma_n+(\rho_1\tau_2-\rho_2\tau_1)^2\sinh^2\kappa_1'\gamma_n\cos^2\kappa_2'\gamma_n\}$$

$$U_0^{\mathrm{III}}=\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}\sinh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n+\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1\tau_2}\sin\kappa_2'\gamma_n\cos\kappa_2'\gamma_n$$

$$T_0^{\mathrm{III}}=-\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n-\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1\tau_2}\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$

$$H_0^{\mathrm{III}}=\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}\sinh^2\kappa_1'\gamma_n\sin^2\kappa_2'\gamma_n-\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}\sinh^2\kappa_1'\gamma_n\cos^2\kappa_2'\gamma_n$$
$$-\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\cosh^2\kappa_1'\gamma_n\sin^2\kappa_2'\gamma_n$$

$$J_0^{\mathrm{III}}=\frac{\tau_1^2+\tau_2^2}{\tau_2}\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$

$$E_0^{\mathrm{III}}=\frac{\tau_1^2+\tau_2^2}{\tau_2}\sinh^2\kappa_1'\gamma_n\sin^2\kappa_2'\gamma_n+\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\sinh^2\kappa_1'\gamma_n\cos^2\kappa_2'\gamma_n$$

$$F_0^{\mathrm{III}}=-\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$

$$P_0^{\mathrm{III}}=-\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\sinh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n-\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\sin\kappa_2'\gamma_n\cos\kappa_2'\gamma_n$$

$$R_0^{\mathrm{III}}=\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n+\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$

$$\rho_1=\kappa_1'(\kappa_1'^2-3\kappa_2'^2-\chi)\,,\qquad \rho_2=\kappa_2'(3\kappa_1'^2-\kappa_2'^2-\chi)\,,$$
$$\tau_1=\kappa_1'^2-\kappa_2'^2-\nu_y\,,\qquad \tau_2=2\kappa_1'\kappa_2'\,.$$

④ Case IV:

$$S_0^{\mathrm{IV}}=(3+\nu)^2\sinh^2\gamma_n-(1-\nu)^2\gamma_n^2$$

$$U_0^{\mathrm{IV}}=-\frac{2(3+\nu)}{(1-\nu)\gamma_n^3}\sinh^2\gamma_n+\frac{1-\nu}{\gamma_n}$$

$$T_0^{\mathrm{IV}}=-\frac{1+\nu}{\gamma_n^2}\sinh\gamma_n$$

$$H_0^{\mathrm{IV}}=-\frac{3+\nu}{\gamma_n^2}\sinh\gamma_n\cosh\gamma_n-\frac{1-\nu}{\gamma_n}$$

$$J_0^{\mathrm{IV}}=\frac{3+\nu}{\gamma_n^2}\sinh\gamma_n+\frac{1-\nu}{\gamma_n}\cosh\gamma_n$$

$$E_0^{\mathrm{IV}}=\frac{2(3+\nu)}{(1-\nu)\gamma_n^3}\sinh\gamma_n\cosh\gamma_n+\frac{2}{\gamma_n^2}$$

$$F_0^{\mathrm{IV}}=-\frac{2(3+\nu)}{(1-\nu)\gamma_n^3}\sinh\gamma_n-\frac{2}{\gamma_n^2}\cosh\gamma_n$$

$$P_0^{\mathrm{IV}}=\frac{3+\nu}{\gamma_n^2}\sinh^2\gamma_n$$

$$R_0^{\mathrm{IV}}=-\frac{1-\nu}{\gamma_n}\sinh\gamma_n\,.$$

On the other hand, the edge-columns at the side AC in flat slab are given such numbers $1, 2,\cdots, i,\cdots,$ and r, and at the side BD, as $1, 2,\cdots, j,\cdots,$ and s, as in Fig. 3. Expanding the redundunt R_i^A, which is the vertical reaction of the edge-column "i" at the side AC, by trigonometric series, we obtain

$$\{R_i^A\}_{SF}=\sum_{n=1}^{\infty}R_{in}^A\sin\beta_n y \tag{17}$$

in which $i=1, 2,\cdots, r$; $R_{in}^A=2/b\cdot R_i^A\cdot\sin n\pi\eta_{iA}$.

$\eta_{iA}b$ is the co-ordinate in y direction of the edge-column "i", and $\{\ \}_{SF}$ means the sine series with dimension $[FL^{-1}]$.

Summing up all of $\{R_i^A\}_{SF}$ given by Eq. (17), the vertical distributed reaction at the side AC in flat slab is found as follows;

$$V_A=\sum_{n=1}^{\infty}V_{An}\sin\beta_n y=\sum_{n=1}^{\infty}\left(\sum_{i=1}^{r}\frac{2}{b}R_i^A\sin n\pi\eta_{iA}\right)\sin\beta_n y$$
$$\therefore\quad V_{An}=\frac{2}{b}\sum_{i=1}^{r}R_i^A\sin n\pi\eta_{iA}\,. \tag{18}$$

Similarly, denoting $(0, \eta_{jB}b)$ and R_j^B for the co-ordinate and the vertical reaction of the edge-column "j" at the side BD in flat slab, the arbitrary constant V_{Bn} in

Eq. (15) will be given as

$$V_{Bn}=-\frac{2}{b}\sum_{j=1}^{s} R_j{}^B \sin n\pi\eta_{jB} \,. \tag{19}$$

After substituting Eqs. (18) and (19) into Eq. (16), the elastic surfaces of plates in flat slab are expressed as follows;

$$w^{\mathrm{I}\sim\mathrm{IV}}=\frac{a^4}{D_x}\Bigg[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{1}{K_{mn}^{\mathrm{I}\sim\mathrm{IV}}}\sin m\pi\xi+(\Phi^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\Omega^{\mathrm{I}\sim\mathrm{IV}})\boldsymbol{R}_{mn}^{\mathrm{I}\sim\mathrm{IV}}\right\}F_{mn}\sin n\pi\eta$$
$$-\frac{2}{\mu a^2}\sum_{n=1}^{\infty}\left(\Phi^{\mathrm{I}\sim\mathrm{IV}}\sum_{i=1}^{r}R_i{}^A\sin n\pi\eta_{iA}-\Omega^{\mathrm{I}\sim\mathrm{IV}}\sum_{j=1}^{s}R_j{}^B\sin n\pi\eta_{jB}\right)\sin n\pi\eta\Bigg] \tag{20}$$

where

$$\xi=x/a\,,\qquad \eta=y/b\,,\qquad \mu=b/a\,,\qquad \boldsymbol{R}_{mn}^{\mathrm{I}\sim\mathrm{IV}}=\{(m\pi)^2+\chi\gamma_n{}^2\}(m\pi)/K_{mn}^{\mathrm{I}\sim\mathrm{IV}}.$$

① Case I:

$$K_{mn}^{\mathrm{I}}=\{(m\pi)^2+\kappa_1{}^2\gamma_n{}^2\}\{(m\pi)^2+\kappa_2{}^2\gamma_n{}^2\}$$

$$\Phi^{\mathrm{I}}=\frac{1}{\gamma_n{}^3S_0^{\mathrm{I}}}\Bigg[\omega_1\{\sinh\kappa_1\gamma_n\xi+\cosh\kappa_2\gamma_n\sinh\kappa_1\gamma_n(1-\xi)\}$$
$$-\frac{\lambda_1}{\lambda_2}\omega_2\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n(1-\xi)$$
$$+\frac{\lambda_1}{\lambda_2}\omega_1\{\sinh\kappa_2\gamma_n\xi+\cosh\kappa_1\gamma_n\sinh\kappa_2\gamma_n(1-\xi)\}$$
$$-\frac{\omega_1{}^2}{\omega_2}\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n(1-\xi)\Bigg]$$

$$\Omega^{\mathrm{I}}=\frac{1}{\gamma_n{}^3S_0^{\mathrm{I}}}\Bigg[-\omega_1\{\sinh\kappa_1\gamma_n(1-\xi)+\cosh\kappa_2\gamma_n\sinh\kappa_1\gamma_n\xi\}$$
$$+\frac{\lambda_1}{\lambda_2}\omega_2\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n\xi$$
$$-\frac{\lambda_1}{\lambda_2}\omega_1\{\sinh\kappa_2\gamma_n(1-\xi)+\cosh\kappa_1\gamma_n\sinh\kappa_2\gamma_n\xi\}$$
$$+\frac{\omega_1{}^2}{\omega_2}\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n\xi\Bigg]$$

② Case II:

$$K_{nn}^{\mathrm{II}}=\{(m\pi)^2+\bar{\kappa}^2\gamma_n{}^2\}^2$$

$$\Phi^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\Bigg[\frac{1}{\gamma_n{}^3}\left\{\frac{4\bar{\kappa}^2}{(\bar{\kappa}^2-\nu_y)}-\frac{2(3\bar{\kappa}^2-\chi)}{(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\cosh\bar{\kappa}\gamma_n(1-\xi)$$
$$+\frac{\xi}{\gamma_n{}^2}\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\sinh\bar{\kappa}\gamma_n(1-\xi)$$
$$+\frac{1-\xi}{\gamma_n}(\bar{\kappa}^2-\nu_y)\sinh\bar{\kappa}\gamma_n\xi+\frac{2\bar{\kappa}}{\gamma_n{}^2}\cosh\bar{\kappa}\gamma_n\xi\Bigg]$$

$$\Omega^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\Bigg[-\frac{1}{\gamma_n^3}\left\{\frac{4\bar{\kappa}^2}{(\bar{\kappa}^2-\nu_y)}-\frac{2(3\bar{\kappa}^2-\chi)}{\bar{\kappa}^2-\chi}\right\}\sinh\bar{\kappa}\gamma_n\cosh\bar{\kappa}\gamma_n\xi$$
$$-\frac{(1-\xi)}{\gamma_n^2}\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\sinh\bar{\kappa}\gamma_n\xi$$
$$-\frac{\xi}{\gamma_n}(\bar{\kappa}^2-\nu_y)\sinh\bar{\kappa}\gamma_n(1-\xi)-\frac{2\bar{\kappa}}{\gamma_n^2}\cosh\bar{\kappa}\gamma_n(1-\xi)\Bigg]$$

③ Case III:

$$K_{mn}^{\mathrm{III}}=(m\pi)^4+2\bar{\kappa}^2\cdot(m\pi)^2\cdot\gamma_n^2+\kappa^2\gamma_n^4$$

$$\Phi^{\mathrm{III}}=\frac{1}{S_0^{\mathrm{III}}}\Bigg[\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}\sinh\kappa_1'\gamma_n\sinh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n\sin\kappa_2'\gamma_n(1-\xi)$$
$$+\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1\tau_2}\cosh\kappa_1'\gamma_n\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n\sin\kappa_2'\gamma_n(1-\xi)$$
$$-\frac{(\tau_1^2+\tau_2^2)}{\tau_2}\sinh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n\sin\kappa_2'\gamma_n(1-\xi)$$
$$+\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}\sinh\kappa_1'\gamma_n\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n\cos\kappa_2'\gamma_n(1-\xi)$$
$$-\frac{(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1}\sinh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n\cos\kappa_2'\gamma_n(1-\xi)$$
$$-\frac{(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1}\cosh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n\cos\kappa_2'\gamma_n(1-\xi)\Bigg]$$

$$\Omega^{\mathrm{III}}=\frac{1}{S_0^{\mathrm{III}}}\Bigg[-\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\tau_2\rho_1}\sinh\kappa_1'\gamma_n\sinh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\sin\kappa_2'\gamma_n\xi$$
$$-\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\tau_2\rho_1}\cosh\kappa_1'\gamma_n\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\sin\kappa_2'\gamma_n\xi$$
$$+\frac{\tau_1^2+\tau_2^2}{\tau_2}\sinh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\sin\kappa_2'\gamma_n\xi$$
$$-\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}\sinh\kappa_1'\gamma_n\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\cos\kappa_2'\gamma_n\xi$$
$$+\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\sinh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\cos\kappa_2'\gamma_n\xi$$
$$+\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\cosh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\cos\kappa_2'\gamma_n\xi\Bigg]$$

④ Case IV:

$$K_{mn}^{\mathrm{IV}}=\{(m\pi)^2+\gamma_n^2\}^2$$

$$\Phi^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}\Bigg[\frac{(3+\nu)\xi}{\gamma_n^2}\sinh\gamma_n\sinh\gamma_n(1-\xi)+\frac{2(3+\nu)}{(1-\nu)\gamma_n^3}\sinh\gamma_n\cosh\gamma_n(1-\xi)$$
$$+\frac{(1-\nu)(1-\xi)}{\gamma_n}\sinh\gamma_n\xi+\frac{2}{\gamma_n^2}\cosh\gamma_n\xi\Bigg]$$

$$\Omega^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}\Bigg[-\frac{(3+\nu)(1-\xi)}{\gamma_n^2}\sinh\gamma_n\sinh\gamma_n\xi-\frac{2(3+\nu)}{(1-\nu)\gamma_n^3}\sinh\gamma_n\cosh\gamma_n\xi$$
$$-\frac{(1-\nu)\xi}{\gamma_n}\sinh\gamma_n(1-\xi)-\frac{2}{\gamma_n^2}\cosh\gamma_n(1-\xi)\Bigg]$$

Then, substituting Eq. (20) into the well-known relationship between the elastic surface of plate, and the other displacements and the stresses of plate, the expressions for the slope, the bending moments and the twisting moments of plate can be found. The following expression for the bending moments of plate in the structure are obtained;

$$M_x^{\mathrm{I}\sim\mathrm{IV}}=a^2\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{(m\pi)^2+\nu_y\gamma_n^2}{K_{mn}^{\mathrm{I}\sim\mathrm{IV}}}\sin m\pi\xi+(\bar{\Phi}_x^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\bar{\Omega}_x^{\mathrm{I}\sim\mathrm{IV}})\boldsymbol{R}_{mn}^{\mathrm{I}\sim\mathrm{IV}}\right\}F_{mn}\sin n\pi\eta\right.$$
$$\left.-\frac{2}{\mu a^2}\sum_{n=1}^{\infty}\left(\bar{\Phi}_x^{\mathrm{I}\sim\mathrm{IV}}\sum_{i=1}^{r}R_i^A\sin n\pi\eta_{iA}-\bar{\Omega}_x^{\mathrm{I}\sim\mathrm{IV}}\sum_{j=1}^{s}R_j^B\sin n\pi\eta_{jB}\right)\sin n\pi\eta\right] \tag{21}$$

in which

① Case I:

$$\bar{\Phi}_x^{\mathrm{I}}=\frac{1}{\gamma_n S_0^{\mathrm{I}}}\Big[\omega_1^2\{\sinh\kappa_1\gamma_n\xi+\cosh\kappa_2\gamma_n\sinh\kappa_1\gamma_n(1-\xi)\}$$
$$-\frac{\lambda_1}{\lambda_2}\omega_1\omega_2\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n(1-\xi)$$
$$+\frac{\lambda_1}{\lambda_2}\omega_1\omega_2\{\sinh\kappa_2\gamma_n\xi+\cosh\kappa_1\gamma_n\sinh\kappa_2\gamma_n(1-\xi)\}$$
$$-\omega_1^2\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n(1-\xi)\Big]$$

$$\bar{\Omega}_x^{\mathrm{I}}=\frac{1}{\gamma_n S_0^{\mathrm{I}}}\Big[-\omega_1^2\{\sinh\kappa_1\gamma_n(1-\xi)+\cosh\kappa_2\gamma_n\sinh\kappa_1\gamma_n\xi\}$$
$$+\frac{\lambda_1}{\lambda_2}\omega_1\omega_2\sinh\kappa_1\gamma_n\cosh\kappa_1\gamma_n\xi$$
$$-\frac{\lambda_1}{\lambda_2}\omega_1\omega_2\{\sinh\kappa_2\gamma_n(1-\xi)+\cosh\kappa_1\gamma_n\sinh\kappa_2\gamma_n\xi\}$$
$$+\omega_1^2\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n\xi\Big]$$

② Case II:

$$\bar{\Phi}_x^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\left[-(\bar{\kappa}^2-\nu_y)\xi\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\sinh\bar{\kappa}\gamma_n(1-\xi)\right.$$
$$\left.-\gamma_n(\bar{\kappa}^2-\nu_y)^2(1-\xi)\sinh\bar{\kappa}\gamma_n\xi\right]$$

$$\bar{\Omega}_x^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\left[(\bar{\kappa}^2-\nu_y)(1-\xi)\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\sinh\bar{\kappa}\gamma_n\xi\right.$$
$$\left.+\gamma_n(\bar{\kappa}^2-\nu_y)^2\xi\sinh\bar{\kappa}\gamma_n(1-\xi)\right]$$

③ Case III:

$$\bar{\Phi}_x^{\mathrm{III}}=\frac{\gamma_n^2}{S_0^{\mathrm{III}}}\left[\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}\{-\tau_1\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n(1-\xi)\right.$$

$$+\tau_2\cosh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n$$
$$+\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1\tau_2}\{-\tau_1\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)$$
$$+\tau_2\cosh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n(1-\xi)\}\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$-\frac{\tau_1^2+\tau_2^2}{\tau_2}\{-\tau_1\cosh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)$$
$$+\tau_2\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$+\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}\{-\tau_1\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$-\tau_2\cosh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$-\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\{-\tau_1\cosh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n$$
$$-\tau_2\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n$$
$$-\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\{-\tau_1\cosh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n(1-\xi)$$
$$\left.-\tau_2\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n(1-\xi)\}\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\right]$$

$$\bar{\Omega}_x^{\mathrm{III}}=\frac{\gamma_n^2}{S_0^{\mathrm{III}}}\left[-\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}(-\tau_1\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\xi\right.$$
$$+\tau_2\cosh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n$$
$$-\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1\tau_2}(-\tau_1\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi$$
$$+\tau_2\cosh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n\xi)\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$+\frac{\tau_1^2+\tau_2^2}{\tau_2}(-\tau_1\cosh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi$$
$$+\tau_2\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$-\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}(-\tau_1\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$-\tau_2\cosh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n$$
$$+\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}(-\tau_1\cosh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n$$
$$-\tau_2\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n$$
$$+\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}(-\tau_1\cosh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\xi$$
$$\left.-\tau_2\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi)\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\right].$$

④ Case IV:

$$\bar{\Phi}_x^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}[-(1-\nu)(3+\nu)\xi\sinh\gamma_n\sinh\gamma_n(1-\xi)-(1-\nu)^2(1-\xi)\gamma_n\sinh\gamma_n\xi]$$

$$\bar{\Omega}_x^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}[(1-\nu)(3+\nu)(1-\xi)\sinh\gamma_n\sinh\gamma_n\xi+(1-\nu)^2\gamma_n\xi\sinh\gamma_n(1-\xi)].$$

And

$$M_y^{\mathrm{I}\sim\mathrm{IV}}=\kappa^2a^2\Bigg[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\gamma_n^2+\nu_x(m\pi)^2}{K_{mn}^{\mathrm{I}\sim\mathrm{IV}}}\sin m\pi\xi+(\bar{\Phi}_y^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\bar{\Omega}_y^{\mathrm{I}\sim\mathrm{IV}})R_{mn}^{\mathrm{I}\sim\mathrm{IV}}\right\}F_{mn}\sin n\pi\eta$$
$$-\frac{2}{\mu a^2}\sum_{n=1}^{\infty}\left(\bar{\Phi}_y^{\mathrm{I}\sim\mathrm{IV}}\sum_{i=1}^{r}R_i^A\sin n\pi\eta_{iA}-\bar{\Omega}_y^{\mathrm{I}\sim\mathrm{IV}}\sum_{j=1}^{s}R_j^B\sin n\pi\eta_{jB}\right)\sin n\pi\eta\Bigg] \tag{22}$$

in which

① Case I:

$$\bar{\Phi}_y^{\mathrm{I}}=\frac{1}{\gamma_nS_0^{\mathrm{I}}}\Big[\omega_1g_1\{\sinh\kappa_1\gamma_n\xi+\cosh\kappa_2\gamma_n\sinh\kappa_1\gamma_n(1-\xi)\}$$
$$-\frac{\lambda_1}{\lambda_2}\omega_2g_1\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n(1-\xi)$$
$$+\frac{\lambda_1}{\lambda_2}\omega_1g_2\{\sinh\kappa_2\gamma_n\xi+\cosh\kappa_1\gamma_n\sinh\kappa_2\gamma_n(1-\xi)\}$$
$$-\frac{\omega_1^2}{\omega_2}g_2\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n(1-\xi)\Big]$$

$$\bar{\Omega}_y^{\mathrm{I}}=\frac{1}{\gamma_nS_0^{\mathrm{I}}}\Big[-\omega_1g_1\{\sinh\kappa_1\gamma_n(1-\xi)+\cosh\kappa_2\gamma_n\sinh\kappa_1\gamma_n\xi\}$$
$$+\frac{\lambda_1}{\lambda_2}\omega_2g_1\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n\xi$$
$$-\frac{\lambda_1}{\lambda_2}\omega_1g_2\{\sinh\kappa_2\gamma_n(1-\xi)+\cosh\kappa_1\gamma_n\sinh\kappa_2\gamma_n\xi\}$$
$$+\frac{\omega_1^2}{\omega_2}g_2\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n\xi\Big]$$

$$g_1=1-\nu_x\kappa_1^2\,,\qquad g_2=1-\nu_x\kappa_2^2\,.$$

② Case II:

$$\bar{\Phi}_y^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\Big[(1-\nu_x\bar{\kappa}^2)\xi\left\{\mu\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\sinh\bar{\kappa}\gamma_n(1-\xi)$$
$$+\frac{2}{\gamma_n}(1-\nu_x\nu_y)\left(\frac{2\bar{\kappa}^2}{\bar{\kappa}^2-\nu_y}-\frac{3\bar{\kappa}^2-\chi}{\bar{\kappa}^2-\chi}\right)\sinh\bar{\kappa}\gamma_n\cosh\bar{\kappa}\gamma_n(1-\xi)$$
$$+\gamma_n(1-\nu_x\bar{\kappa}^2)(\bar{\kappa}^2-\nu_y)(1-\xi)\sinh\bar{\kappa}\gamma_n\xi+2\bar{\kappa}(1-\nu_x\nu_y)\cosh\bar{\kappa}\gamma_n\xi\Big]$$

$$\bar{\Omega}_y^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\Big[-(1-\nu_x\bar{\kappa}^2)(1-\xi)\left\{2\bar{\kappa}-\frac{(\bar{\kappa}^2-\nu_y)(3\bar{\kappa}^2-\chi)}{\bar{\kappa}(\bar{\kappa}^2-\chi)}\right\}\sinh\bar{\kappa}\gamma_n\sinh\bar{\kappa}\gamma_n\xi$$
$$-\frac{2}{\gamma_n}(1-\nu_x\nu_y)\left(\frac{2\bar{\kappa}^2}{\bar{\kappa}^2-\nu_y}-\frac{3\bar{\kappa}^2-\chi}{\bar{\kappa}^2-\chi}\right)\sinh\bar{\kappa}\gamma_n\cosh\bar{\kappa}\gamma_n\xi$$
$$-\gamma_n(1-\nu_x\bar{\kappa}^2)(\bar{\kappa}^2-\nu_y)\xi\sinh\bar{\kappa}\gamma_n(1-\xi)-2\bar{\kappa}(1-\nu_x\nu_y)\cosh\bar{\kappa}\gamma_n(1-\xi)\Big]$$

③ Case III:

$$\begin{aligned}\bar{\Phi}_y^{\mathrm{III}}=\frac{\gamma_n^2}{S_0^{\mathrm{III}}}\Bigg[&\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}\{f_1\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\\&-f_2\cosh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n\\&+\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1\tau_2}\{f_1\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\\&-f_2\cosh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n(1-\xi)\}\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\\&-\frac{\tau_1^2+\tau_2^2}{\tau_2^2}\{f_1\cosh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\\&-f_2\sinh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\\&+\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}\{f_1\sinh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n(1-\xi)\\&+f_2\cosh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\\&-\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\{f_1\cosh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n(1-\xi)\\&+f_2\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\}\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n\\&-\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\{f_1\cosh\kappa_1'\gamma_n(1-\xi)\cos\kappa_2'\gamma_n(1-\xi)\\&+f_2\sinh\kappa_1'\gamma_n(1-\xi)\sin\kappa_2'\gamma_n(1-\xi)\}\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\Bigg]\end{aligned}$$

$$\begin{aligned}\bar{\Omega}_y^{\mathrm{III}}=\frac{\gamma_n^2}{S_0^{\mathrm{III}}}\Bigg[&-\frac{\tau_1(\rho_1\tau_2-\rho_2\tau_1)}{\rho_1\tau_2}(f_1\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi\\&-f_2\cosh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n\\&-\frac{\tau_1(\rho_1\tau_1+\rho_2\tau_2)}{\rho_1\tau_2}(f_1\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi\\&-f_2\cosh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\xi)\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\\&+\frac{\tau_1^2+\tau_2^2}{\tau_2}(f_1\cosh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi\\&-f_2\sinh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\\&-\frac{\rho_2(\tau_1^2+\tau_2^2)}{\rho_1\tau_2}(f_1\sinh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\xi\\&+f_2\cosh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\\&+\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}(f_1\cosh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\xi\\&+f_2\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi)\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n\\&+\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}(f_1\cosh\kappa_1'\gamma_n\xi\cos\kappa_2'\gamma_n\xi\\&+f_2\sinh\kappa_1'\gamma_n\xi\sin\kappa_2'\gamma_n\xi)\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\Bigg]\end{aligned}$$

$$f_1=1-\nu_x(\kappa_1^2-\kappa_2^2)\,,\qquad f_2=2\nu_x\kappa_1\kappa_2\ .$$

④ Case IV:

$$\bar{\Phi}_y^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}\left[\frac{2(1+\nu)(3+\nu)}{\gamma_n}\sinh\gamma_n\cosh\gamma_n(1-\xi)+(1-\nu)(3+\nu)\xi\sinh\gamma_n\sinh\gamma_n(1-\xi)\right.$$
$$\left.+(1-\nu)^2(1-\xi)\gamma_n\sinh\gamma_n\xi+2(1-\nu^2)\cosh\gamma_n\xi\right]$$

$$\bar{\Omega}_y^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}\left[-\frac{2(1+\nu)(3+\nu)}{\gamma_n}\sinh\gamma_n\cosh\gamma_n\xi-(1-\nu)(3+\nu)(1-\xi)\sinh\gamma_n\sinh\gamma_n\xi\right.$$
$$\left.-(1-\nu)^2\gamma_n\xi\sinh\gamma_n(1-\xi)-2(1-\nu^2)\cosh\gamma_n(1-\xi)\right]$$

(2) Fundamental Simultaneous Equations

Consider that the vertical displacements of edge-columns in flat slab yield. Assuming that no up-lift of the plate at edge-columns occures, it follows that the deflection of the point on plate, at which the plate is connected with the edge-column, should be equivalent to the displacement of edge-column. Hence, a series of compatibility equations are given as follows:

$$\left.\begin{aligned}&(w^{\mathrm{I}\sim\mathrm{IV}})_{\xi=0,\eta=\eta_{kA}}=d_{kA} \qquad (k=1,2,\cdots,r)\\&(w^{\mathrm{I}\sim\mathrm{IV}})_{\xi=1,\eta=\eta_{lB}}=d_{lB} \qquad (l=1,2,\cdots,s)\end{aligned}\right\} \tag{23}$$

d_{kA}: vertical displacement of edge-column "k" at AC side,
d_{lB}: vertical displacement of edge-column "l" at BD side.

Substituting Eq. (20) into Eq. (23), the fundamental simultaneous equation are obtained as:

$$\left.\begin{aligned}&\sum_{i=1}^{r}\boldsymbol{\Gamma}_{ik}^{\mathrm{I}\sim\mathrm{IV}}R_i{}^A-\sum_{j=1}^{s}\boldsymbol{\Lambda}_{jk}^{\mathrm{I}\sim\mathrm{IV}}R_j{}^B=a^2\boldsymbol{G}_k^{\mathrm{I}\sim\mathrm{IV}}-\frac{\mu D_x}{2a^2}d_{kA}\\&\sum_{i=1}^{r}\bar{\boldsymbol{\Gamma}}_{il}^{\mathrm{I}\sim\mathrm{IV}}R_i{}^A-\sum_{j=1}^{s}\bar{\boldsymbol{\Lambda}}_{jl}^{\mathrm{I}\sim\mathrm{IV}}R_j{}^B=a^2\bar{\boldsymbol{G}}_l^{\mathrm{I}\sim\mathrm{IV}}-\frac{\mu D_x}{2a^2}d_{lB}\end{aligned}\right\} \tag{24}$$

in which $k=1,2,\cdots,r$; $l=1,2,\cdots,s$;

$$\boldsymbol{\Gamma}_{ik}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Phi_0^{\mathrm{I}\sim\mathrm{IV}}\sin n\pi\eta_{iA}\sin n\pi\eta_{kA}$$

$$\boldsymbol{\Lambda}_{jk}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Omega_0^{\mathrm{I}\sim\mathrm{IV}}\sin n\pi\eta_{jB}\sin n\pi\eta_{kA}$$

$$\bar{\boldsymbol{\Gamma}}_{il}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Phi_a^{\mathrm{I}\sim\mathrm{IV}}\sin n\pi\eta_{iA}\sin n\pi\eta_{lB}$$

$$\bar{\boldsymbol{\Lambda}}_{jl}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Omega_a^{\mathrm{I}\sim\mathrm{IV}}\sin n\pi\eta_{jB}\sin n\pi\eta_{lB}$$

$$\boldsymbol{G}_k^{\mathrm{I}\sim\mathrm{IV}}=\frac{\mu}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\Phi_0^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\Omega_0^{\mathrm{I}\sim\mathrm{IV}}\}\boldsymbol{R}_{mn}^{\mathrm{I}\sim\mathrm{IV}}F_{mn}\sin n\pi\eta_{kA}$$

$$\bar{\boldsymbol{G}}_l^{\mathrm{I}\sim\mathrm{IV}}=\frac{\mu}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\Phi_a^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\Omega_a^{\mathrm{I}\sim\mathrm{IV}}\}\boldsymbol{R}_{mn}^{\mathrm{I}\sim\mathrm{IV}}F_{mn}\sin n\pi\eta_{lB}$$

① Case I:

$$\Phi_0^{\mathrm{I}}=\frac{1}{S_0^{\mathrm{I}}}\left\{\frac{\omega_1-\omega_2}{\gamma_n^3}\left(-\frac{\omega_1}{\omega_2}\sinh\kappa_1\gamma_n\cosh\kappa_2\gamma_n+\frac{\lambda_1}{\lambda_2}\sinh\kappa_2\gamma_n\cosh\kappa_1\gamma_n\right)\right\}$$

$$\Omega_0^{\mathrm{I}}=\frac{1}{S_0^{\mathrm{I}}}\left\{-\frac{\omega_1-\omega_2}{\gamma_n^3}\left(\frac{\omega_1}{\omega_2}\sinh\kappa_1\gamma_n-\frac{\lambda_1}{\lambda_2}\sinh\kappa_2\gamma_n\right)\right\}$$

$$\Phi_a^{\mathrm{I}}=-\Omega_0^{\mathrm{I}} \qquad \Omega_a^{\mathrm{I}}=-\Phi_0^{\mathrm{I}}\,.$$

② Case II:

$$\Phi_0^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\left\{\frac{1}{\gamma_n^3}\left(\frac{4\bar\kappa^2}{\bar\kappa^2-\nu_y}-\frac{2(3\bar\kappa^2-\chi)}{\bar\kappa^2-\chi}\right)\sinh\bar\kappa\gamma_n\cosh\bar\kappa\gamma_n+\frac{2\bar\kappa}{\gamma_n^2}\right\}$$

$$\Omega_0^{\mathrm{II}}=\frac{1}{S_0^{\mathrm{II}}}\left\{\frac{-1}{\gamma_n^3}\left(\frac{4\bar\kappa^2}{\bar\kappa^2-\nu_y}-\frac{2(3\bar\kappa^2-\chi)}{\bar\kappa^2-\chi}\right)\sinh\bar\kappa\gamma_n-\frac{2\bar\kappa}{\gamma_n^2}\cosh\bar\kappa\gamma_n\right\}$$

$$\Phi_a^{\mathrm{II}}=-\Omega_0^{\mathrm{II}} \qquad \Omega_a^{\mathrm{II}}=-\Phi_0^{\mathrm{II}}\,.$$

③ Case III:

$$\Phi_0^{\mathrm{III}}=\frac{1}{S_0^{\mathrm{III}}}\left\{-\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\sinh\kappa_1'\gamma_n\cosh\kappa_1'\gamma_n-\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\sin\kappa_2'\gamma_n\cos\kappa_2'\gamma_n\right\}$$

$$\Omega_0^{\mathrm{III}}=\frac{1}{S_0^{\mathrm{III}}}\left\{\frac{\rho_1\tau_2-\rho_2\tau_1}{\rho_1}\sinh\kappa_1'\gamma_n\cos\kappa_2'\gamma_n+\frac{\rho_1\tau_1+\rho_2\tau_2}{\rho_1}\cosh\kappa_1'\gamma_n\sin\kappa_2'\gamma_n\right\}$$

$$\Phi_a^{\mathrm{III}}=-\Omega_0^{\mathrm{III}} \qquad \Omega_a^{\mathrm{III}}=-\Phi_0^{\mathrm{III}}\,.$$

④ Case IV:

$$\Phi_0^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}\left\{\frac{2(3+\nu)}{(1-\nu)\gamma_n^3}\sinh\gamma_n\cosh\gamma_n+\frac{2}{\gamma_n^2}\right\}$$

$$\Omega_0^{\mathrm{IV}}=\frac{1}{S_0^{\mathrm{IV}}}\left\{-\frac{2(3+\nu)}{(1-\nu)\gamma_n^3}\sinh\gamma_n-\frac{2}{\gamma_n^2}\cosh\gamma_n\right\}$$

$$\Phi_a^{\mathrm{IV}}=-\Omega_0^{\mathrm{IV}} \qquad \Omega_a^{\mathrm{IV}}=-\Phi_0^{\mathrm{IV}}\,.$$

Solving Eq. (24), the redundants R_i^A and R_j^B can be found. Then, using these results, the displacements and the stresses of plate and columns in flat slab can be given.

(3) Flexibility Coefficients and Loading Terms

The coefficients $\Gamma_{ik}^{\mathrm{I}\sim\mathrm{IV}}$, $\Lambda_{jk}^{\mathrm{I}\sim\mathrm{IV}}$, $\bar\Gamma_{il}^{\mathrm{I}\sim\mathrm{IV}}$ and $\bar\Lambda_{jl}^{\mathrm{I}\sim\mathrm{IV}}$ in Eq. (24) are called the flexibility coefficients, which depend on the sizes of plate and the co-ordinates of edge-columns in the structures. The constants $G_k^{\mathrm{I}\sim\mathrm{IV}}$ and $\bar G_l^{\mathrm{I}\sim\mathrm{IV}}$ in Eq. (24) are called the loading terms, which depend on given loading on the plate.

a) Flexibility Coefficients

Removing from the actual structure the edge-columns at sides AC and BD, the statically determinate and stable plate is obtained and is called the primary structure (see Fig. 4). This primary structure may be now subjected to the unit

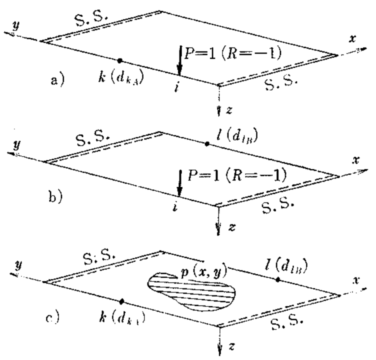

Fig. 4

load at the point "i" on the side AC of plate, hereafter called "condition $P_{iA}=1$" (see Fig. 4(a)). Assuming the positive direction of the load P_{iA} to be downward, the following notation is introduced;

d_{kA}=downward deflection of point "k" at the side AC of primary structure under "condition $P_{iA}=1$".

On the other hand, if the deflection "d_{kA}" is supposed to be the vertical displacement of point "k" at the side AC of the statically indeterminate structure under the settlement of edge-column "k", which is simply supported by the edge-columns "i" and "k" at points "i" and "k" of the primary structure and hereafter is called "structure with edge-columns "i" and "k"", then the unit load P_{iA} in primary structure may be too considered as follows:

$-P_{iA}$=vertical reaction of edge-column "i" in "structure with edge-columns "i" and "k"", due to the settlement of edge-column "k", d_{kA} (i.e. $R_i^A=-1$).

And the vertical reaction of edge-column "k" is equal to zero. For this statically indeterminate structure with edge-columns "i" and "k", the fundamental simultaneous equations are given by (see Eq. (24))

$$\Gamma_{ik}^{\mathrm{I}\sim\mathrm{IV}}\times(-1)=\frac{D_x\mu}{2a^2}\times(-d_{kA}) \qquad \therefore \quad \frac{2a^2}{D_x\mu}\Gamma_{ik}^{\mathrm{I}\sim\mathrm{IV}}=d_{kA}\,.$$

Hence, it is said that $(2a^2/D_x\mu)\Gamma_{ik}^{\mathrm{I}\sim\mathrm{IV}}$ in Eq. (24) is equal to the downward deflection of point "k" at the side AC of primary structure subjected to the unit load $P_{iA}=1$ at point "i".

In a similar manner, the following expression for the flexibility coefficient $\bar{\Gamma}_{il}^{\mathrm{I}\sim\mathrm{IV}}$ can be obtained

$$\frac{2a^2}{\mu D_x}\bar{\Gamma}_{il}^{\mathrm{I}\sim\mathrm{IV}}=d_{lB}$$

where d_{lB} is the vertical displacement at point "l" in the side BD of primary structure under the unit load, $P_{iA}=1$ (see Fig. 4(b)).

Hence, it is found that $(2a^2/\mu D_x)\bar{\Gamma}_{il}^{\mathrm{I}\sim\mathrm{IV}}$ in Eq. (24) is equal to the downward deflection of point "l" at BD side of primary structure subjected to the unit load $P_{iA}=1$. The expressions for the coefficients $\Lambda_{jk}^{\mathrm{I}\sim\mathrm{IV}}$ and $\bar{\Lambda}_{jl}^{\mathrm{I}\sim\mathrm{IV}}$ can be written as follows for "condition $P_{jB}=1$", when the primary structure is subjected to the unit load at the point "j" on the side BD of the plate.

$$\frac{2a^2}{\mu D_x}\Lambda_{jk}^{\mathrm{I}\sim\mathrm{IV}}=d_{kA}\,, \qquad \frac{2a^2}{\mu D_x}\bar{\Lambda}_{jl}^{\mathrm{I}\sim\mathrm{IV}}=d_{lB}\,.$$

Hence, it is found that

$\frac{2a^2}{\mu D_x}\Lambda_{jk}^{\mathrm{I}\sim\mathrm{IV}}$ = downward deflection of point "k" at AC side of primary structure under "condition $P_{jB}=1$",

$\frac{2a^2}{\mu D_x}\bar{\Lambda}_{jl}^{\mathrm{I}\sim\mathrm{IV}}$ = downward deflection of point "l" at BD side of primary structure under "condition $P_{jB}=1$".

b) Loading Terms

When the primary structure is subjected to a given load, as in flat slab, the fundamental simultaneous equations of Eq. (24) are rewritten as follows (see Fig. 4 (c)):

$$a^2 G_k^{\mathrm{I}\sim\mathrm{IV}} - \frac{\mu D_x}{2a^2} d_{kA} = 0 \qquad \therefore \quad \frac{2a^4}{\mu D_x} G_k^{\mathrm{I}\sim\mathrm{IV}} = d_{kA}$$

$$a^2 \bar{G}_l^{\mathrm{I}\sim\mathrm{IV}} - \frac{\mu D_x}{2a^2} d_{lB} = 0 \qquad \therefore \quad \frac{2a^4}{\mu D_x} \bar{G}_l^{\mathrm{I}\sim\mathrm{IV}} = d_{lB}\,.$$

Hence, it can be said that

$\frac{2a^4}{\mu D_x} G_k^{\mathrm{I}\sim\mathrm{IV}}$ = downward deflection of point "k" at AC side of primary structure, due to a given load,

$\frac{2a^4}{\mu D_x} \bar{G}_l^{\mathrm{I}\sim\mathrm{IV}}$ = downward deflection of point "l" at BD side of primary structure, due to a given load.

III. NUMERICAL EXAMPLE

(1) Flat Slab Subjected to Uniformly Distributed Loading

Numerical example of the flat slab, as shown in Fig. 5, is illustrated in this article.

If it is assumed that no settlement takes place for all of the edge-columns, d_{kA} and d_{lB} are equal to zero and the following simultaneous equations can be obtained;

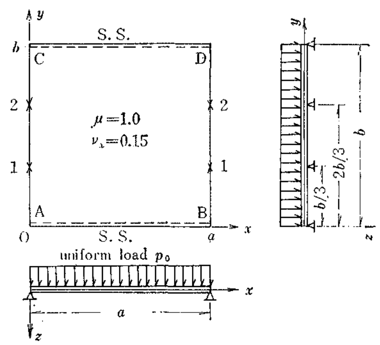

Fig. 5

$$\begin{pmatrix} \Gamma_{11}^{\mathrm{I}\sim\mathrm{IV}} & \Gamma_{21}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{11}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{21}^{\mathrm{I}\sim\mathrm{IV}} \\ \Gamma_{12}^{\mathrm{I}\sim\mathrm{IV}} & \Gamma_{22}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{12}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{22}^{\mathrm{I}\sim\mathrm{IV}} \\ \bar{\Gamma}_{11}^{\mathrm{I}\sim\mathrm{IV}} & \bar{\Gamma}_{21}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{11}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{21}^{\mathrm{I}\sim\mathrm{IV}} \\ \bar{\Gamma}_{12}^{\mathrm{I}\sim\mathrm{IV}} & \bar{\Gamma}_{22}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{12}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{22}^{\mathrm{I}\sim\mathrm{IV}} \end{pmatrix} \cdot \begin{pmatrix} R_1{}^A \\ R_2{}^A \\ R_1{}^B \\ R_2{}^B \end{pmatrix} = a^2 \begin{pmatrix} G_1^{\mathrm{I}\sim\mathrm{IV}} \\ G_2^{\mathrm{I}\sim\mathrm{IV}} \\ \bar{G}_1^{\mathrm{I}\sim\mathrm{IV}} \\ \bar{G}_2^{\mathrm{I}\sim\mathrm{IV}} \end{pmatrix} \tag{25}$$

in which

flexibility coefficients:

$$\Gamma_{11}^{\mathrm{I}\sim\mathrm{IV}} = \sum_{n=1}^{\infty} \Phi_0^{\mathrm{I}\sim\mathrm{IV}} \sin^2 \frac{n\pi}{3}\,,$$

$$\Gamma_{21}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Phi_0^{\mathrm{I}\sim\mathrm{IV}}\sin\frac{2n\pi}{3}\sin\frac{n\pi}{3}\quad(=\Gamma_{12}^{\mathrm{I}\sim\mathrm{IV}})\,,$$

$$\Gamma_{22}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Phi_0^{\mathrm{I}\sim\mathrm{IV}}\sin^2\frac{2n\pi}{3}\,,$$

$$\Lambda_{11}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Omega_0^{\mathrm{I}\sim\mathrm{IV}}\sin^2\frac{n\pi}{3}\,,$$

$$\Lambda_{21}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Omega_0^{\mathrm{I}\sim\mathrm{IV}}\sin\frac{2n\pi}{3}\sin\frac{n\pi}{3}\quad(=\Lambda_{12}^{\mathrm{I}\sim\mathrm{IV}})\,,$$

$$\Lambda_{22}^{\mathrm{I}\sim\mathrm{IV}}=\sum_{n=1}^{\infty}\Omega_0^{\mathrm{I}\sim\mathrm{IV}}\sin^2\frac{2n\pi}{3}\,,$$

$$\bar{\Gamma}_{11}^{\mathrm{I}\sim\mathrm{IV}}=-\Lambda_{11}^{\mathrm{I}\sim\mathrm{IV}}\,,\quad \bar{\Gamma}_{21}^{\mathrm{I}\sim\mathrm{IV}}=-\Lambda_{21}^{\mathrm{I}\sim\mathrm{IV}}\,,\quad \bar{\Lambda}_{11}^{\mathrm{I}\sim\mathrm{IV}}=-\Gamma_{11}^{\mathrm{I}\sim\mathrm{IV}}\,,\quad \bar{\Lambda}_{21}^{\mathrm{I}\sim\mathrm{IV}}=-\Gamma_{21}^{\mathrm{I}\sim\mathrm{IV}}$$

$$\bar{\Gamma}_{12}^{\mathrm{I}\sim\mathrm{IV}}=-\Lambda_{12}^{\mathrm{I}\sim\mathrm{IV}}\,,\quad \bar{\Gamma}_{22}^{\mathrm{I}\sim\mathrm{IV}}=-\Lambda_{22}^{\mathrm{I}\sim\mathrm{IV}}\,,\quad \bar{\Lambda}_{12}^{\mathrm{I}\sim\mathrm{IV}}=-\Gamma_{12}^{\mathrm{I}\sim\mathrm{IV}}\,,\quad \bar{\Lambda}_{22}^{\mathrm{I}\sim\mathrm{IV}}=-\Gamma_{22}^{\mathrm{I}\sim\mathrm{IV}}$$

loading terms:

$$G_1^{\mathrm{I}\sim\mathrm{IV}}=\frac{\mu}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\Phi_0^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\Omega_0^{\mathrm{I}\sim\mathrm{IV}}\}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}F_{mn}\sin\frac{n\pi}{3}$$

$$G_2^{\mathrm{I}\sim\mathrm{IV}}=\frac{\mu}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\Phi_0^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\Omega_0^{\mathrm{I}\sim\mathrm{IV}}\}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}F_{mn}\sin\frac{2n\pi}{3}$$

$$\bar{G}_1^{\mathrm{I}\sim\mathrm{IV}}=\frac{\mu}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\Phi_a^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\Omega_a^{\mathrm{I}\sim\mathrm{IV}}\}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}F_{mn}\sin\frac{n\pi}{3}$$

$$\bar{G}_2^{\mathrm{I}\sim\mathrm{IV}}=\frac{\mu}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\Phi_a^{\mathrm{I}\sim\mathrm{IV}}+(-1)^m\Omega_a^{\mathrm{I}\sim\mathrm{IV}}\}R_{mn}^{\mathrm{I}\sim\mathrm{IV}}F_{mn}\sin\frac{2n\pi}{3}$$

$$F_{mn}=\frac{16p_0}{mn\pi^2}\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}\,.$$

a) In calculation of the flexibility coefficients and the loading terms in Eq. (25), it is assumed that

$$G=\frac{(1-\nu_x)E_y+(1-\nu_y)E_x}{4(1-\nu_x\nu_y)}\,. \tag{26}$$

Then, the value of $(\bar{\kappa}^4-\kappa^2)$ becomes positive and Case I in the analysis is applied to this example.

The flexibility coefficients and the loading terms in Eq. (25) are calculated for the case that the side length ratio, $\mu=b/a$, is 1.0 and the Poisson's ratio in x direction, ν_x, is equal to 0.15, and the results are shown in Table 1 for each value of λ $(=\nu_y/\nu_x)$. Substituting the results into the simultaneous equations of Eq. (25), we can find the redundants of edge-column in flat slab, which are shown by the solid line in Fig. 6. It is noted that all of the redundants of edge-columns decrease when increasing λ. For example, the redundants in $\lambda=1.5$ would be less 11.2% than these in $\lambda=0.7$.

Table 1

λ	Flexibility coefficient				Loading term
	Γ^{I}_{11}, $-\bar{\Lambda}^{\mathrm{I}}_{11}$ Γ^{I}_{22}, $-\bar{\Lambda}^{\mathrm{I}}_{22}$	Γ^{I}_{21}, $-\bar{\Lambda}^{\mathrm{I}}_{21}$ Γ^{I}_{12}, $-\bar{\Lambda}^{\mathrm{I}}_{12}$	$\Lambda^{\mathrm{I}}_{11}$, $-\bar{\Gamma}^{\mathrm{I}}_{11}$ $\Lambda^{\mathrm{I}}_{22}$, $-\bar{\Gamma}^{\mathrm{I}}_{22}$	$\Lambda^{\mathrm{I}}_{21}$, $-\bar{\Gamma}^{\mathrm{I}}_{21}$ $\Lambda^{\mathrm{I}}_{12}$, $-\bar{\Gamma}^{\mathrm{I}}_{12}$	G^{I}_1 $\bar{G}^{\mathrm{I}}_1$ G^{I}_2 $\bar{G}^{\mathrm{I}}_2$
.7	$.2728\times10^{-1}$	$.2063\times10^{-1}$	$-.4818\times10^{-2}$	$-.4726\times10^{-2}$	$.8382\times10^{-2}p_0$
.8	.2481	.1873	−.4013	−.3941	.7376
.9	.2283	.1721	−.3406	−.3349	.6576
1.0	—	—	—	—	—
1.1	.1982	.1492	−.2559	−.2522	.5422
1.2	.1864	.1403	−.2254	−.2224	.4988
1.3	.1762	.1326	−.2002	−.1978	.4620
1.4	.1673	.1259	−.1782	−.1772	.4305
1.5	.1592	.1199	−.1614	−.1598	.4030
1.6	.1521	.1145	−.1461	−.1448	.3790
1.7	.1457	.1097	−.1330	−.1319	.3578
1.8	.1399	.1054	−.1216	−.1197	.3388
1.9	.1346	.1014	−.1116	−.1109	.3219
2.0	.1297	.0978	−.1027	−.1022	.3066

Substituting the redundants thus obtained into Eq. (20), the elastic surface of plate in flat slab will be found. The curves in Fig. 7 show the elastic deflections of plate at $\xi=0.5$ section for each value of λ. The maximum deflection of plate takes place at the center of plate, which is shown by the solid line in Fig. 8. It is noticed that the maximum deflection of plate decrease with increasing λ.

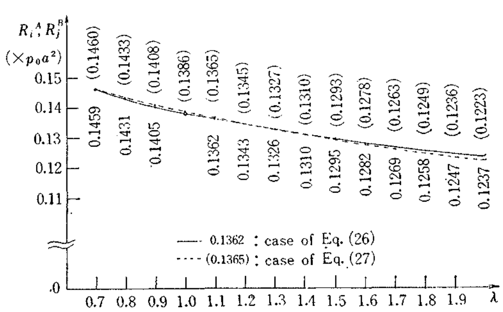

Fig. 6 Redundants of edge-columns

The relationships between λ and the moments, M_x and M_y, at the center of plate obtained from Eqs. (21) and (22), are shown in Fig. 9. The moment M_x decreases with increasing λ and the moment M_y increases.

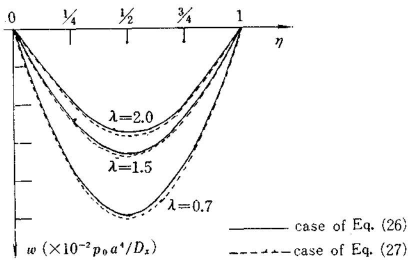

Fig. 7 Deflection at $\xi=1/2$-section

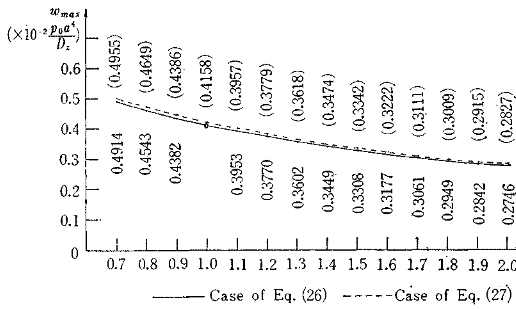

Fig. 8 w_{max}

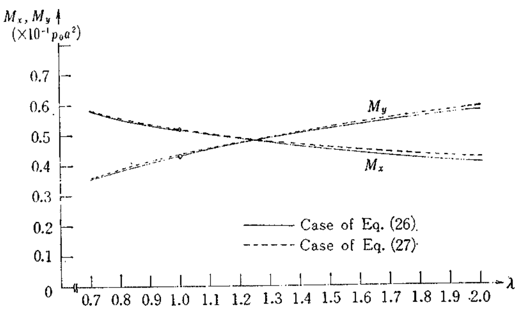

Fig. 9 Bending moments M_x and M_y at the center of plate

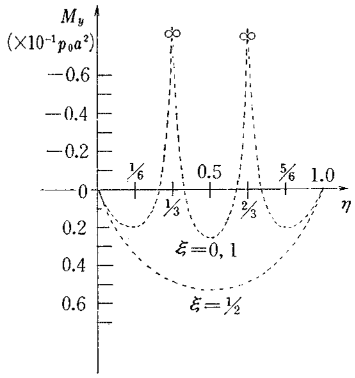

Fig. 10

b) In a concrete slab with two-way reinforcements in the directions x and y, we can assume the following expression for G recommended by M. T. Huber;

$$G=\frac{\sqrt{E_xE_y}}{2(1+\sqrt{\nu_x\nu_y})}\,. \tag{27}$$

Then, the value of $(\bar{\kappa}^4-\kappa^2)$ is equal to zero and the problem falls into Case II.

Using Eq. (27) and solving the same problem as the preceeding article, we have the flexibility coefficients and the loading terms in the fundamental simultaneous equations for the redundants of edge-columns in flat slab, as in Table 2. And, the curves for redundants of edge-columns, deflections and moments of plates are found as shown by the dotted lines in Figs. 6, 7, 8 and 9. Fig. 10 shows moments of the plate in the y direction.

Table 2

	Flexibility coefficient				Loading term
	$\Gamma^{II}_{11}, -\bar{A}^{II}_{11}$ $\Gamma^{II}_{22}, -\bar{A}^{II}_{22}$	$\Gamma^{II}_{21}, -\bar{A}^{II}_{21}$ $\Gamma^{II}_{12}, -\bar{A}^{II}_{12}$	$A^{II}_{11}, -\bar{\Gamma}^{II}_{11}$ $A^{II}_{22}, -\bar{\Gamma}^{II}_{22}$	$A^{II}_{21}, -\bar{\Gamma}^{II}_{21}$ $A^{II}_{12}, -\bar{\Gamma}^{II}_{12}$	$G^{II}_1, \bar{G}^{II}_1$ $G^{II}_2, \bar{G}^{II}_2$
.7	.2736×10⁻¹	.2066×10⁻¹	−.4831×10⁻²	−.4735×10⁻²	.8406×10⁻² p_0
.8	.2484	.1874	−.4017	−.3944	.7386
.9	.2284	.1722	−.3407	−.3350	.6593
1.0	.2120	.1597	−.2934	−.2889	.5957
1.1	.1982	.1493	−.2559	−.2522	.5436
1.2	.1866	.1404	−.2256	−.2226	.5002
1.3	.1766	.1328	−.2007	−.1981	.4634
1.4	.1678	.1262	−.1799	−.1777	.4319
1.5	.1601	.1204	−.1623	−.1605	.4045
1.6	.1533	.1152	−.1473	−.1457	.3806
1.7	.1472	.1106	−.1344	−.1330	.3594
1.8	.1417	.1064	−.1231	−.1219	.3406
1.9	.1367	.1027	−.1133	−.1123	.3237
2.0	.1322	.0993	−.1047	−.1038	.3085

(2) Settlement of Edge-Column

Using the method proposed in this paper, the problem of unequal settlement of edge-columns in the structure can be also analyzed easily and exactly.

When the edge-column "1" at the side AC of the plate as in Fig. 5 undergoes a vertical displacement of "d_{1A}" and the rest of the edge-columns undergo no displacements, the fundamental simultaneous equations for the redundants of edge-columns in the structure can be written as follows;

$$\begin{bmatrix} \Gamma_{11}^{\mathrm{I}\sim\mathrm{IV}} & \Gamma_{21}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{11}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{21}^{\mathrm{I}\sim\mathrm{IV}} \\ \Gamma_{12}^{\mathrm{I}\sim\mathrm{IV}} & \Gamma_{22}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{12}^{\mathrm{I}\sim\mathrm{IV}} & -\Lambda_{22}^{\mathrm{I}\sim\mathrm{IV}} \\ \bar{\Gamma}_{11}^{\mathrm{I}\sim\mathrm{IV}} & \bar{\Gamma}_{21}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{11}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{21}^{\mathrm{I}\sim\mathrm{IV}} \\ \bar{\Gamma}_{12}^{\mathrm{I}\sim\mathrm{IV}} & \bar{\Gamma}_{22}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{12}^{\mathrm{I}\sim\mathrm{IV}} & -\bar{\Lambda}_{22}^{\mathrm{I}\sim\mathrm{IV}} \end{bmatrix} \cdot \begin{bmatrix} R_1^A \\ R_2^A \\ R_1^B \\ R_2^B \end{bmatrix} = -\frac{\mu D_x}{2a^2} d_{1A} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \tag{28}$$

Assigning as an example $\mu=1.0$, $\nu_x=0.15$ and $\lambda=1.5$, and using Eq. (27) for G, there results

$$\left. \begin{aligned} R_1^A &= -71.942 \times (D_x d_{1A}/a^2), & R_2^A &= 53.878 \times (D_x d_{1A}/a^2) \\ R_1^B &= 1.327 \times (D_x d_{1A}/a^2), & R_2^B &= 0.751 \times (D_x d_{1A}/a^2). \end{aligned} \right\} \tag{29}$$

Substituting Eq. (29) into Eq. (20), the elastic surface of plate in flat slab can be expressed as

$$\begin{aligned} w^{\mathrm{II}} = -2d_{1A}\Bigg[\sum_{n=1}^{\infty} \Bigg\{ & \Phi^{\mathrm{II}}\left(-71.942 \sin\frac{n\pi}{3} + 53.878 \sin\frac{2n\pi}{3}\right) \\ & -\Omega^{\mathrm{II}}\left(1.327 \sin\frac{n\pi}{3} + 0.751 \sin\frac{2n\pi}{3}\right)\Bigg\} \sin n\pi\eta\Bigg] . \end{aligned} \tag{30}$$

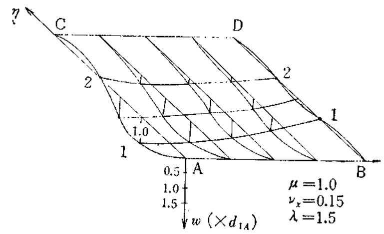

Fig. 11

Numerical results for each values of ξ and η are shown in Fig. 11.

IV. SUMMARY AND CONCLUSION

The elastic constants for orthotropic plate are E_x, E_y, ν_x, ν_y and G. From the reciprocal property of elastic material, the following relation exists;

$$E_x/E_y = \nu_x/\nu_y .$$

As the result, it is necessary to determine experimentally four independent constants to specify the complete material properties for an orthotropic plate.

Orthotropic plates shall be classified into a natural orthotropic plate and a plate whose orthotropy is introduced artificially. An example of the former is a wooden plate and the latter covers reinforced concrete slab, a plywood, a slab with ribs, a corrugated sheet, a lattice structure and so on. Case I in analysis is applied to a celular deck plate with large twisting rigidity, and Case II, to a reinforced concrete slab. A plate with smaller ribs and a corrugated sheet should be solved by using Case III in analysis.

The paper presents the result of an analysis for a flat slab, which is simply supported by edge-columns at two opposite sides on plate and is subjected to an arbitrary transverse load. The merits of this method are as follows:

(1) The redandants of edge-columns can be found easily by solving the fundamental simultaneous equations of Eq. (24).

(2) When the displacements and stresses of the primary structure which is obtained by removing from a given structure the edge-columns are known, the necessary effort in the analysis of a given structure will be the same as that for frames.

(3) This method can be applied to such a problem as the unequal settlement of edge-columns in flat slab, as shown in the numerical example, and may be developed into obtaining the influence surface of the displacements and stresses of a given structure.

The bending moments of plate on the edge-column in the structure are essentially infinite, since the edge-column is regarded as the point-supports (see Fig. 10). On the other hand, the plate in such a structure is supported by the columns with the finite width, in practice. Therefore, it will be necessary to consider that the bending moments at the edge of the cross section of column should be used for the design of the structure, and so on; on which problem may be investigated with the research on the analysis of the flat slabs considered the flexibility of columns, after this report.

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