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A NUMERICAL METHOD FOR THE LIMIT ANALYSIS OF GRILLAGE GIRDERS AND PLATES

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1. INTRODUCTION

Many papers on the limit analysis of grillage girders and plates have been published. The exact evaluation of limit loads of such structures remains, however, as a difficult problem due to its nonlinear characteristic and then the general method of analysis for the structures has not yet been formulated. The lower and upper bound theorems of limit analysis provide a powerful weapon for the determination of limit load. According to the bound theorems the numerical determination of limit load can be reduced to an optimization problem. Since 1956, for the frame structures such as trusses, continuous girders, ridgidly connected frames and so on, the numerical solutions have been studied by the method of transformation of the limit analysis problems into linear programing ones[1),2),3),4)]. While, for plate problems the similar method was studied by Koopmann and Rance[5)] in 1965 who dealt with axially symmetrically loaded circular plates and uniformly loaded square plates in which the linearlized yield criterion of Tresca type was used. Recently, for the variously loaded rectangular plates obeying Mises' yield criterion P. G. Hodge and T. Belytschko[6)] investigated the limit analysis problems by transforming to the nonlinear programming ones that could be solved by the Sequential Unconstrained Minimization Techniques developed by Fiacco and McCormick, and gave the most reliable values among the already known ones of the limit load.

In this paper, a general method for the determination of the limit loads for grillage girders, circular plates and rectangular ones which are made of the perfectly plastic material obeying Mises' yield criterion is studied. The method is based on the lower bound theorem of limit analysis. If the stress fields are expressed by the stresses on the definite points of structures and the equilibrium conditions are expressed by suitable linear algebraic expressions, the limit analysis problems can be transformed into mathematical programming ones in which linear equalities representing the equilibrium conditions and quadratic inequalities required by reason that the stress fields must satisfy the yield condition are taken as constraints and the factor of load expressed by a statically admissible multiplier is taken as an objective function. On account of the convexity of yield surface the problems become convex programming ones which may be solved by the cutting plane method developed by J. E. Kelley[7)]. The method presented herein is developed by replacing the limit analysis problems by such the convex programming ones. The method is justified by several numerical examples in which the calculated limit loads by this method are compared with the already known ones.

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2. GRILLAGE GIRDERS

2.1 Equilibrium conditions

Assuming that the cross section of each member between two adjacent joints of a grid is uniform and external loads are applied only at the joints of a grid, the plastic hinges may be formed at either end of members. Hence it is sufficient to make equilibrium equations that the bending moments and the twisting ones acting at the joints of a grid are assumed as variables. Denoting the fully plastic moment of member by M_0, the fully plastic torque by T_0, the panel length by Λ, the length of reference member by L and the fully plastic moment of reference member by M_b, and introducing the non-dimensional quantities such as $m'=M/M_0$, $t'=T/T_0$, $\beta=T_0/M_0$, $\mu=M_0/M_b$ and $\lambda=L/\Lambda$, the equilibrium equations at an arbitrary joint of a grid as shown in Fig. 1 are given as follows:

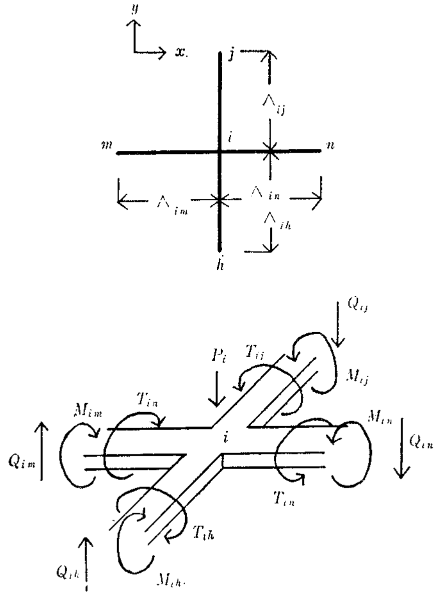

Fig. 1

$$\left.\begin{aligned} &\lambda_{ih}\mu_{ih}(m_{ih}'-m_{hi}')+\lambda_{ij}\mu_{ij}(m_{ij}'-m_{ji}') \\ &\quad+\lambda_{im}\mu_{im}(m_{im}'-m_{mi}') \\ &\quad+\lambda_{in}\mu_{in}(m_{in}'-m_{ni}')=P_iL/M_b\ , \\ &\beta_{ih}\mu_{ih}t_{ih}'-\beta_{ij}\mu_{ij}t_{ij}'+\mu_{im}m_{im}' \\ &\quad-\mu_{in}m_{in}'=0\ , \\ &\beta_{im}\mu_{im}t_{im}'-\beta_{in}\mu_{in}t_{in}'+\mu_{ih}m_{ih}' \\ &\quad-\mu_{ij}m_{ij}'=0\ , \\ &t_{ih}=t_{hi}\,,\ t_{ij}=t_{ji}\,,\cdots, \end{aligned}\right\} \quad (1)$$

where the position of member is identified by the double subscripts and the first one indicates the relevant joint of a grid.

In general it is advantageous to use non-negative variables for the application of mathematical programming. Since it is evident from the yield inequalities (3) that the variables m' and t' can take the values between -1 and $+1$ respectively, Eq. (1) can be rewritten by introducing the new variables $m=m'+1$ and $t=t'+1$ as follows:

$$\left.\begin{aligned} &\lambda_{ih}\mu_{ih}(m_{ih}-m_{hi})+\lambda_{ij}\mu_{ij}(m_{ij}-m_{ji})+\lambda_{im}\mu_{im}(m_{im}-m_{mi}) \\ &\quad+\lambda_{in}\mu_{in}(m_{in}-m_{ni})-q_iP_0=0\ , \\ &\beta_{ih}\mu_{ih}t_{ih}-\beta_{ij}\mu_{ij}t_{ij}+\mu_{im}m_{im}-\mu_{in}m_{in}=\mu_{im}-\mu_{in}+\beta_{ih}\mu_{ih}-\beta_{ij}\mu_{ij}\ , \\ &\beta_{im}\mu_{im}t_{im}-\beta_{in}\mu_{in}t_{in}+\mu_{ih}m_{ih}-\mu_{ij}m_{ij}=\mu_{ih}-\mu_{ij}+\beta_{im}\mu_{im}-\beta_{in}\mu_{in}\ , \end{aligned}\right\} \quad (2)$$

where q_i denotes the ratio of the load P_iL/M_b to a certain reference load P_0 and the proportional loading is assumed for the present.

2.2 Yield condition

The yield condition of a girder subjected to the actions of both bending and twisting moments may be expressed approximately as follows:

$$m'^2+t'^2=1 \,.$$

Though this expression represents merely a lower bound yield condition for the various cross sectional shapes, in the subsequent discussion it is used as a proper yield condition. Then in order that the yield condition is not violated at the joint i shown in Fig. 1, the following four inequalities must be satisfied:

$$\left.\begin{aligned}&(m_{ih}-1)^2+(t_{ih}-1)^2\leq 1\,,\\&(m_{im}-1)^2+(t_{im}-1)^2\leq 1\,,\\&(m_{ij}-1)^2+(t_{ij}-1)^2\leq 1\,,\\&(m_{in}-1)^2+(t_{in}-1)^2\leq 1\,.\end{aligned}\right\}\qquad(3)$$

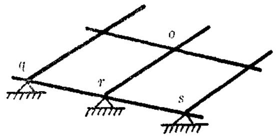

Fig. 2

2.3 Boundary conditions of stresses

The support conditions of grillage girder are considered in two cases, the first is the condition without a transverse girder at the supported ends and the second with a transverse girder.

(1) The case without a transverse girder.
Simply supported end: $m=1,\ \ 0\leq t\leq 2.$
Fixed end: $(m-1)^2+(t-1)^2\leq 1.$

(2) The case with a transverse girder
Simply supported end (referring to Fig. 2):

$$\begin{aligned}&-\beta_{r0}\mu_{r0}t_{r0}+\mu_{rq}m_{rq}-\mu_{rs}m_{rs}=-\beta_{r0}\mu_{r0}+\mu_{rq}-\mu_{rs}\,,\\&\beta_{rq}\mu_{rq}t_{rq}-\beta_{rs}\mu_{rs}t_{rs}-\mu_{r0}m_{r0}=\beta_{rq}\mu_{rq}-\beta_{rs}\mu_{rs}-\mu_{r0}\,,\\&(m_{r0}-1)^2+(t_{r0}-1)^2\leq 1\,,\\&(m_{rs}-1)^2+(t_{rs}-1)^2\leq 1\,,\\&(m_{rq}-1)^2+(t_{rq}-1)^2\leq 1\,.\end{aligned}\qquad(4)$$

2.4 Application of mathematical programming

In the theorem of limit analysis any statically admissible and safe stress fields must satisfy the eqilibrium conditions (2), the boundary conditions (4) and the yield inequalities (3) at all joints of a grid (including the boundary points). The theorem of limit analysis shows the fact that the actual moment stress field always corresponds to a larger value of the collapse load than the other admissible and safe moment stress fields. Hence the problem of searching the actual moment stress field can be regarded as a problem of mathematical programming in which the expressions (2), (3) and (4) are taken as constraints and the reference load P_0 as an objective function.

Now denoting the number of members of the grillage girder by w and the number of the eqilibrium equations consisting of (2) and (4) by v, the constraints may be expressed in the following matrix forms:

$$\underset{v\times 2w}{[a_{ij}]}\,\underset{2w\times 1}{\{m_j\}}+\underset{v\times w}{[b_{ik}]}\,\underset{w\times 1}{\{t_k\}}+P_0\underset{v\times 1}{\{q_i\}}=\underset{v\times 1}{\{c_i\}}\,,\qquad(5)$$

$$\underset{2w\times1}{\{m_j^2\}}+\underset{2w\times w}{[T_{jk}]}\underset{w\times1}{\{t_k^2\}}-2\underset{2w\times1}{\{m_j\}}-2\underset{2w\times w}{[T_{jk}]}\underset{w\times1}{\{t_k\}}+\underset{2w\times1}{\{I\}}\leq\underset{2w\times1}{\{0\}}\ , \tag{6}$$

where

$[a_{ij}]$, $[b_{ik}]$ and $\{c_i\}$ are the matrices determined by the geometric configuration of the structure and the cross sectional shape of the members,

$[T_{jk}]$ is a transformation matrix necessary to make the orders of the matrices equal since the number of twisting moments is reduced to one half of that of bending moments by the relations $t_{ij}=t_{ji}, t_{ih}=t_{hi}, \cdots$ in Eq. (1),

$\{I\}$ is a column vector, all elements of which are unity,

$\{0\}$ is zero vector.

Thus, the problem reduces to the determination of the maximum value of P_0 under the constraints (5) and (6). Since the constraints (6) mean that the interior of unit circles in m, t-coordinates is taken as the permissible domain, this problem may be considered as a problem of convex programming whose solution may be obtained by the cutting plane method developed by J. E. Kelley[7]. In the cutting plane method a linear programming problem is utilized repeatedly by replacing the nonlinear constraints by the suitable linear ones. At present the convex polyhedron $\{S=(m, t)\ 0\leq m\leq 2,\ 0\leq t\leq 2\}$ which encloses the actual permissible domain may be chosen as the first approximation of the permissible domain. The maximum value of load P_0 in the first approximate permissible domain can be searched by means of solving the following linear programming problem:

$$\left.\begin{array}{l}[a_{ij}]\{m_j\}+[b_{ik}]\{t_k\}+P_0\{q_i\}=\{c_i\}\ ,\\ \{0\}\leq\{m_j\}\leq 2\{I\}\ ,\\ \{0\}\leq\{t_k\}\leq 2\{I\}\ ,\\ \text{max. } P_0 \text{ should be required.}\end{array}\right\} \tag{7}$$

Thus the optimal solution $\{m_j\}^1$, $\{t_k\}^1$* and the maximum value of load P_0, which is denoted by P_{01}, in the calculation of the first step can be evaluated by the usual simplex method. And in order to know whether the moment stress field given by the solution $\{m_j\}^1$, $\{t_k\}^1$ satisfies the actual yield conditions or not, the following expression must be examined:

$$\{\delta_j\}^1=\{m_j^2\}^1+[T_{jk}]\{t_k^2\}^1-2\{m_j\}^1-2[T_{jk}]\{t_k\}^1+\{I\}\ . \tag{8}$$

If $\{\delta_j\}^1\leq\{0\}$ is satisfied, the load P_{01} is the maximum lower bound and proved to be the actual limit load due to the uniqueness theorem of limit analysis. Whereas if $\{\delta_j\}^1\leq\{0\}$ is not satisfied, $\{m_j\}^1$, $\{t_k\}^1$ is not the actual solution and then the new linear programming problem must be solved with the following additional constraints called as the cutting planes (the second step):

$$[d_{\lambda\lambda}]\{m_\lambda\}+[T_{\lambda\gamma}'][\bar{d}_{\gamma\gamma}]\{t_\gamma\}\leq\frac{1}{2}\{e_\lambda\}$$

where

$\{m_\lambda\}$ is the column vector consisting of the elements of $\{m_j\}$, which corre-

* The optimal solutions are distinguished from the variables by marking with a numeral letter on the right shoulders.

spond to the positive components δ_λ^1 ($\lambda=1, 2, 3, \cdots, r$, $r\leq 2w$) of the matrix $\{\delta_j\}^1$ in (8),

$\{t_\gamma\}$ is the column vector consisting of the elements of $\{t_k\}$, which correspond to the positive components δ_λ^1 of $\{\delta_j\}^1$ in (8),

$[d_{\lambda\lambda}]$ is the diagonal matrix, its diagonal elements $d_{\lambda\lambda}=m_\lambda^1-1$ where m_λ^1 shows the components of the optimal solution $\{m_\lambda\}^1$,

$[\bar{d}_{\gamma\gamma}]$ is the diagonal matrix, its diagonal elements $\bar{d}_{\gamma\gamma}=t_\gamma^1-1$ where t_γ^1 shows the components of $\{t_\gamma\}^1$,

$[T_{\lambda\gamma}']$ is the transformation matrix with the same character as $[T_{jk}]$ in the expression (6),

$\{e_\lambda\}=\{m_\lambda^2\}^1+[T_{\lambda\gamma}']\{t_\gamma^2\}^1-\{I\}$.

By using the optimal solution $\{m_j\}^2$, $\{t_k\}^2$ at the second step which may be evaluated easily by the dual simplex method, $\{\delta_j\}^2$ is evaluated in the similar manner to (8). And if $\{\delta_j\}^2\leq\{0\}$ is satisfied, $\{m_j\}^2$, $\{t_k\}^2$ represent the actual moment stress field and the corresponding load P_{02} the actual limit load. If $\{\delta_j\}^2\leq\{0\}$ is not satisfied, $\{m_j\}^2$, $\{t_k\}^2$ does not represent the actual solution and hence another solution must be searched in the new linear programming problem in which the new constraints analogous to (9) concerning the positive components of $\{\delta_j\}^2$ are added further, and consequently $\{m_j\}^3$, $\{t_k\}^3$ and P_{03} at the third step are obtained. Repetition of the similar computation gives the following relations[7]: $P_{01}\geq P_{02}\geq P_{03}\cdots\geq P_{0n}\geq P_0$, and P_{0n} approaches the actual limit load P_0 as the iteration process increases in number.

2.5 Estimation of error in the numerical analysis

Continuing the iteration infinitely, P_{0n} tends to the actual limit load P_0. In practical calculation, however, the iteration must be terminated at finite times and it is, therefore, necessary to estimate the difference between P_{0n} and P_0.

Now calculating $\{\delta_j\}^n$ from the optimal solution $\{m_j\}^n$, $\{t_k\}^n$ found at the nth iteration and designating the maximum component of $\{\delta_j\}^n$ as δ_n^* and the bending moment and twisting moment corresponding to δ_n^* as m_n^*, t_n^* respectively, the difference between P_{0n} and P_0 can be estimated as follows:

Substituting m_n^* and t_n^* into the left side of the corresponding inequality of (3), one obtains

$$(m_n^*-1)^2+(t_n^*-1)^2=1+\delta_n^* .$$

From the relations $m_n^{*\prime}=m_n^*-1$ and $t_n^{*\prime}=t_n^*-1$, the moment stress vector ($m_n^{*\prime}$, $t_n^{*\prime}$) is placed on the circle with the radius $\sqrt{1+\delta_n^*}$ and the center at the origin. Hence the moment stress field $(1/\sqrt{1+\delta_n^*})\{m_j\}^n$, $(1/\sqrt{1+\delta_n^*})\{t_k\}^n$ satisfies the yield inequalities (3) and the corresponding load $(1/\sqrt{1+\delta_n^*})P_{0n}$ is proved to be less than or equal to the actual limit load P_0 according to the lower bound theorem of limit analysis. Since $P_{0n}\geq P_0$ as shown in the previous article, the evaluated load P_{0n} is larger than the actual limit load P_0 by $\{1-1/\sqrt{1+\delta_n^*}\}P_{0n}\doteqdot(\delta_n^*/2)\times P_{0n}$ at the most. Thus, for the practical calculation of limit load, if the error is required to be less than or equal to $\varepsilon\%$, it is sufficient that the iteration is terminated when $\delta_n^*\leq 2\varepsilon/100$.

3. CIRCULAR PLATE SUBJECTED TO AN AXIALLY SYMMETRIC LOAD

3.1 Equilibrium conditions

The equilibrium equation of circular plate subjected to an axially symmetric load is given as

$$\frac{d}{dr}(rM)-N=-\int_0^r rPdr \tag{10}$$

where

r is a distance from the center of plate,
M is the bending moment in the radial direction,
N is the bending moment in the circumferential direction,
P is the intensity of load.

Introducing the non-dimensional quantities such as $p=PR^2/M_0$, $m'=M/M_0$, $n'=N/M_0$ and $x=r/R$ where M_0 indicates the fully plastic moment per unit length of plate and R the radius of plate, Eq. (10) can be rewritten as

$$\frac{d}{dx}(xm')-n'=-\int_0^x xpdx\ . \tag{11}$$

In the application of mathematical programming the equilibrium conditions must be represented by a suitable algebraical expressions. In this paper, the differential equation (11) is approximated by a finite difference one. Denoting the spacing of mesh by Δ and the total number of mesh by $s+1$, the following finite difference formula with the terms of order $O(h^4)$ is used:

$$jh\frac{-m_{j+2}'+8m_{j+1}'-8m_{j-1}'+m_{j-2}'}{12h}+m_j'-n_j'+h^2\sum_{k=1}^{j}kp_{k-1}=0 \tag{12}$$
$$\text{at } j=0,1,2,3,\cdots,s-2$$

where $h=\Delta/R$.

At the mesh points near the boundary of plate, i.e., s and $s-1$, the above formula requires the fictitious points situated in the exterior side of the boundary. Since this problem has originally more variables than the number of independent equilibrium equations (including the boundary conditions), the introduction of the fictitious points must be avoided. Then the following formulas with the terms $O(h^3)$ may be used for the mesh points near the boundary:

$$jh\frac{-2m_{j+1}'-3m_j'+6m_{j-1}'-m_{j-2}'}{6h}+m_j'-n_j'+h^2\sum_{k=1}^{j}kp_{k-1}=0 \qquad \text{at } j=s-1\ , \tag{13}$$

$$jh\frac{-11m_j'+18m_{j-1}'-9m_{j-2}'+2m_{j-3}'}{6h}+m_j'-n_j'+h^2\sum_{k=1}^{j}kp_{k-1}=0 \qquad \text{at } j=s\ . \tag{14}$$

The yield inequalities (16) show that the variables m' and n' take the values between $-2/\sqrt{3}$ and $2/\sqrt{3}$. Then by introducing the non-negative variables $m=m'+2/\sqrt{3}$ and $n=n'+2/\sqrt{3}$, Eqs. (12), (13) and (14) can be expressed collectively as follows:

$$\underset{s'\times s'}{[a_{ij}]}\underset{s'\times 1}{\{m_j\}}-\underset{s'\times s'}{[b_{ij}]}\underset{s'\times 1}{\{n_j\}}+p_0\underset{s'\times 1}{\{q_i\}}=\underset{s'\times 1}{\{0\}} \tag{15}$$

where p_0 means the reference load to be used for non-dimensional expressions, and $s'=s+1$.

3.2 Yield conditions

Assuming a sandwich plate obeying Mises' yield criterion, the yield condition for the element of plates is expressed as

$$M^2-MN+N^2=M_0^2\ .$$

By introducing the non-dimensional variables m and n, the above expression is reduced to

$$\left(m-\frac{2}{\sqrt{3}}\right)^2-\left(m-\frac{2}{\sqrt{3}}\right)\left(n-\frac{2}{\sqrt{3}}\right)+\left(n-\frac{2}{\sqrt{3}}\right)^2=1\ .$$

Hence in order that the yield condition is not violated at every mesh point, the following inequalities must be satisfied:

$$\{m_j^2\}-\{m_jn_j\}+\{n_j^2\}-\frac{2}{\sqrt{3}}\{m_j\}-\frac{2}{\sqrt{3}}\{n_j\}+\frac{1}{3}\{I\}\leq\{0\}\ . \qquad (16)$$

3.3 Boundary conditions of stresses

(1) Along the simply supported edge

$$m=\frac{2}{\sqrt{3}}\ ,\qquad \frac{2}{\sqrt{3}}-1\leq n\leq 1+\frac{2}{\sqrt{3}}\ .$$

(2) Along the built-in edge

$$\left(m-\frac{2}{\sqrt{3}}\right)^2-\left(m-\frac{2}{\sqrt{3}}\right)\left(n-\frac{2}{\sqrt{3}}\right)+\left(n-\frac{2}{\sqrt{3}}\right)^2\leq 1\ . \qquad (17)$$

3.4 Application of convex programming

Determination of the limit load of circular plates may be regarded as the problem of convex programming having (15), (16) and (17) as constraints and the reference load p_0 as an objective function. The application of the cutting plane method to solve this problem is intended. The following linear programming problem is dealt with at first:

$$[a_{ij}]\{m_j\}-[b_{ij}]\{n_j\}+p_0\{q_j\}=\{0\}\ ,\qquad \{0\}\leq\{m_j\}\leq\frac{4}{\sqrt{3}}\{I\}\ ,$$

$$\{0\}\leq\{n_j\}\leq\frac{4}{\sqrt{3}}\{I\}\ ,\qquad \max. P_0 \text{ should be required,}$$

where the above equalities include both (15) and (17).

The optimal solution $\{m_j\}^1$, $\{n_j\}^1$ can be obtained by use of the usual simplex method and consequently $\{\delta_j\}^1$ is evaluated as follows:

$$\{\delta_j\}^1=\{m_j^2\}^1-\{m_jn_j\}^1+\{n_j^2\}^1-\frac{2}{\sqrt{3}}\{m_j\}^1-\frac{2}{\sqrt{3}}\{n_j\}^1+\frac{1}{3}\{I\}\ . \qquad (18)$$

If $\{\delta_j\}^1 \leq \{0\}$ is satisfied, $\{m_j\}^1$, $\{n_j\}^1$ is the actual solution and p_{01} ($=\max. p_0$) becomes the actual limit load. But, if $\{\delta_j\}^1 \leq \{0\}$ is not satisfied, the new linear programming problem must be solved with the following additional constraints concerning the positive components $\delta_\lambda{}^1$ ($\lambda = 1, 2, 3, \cdots, r,\ r \leq s+1$) of $\{\delta_j\}^1$.

$$[d_{\lambda\lambda}]\{m_\lambda\} + [\bar{d}_{\lambda\lambda}]\{n_\lambda\} \leq \{f\}$$

where

$\{m_\lambda\}$ is the column vector consisting of the elements of $\{m_j\}$, which correspond to $\delta_\lambda{}^1$ in (18),

$\{n_\lambda\}$ is the column vector consisting of the elements of $\{n_j\}$, which correspond to $\delta_\lambda{}^1$ in (18),

$[d_{\lambda\lambda}]$ is the diagonal matrix, its diagonal elements $d_{\lambda\lambda} = 2m_\lambda{}^1 - n_\lambda{}^1 - \frac{2}{\sqrt{3}}$ where $m_\lambda{}^1$, $n_\lambda{}^1$ show the elements of the optimal solutions $\{m_\lambda\}^1$ and $\{n_\lambda\}^1$ respectively,

$[\bar{d}_{\lambda\lambda}]$ is the diagonal matrix, its diagonal elements $\bar{d}_{\lambda\lambda} = 2n_\lambda{}^1 - m_\lambda{}^1 - \frac{2}{\sqrt{3}}$,

$\{f\} = \{m_\lambda{}^2\}^1 + \{n_\lambda{}^2\}^1 - \{m_\lambda n_\lambda\}^1 - \frac{1}{3}\{I\}$.

Continuation of the iteration is required similarly as in the case of grillage girder problems until a stable solution is obtained.

4. RECTANGULAR PLATES

4.1 Equilibrium conditions

The equilibrium equation of rectangular plates subjected to a uniform load P is given as

$$\frac{\partial^2 M_x}{\partial X^2} - 2\frac{\partial^2 M_{xy}}{\partial X \partial Y} + \frac{\partial^2 M_y}{\partial Y^2} = -P$$

where M_x, M_y and M_{xy} are the bending moments in X-direction, Y-direction and the twisting moment respectively. Denoting the fully plastic moment per unit length of plate by M_0, the fully plastic twisting moment per unit length of plate by T_0 and the reference length of plate by L, and introducing the non-dimensional quantities $m' = M_x/M_0$, $n' = M_y/M_0$, $t' = M_{xy}/T_0$, $x = X/L$, $y = Y/L$, $p = PL^2/M_0$ and $\beta = T_0/M_0$, the above equation is rewritten as

$$\frac{\partial^2 m'}{\partial x^2} - 2\beta\frac{\partial^2 t'}{\partial x \partial y} + \frac{\partial^2 n'}{\partial y^2} + p = 0\ .$$

In the same manner for circular plates the differential equation can be approximately expressed by a finite difference equation as follows:

$$\frac{1}{12\Delta_x{}^2}\Big\{ -m'_{i,j-2} + 16m'_{i,j-1} - 30m'_{i,j} + 16m'_{i,j+1} - m'_{i,j+2} + \alpha^2(-n'_{i-2,j} + 16n'_{i-1,j} - 30n'_{i,j}$$
$$+ 16n'_{i+1,j} - n'_{i+2,j}) - \alpha\beta(-10t'_{i+1,j+1} + 10t'_{i-1,j+1} - 10t'_{i-1,j-1}$$

$$+10t'_{i+1,j-1}+t'_{i+1,j+2}+t'_{i+2,j+1}-t'_{i-1,j+2}-t'_{i-2,j+1}+t'_{i-1,j-2}+t'_{i-2,j-1}$$

$$-t'_{i+1,j-2}-t'_{i+2,j-1})\Big\}+p_{i,j}=0\ ,$$

$$\text{at}\quad i=2,3,4,\cdots,q-2,\quad j=2,3,4,\cdots,r-2\ , \tag{19}$$

$$\frac{1}{\Delta_x^2}\Big\{m'_{i,j-1}-2m'_{i,j}+m'_{i,j+1}+\alpha^2(n'_{i-1,j}-2n'_{i,j}+n'_{i+1,j})$$

$$-\frac{\alpha\beta}{2}(-t'_{i+1,j+1}+t'_{i-1,j+1}-t'_{i-1,j-1}+t'_{i+1,j-1})\Big\}+p_{i,j}=0$$

$$\text{at}\quad i=1,q-1,\quad j=1,r-1\ , \tag{20}$$

where r and q are the numbers of mesh in X and Y-directions respectively and $\alpha=\Delta_x/\Delta_y$ as shown in Fig. 3. In the finite difference formulas of (19) and (20), the smaller terms than the term $O(h^4)$ and $O(h^2)$ respectively in Taylor's expansion have been omitted where $h=\Delta_x/L_x$ or Δ_y/L_y. The two types of approximation like above must be used by the same reason for the circular plate problem*. By introducing the non-negative variables $m=m'+2/\sqrt{3}$, $n=n'+2/\sqrt{3}$ and $t=t'+1$, the equality constraints corresponding to Eq. (5) of the grillage girder problem may be made.

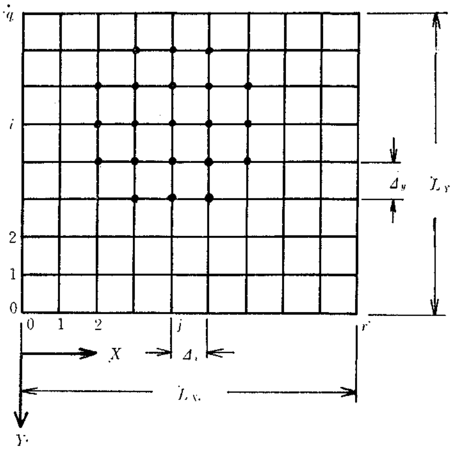

Fig. 3

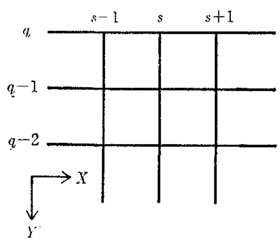

Fig. 4

4.2 Yield conditions

Assuming that the plate considered now consists of a sandwich plate obeying Mises' yield criterion, the yield conditions in terms of bending and twisting moments are expressed as follows:

$$\left(m_{i,j}-\frac{2}{\sqrt{3}}\right)^2-\left(m_{i,j}-\frac{2}{\sqrt{3}}\right)\left(n_{i,j}-\frac{2}{\sqrt{3}}\right)+\left(n_{i,j}-\frac{2}{\sqrt{3}}\right)^2+(t_{i,j}-1)^2\leq 1$$

$$\text{at}\quad i=1,2,3,\cdots,q-1,\quad j=1,2,3,\cdots,r-1\ . \tag{21}$$

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* Mixed using of such two finite approximations of different accuracy may seem to be questionable. However, some numerical examples for the square and the rectangular plates with simply supported or clamped edges show that the results obtained by using partially the finite difference formula with the terms of higher orders such as (19) are more excellent than ones obtained by using the usual formula (20) alone.

4.3 Boundary conditions of stresses

(1) Along the simply supported edges (perpendicular to X-axis)

$$m=\frac{2}{\sqrt{3}}, \quad \left(n-\frac{2}{\sqrt{3}}\right)^2+(t-1)^2\leq 1 .$$

(2) Along the built-in edges

$$\left(m-\frac{2}{\sqrt{3}}\right)^2-\left(m-\frac{2}{\sqrt{3}}\right)\left(n-\frac{2}{\sqrt{3}}\right)+\left(n-\frac{2}{\sqrt{3}}\right)^2+(t-1)^2\leq 1 .$$

(3) Along the free edge (referring to Fig. 4)

$$n_s=\frac{2}{\sqrt{3}}, \quad \left(\frac{\partial n}{\partial y}-2\frac{\partial t}{\partial x}\right)_s=0 .$$

Transforming the above differential equation into a finite difference one in which the terms of $O(h^2)$ are taken in consideration, the following expression is obtained at a mesh point s in Fig. 4:

$$2\alpha n_{q-1,s}-0.5n_{q-2,s}+\beta t_{q,s-1}-\beta t_{q,s+1}=\sqrt{3}\,\alpha .$$

While the equilibrium equation at the point s is given as

$$m_{q,s-1}-2m_{q,s}+m_{q,s+1}+2\alpha^2 n_{q-1,s}+0.5\alpha\beta \cdot (t_{q,s-1}-t_{q,s+1}+4t_{q-1,s-1}-4t_{q-1,s+1}-t_{q-2,s-1}+t_{q-2,s+1})=\frac{4}{\sqrt{3}}\alpha^2 .$$

And the yield condition at the same mesh point is expressed as

$$\left(m_{m,s}-\frac{2}{\sqrt{3}}\right)^2+(t_{m,s}-1)^2\leq 1 .$$

4.4 Application of convex programming

In quite similar manner to circular plate problems, the nonlinear inequalities are replaced firstly by the following linear inequalities and the iterative approach by means of solving the linear programming problem is dealt with as before:

$$\{0\}\leq\{m_j\}\leq\frac{4}{\sqrt{3}}\{I\},$$

$$\{0\}\leq\{n_j\}\leq\frac{4}{\sqrt{3}}\{I\},$$

$$\{0\}\leq\{t_j\}\leq 2\{I\} .$$

From the optimal solution $\{m_j\}^1$, $\{n_j\}^1$ and $\{t_j\}^1$ at the first step of calculation, one have

$$\{\delta_j\}^1=\{m_j^2\}^1+\{n_j^2\}^1-\{m_j n_j\}^1+\{t_j^2\}^1 -\frac{2}{\sqrt{3}}\{m_j\}^1-\frac{2}{\sqrt{3}}\{n_j\}^1-2\{t_j\}^1+\frac{3}{4}\{I\} . \tag{22}$$

If $\{\delta_j\}^1\leq\{0\}$ is not satisfied, the new linear programming problem must be solved with the following additional constraints concerning the positive components δ_λ^1 in the matrix $\{\delta_j\}^1$:

$$\left(2m_{1\lambda}-n_{1\lambda}-\frac{2}{\sqrt{3}}\right)m_\lambda+\left(2n_{1\lambda}-m_{1\lambda}-\frac{2}{\sqrt{3}}\right)n_\lambda+2(t_{1\lambda}-1)t_\lambda$$
$$\leq m_{1\lambda}(m_{1\lambda}-n_{1\lambda})+n_{1\lambda}^2+t_{1\lambda}^2+\frac{4}{3}\ .$$

where $m_{1\lambda}$, $n_{1\lambda}$ and $t_{1\lambda}$ represent the components of $\{m_j\}^1$, $\{n_j\}^1$ and $\{t_j\}^1$ corresponding to δ_λ^1 in (22) respectively, and m_λ, n_λ and t_λ show the variables corresponding to $m_{1\lambda}$, $n_{1\lambda}$ and $t_{1\lambda}$ respectively.

4.5 Errors in the numerical analysis

In the numerical analysis of the limit loads of circular and rectangular plates, the errors may occur mainly during the following processes: (1) an elimination procedure in the simplex method, (2) using a finite difference equation instead of the differential equation, and (3) terminating the iteration at certain finite times. The error by (1) may be removed almost by performing the operation with many figures, for instance, a double precision-operation in a digital computer. The error by (2) decreases as the spacing of mesh in the finite difference approximation becomes smaller. In practice, however, the considerably coarse mesh must be used on account of the restricted capacities of the memory of computer. The magnitude of error due to the coarse mesh can not be estimated precisely. From some results of the numerical examples, however, the considerably refined solutions may be expected by use of the finite difference formula with the terms of higher order such as Eq. (12) or Eq. (19). Finally the error by (3) may be estimated in the same manner as in the case of grillage girder problems. Denoting the maximum value of the components of $\{\delta_j\}^n$, which is obtained after the n times of iteration, by δ_n^*, the corresponding load p_{0n} is larger than the actual limit load p_0 by $\delta_n^*/2\times 100\%$ at most.

5. DETAILED PROCEDURE OF CALCULATION

If the sufficiently reliable limit load is required in the problem, it can not be avoided that the number of iteration becomes large and the number of rows of the simplex tableau becomes much larger. By reason of the capacities of the memory of computer the size of simplex tableau must remain constant during the iteration, and then some advices must be given in the practical calculation. For the simple case of a square grillage girder with fixed ends under a uniform load as shown in Fig. 5, the detailed procedure of calculation is illustrated.

Assuming that the members are of the uniform cross section of square type and

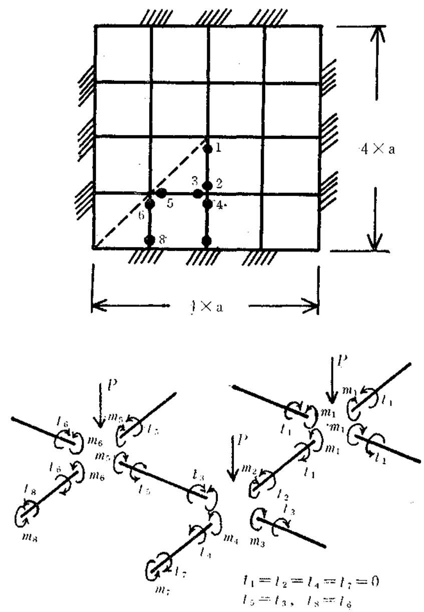

Fig. 5

consist of the material obeying Mises' yield criterion.

$$\beta=\frac{T_0}{M_0}=\frac{4}{3\sqrt{3}}\,.$$

The expression (7) gives

$$\left.\begin{aligned}
&\begin{bmatrix} -5 & 5 & 2 & 1 & -2 & 0 & 1 & 0 \\ -2 & 2 & -1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{Bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{8}{3\sqrt{3}} & 0 \\ \frac{4}{3\sqrt{3}} & -\frac{4}{3\sqrt{3}} \end{bmatrix}\begin{Bmatrix} t_3 \\ t_6 \end{Bmatrix}=\begin{Bmatrix} 2 \\ 2 \\ \frac{8}{3\sqrt{3}} \\ 0 \end{Bmatrix}, \\
&\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\leq\begin{Bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{Bmatrix}\leq 2\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\leq\begin{Bmatrix} t_3 \\ t_6 \end{Bmatrix}\leq 2\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \\
&\max.\left(P_0=\frac{Pa}{4M_0}=m_1-m_2\right).
\end{aligned}\right\} \tag{24}$$

Since in the above linear programming problem some equalities are included, it must be solved by the so-called two-phase method[8] in the simplex method. Table 1 shows the basic tableau* for calculating a feasible solution, where $\lambda_1\sim\lambda_{10}$ are slack variables, $\mu_1\sim\mu_4$ are artificial variables and $\zeta=-\mu_1-\mu_2-\mu_3-\mu_4$ is taken as the auxiliary objective function. By means of the simplex algorithm starting from the basic tableau, the relations $\mu_1=\mu_2=\mu_3=\mu_4=0$ and a feasible solution are obtained. Since these relations must be kept in the subsequent processes, the four columns belonging to μ_1, μ_2, μ_3 and μ_4 are excepted from the tableau when these artificial variables are in the set of the non-basic variables. And by the simplex algorithm at the second phase the first optimal solution is obtained and consequently $\{\delta_j\}^1$ is calculated. These results are shown in Table 2 and the first row of Table 5. If the error in the final result is required to be within 0.25% in magnitude, the condition $\delta^*\leq 0.005$ must be satisfied. The values in the third, the sixth and the eighth rows in $\{\delta_j\}^1$ do not satisfy this condition and the following additional constraints corresponding to (9), must be added:

$$\begin{bmatrix} 1.0000 & 0.0 & 0.0 \\ 0.0 & 0.8619 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}\begin{Bmatrix} m_3 \\ m_6 \\ m_8 \end{Bmatrix}+\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 1.0 \end{bmatrix}\begin{bmatrix} 0.4131 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}\begin{Bmatrix} t_3 \\ t_6 \end{Bmatrix}\leq\frac{1}{2}\begin{Bmatrix} 4.9969 \\ 6.4667 \\ 7.0000 \end{Bmatrix} \tag{25}$$

* In this tableau the objective function transformed into one of standard type is presented and the part of unit matrix is omitted from the tableau.

Table 1

		Nonbasic variables										
		s_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	t_3	t_6
Basic variables	ζ	−5.5396	7.0	−8.0	−1.0	0.0	2.0	−2.0	−1.0	−1.0	−2.3094	0.76980
	μ_1	2.0	−5.0	5.0	2.0	1.0	−2.0	0.0	1.0	0.0	0.0	0.0
	μ_2	2.0	−2.0	2.0	−1.0	0.0	1.0	1.0	0.0	1.0	0.0	0.0
	μ_3	1.5396	0.0	1.0	0.0	−1.0	0.0	0.0	0.0	0.0	1.5396	0.0
	μ_4	0.0	0.0	0.0	0.0	0.0	−1.0	1.0	0.0	0.0	0.76980	−0.76980
	λ_1	2.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	λ_2	2.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	λ_3	2.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	λ_4	2.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
	λ_5	2.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0
	λ_6	2.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0
	λ_7	2.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0
	λ_8	2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0
	λ_9	2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0
	λ_{10}	2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

Note: So means the constant terms in expressions (24).

Table 2

		Nonbasic variables						
		s_0	λ_8	λ_{10}	λ_3	λ_1	λ_7	λ_4
Basic variables	P_0	0.63597	0.13333	0.10264	0.13333	0.06667	0.13333	0.06667
	m_7	2.0000	−0.00000	0.00000	0.00000	0.00000	1.0000	−0.00000
	m_6	1.8619	−0.40000	0.46188	0.60000	0.30000	0.10000	−0.20000
	m_5	1.4101	−0.33333	−0.25660	0.66667	−0.16666	0.16667	0.33333
	m_2	1.3640	−0.13333	−0.10264	−0.13333	0.93334	−0.13333	−0.06667
	m_4	2.0000	0.00000	0.00000	0.00000	−0.00000	0.00000	1.00000
	λ_2	0.63597	0.13333	0.10264	0.13333	−0.93333	0.13333	0.06667
	m_8	2.0000	1.0000	0.00000	0.00000	−0.00000	0.00000	0.00000
	λ_6	0.13812	0.40000	−0.46188	−0.60000	−0.30000	−0.10000	0.20000
	λ_5	0.58993	0.33333	0.25660	−0.66667	0.16667	−0.16667	−0.33333
	m_3	2.0000	−0.00000	0.00000	1.0000	0.00000	0.00000	0.00000
	t_3	1.4131	0.08660	0.06667	0.08660	−0.60622	0.08660	0.69282
	m_1	2.0000	−0.00000	−0.00000	−0.00000	1.0000	−0.00000	−0.00000
	λ_9	0.58693	−0.08660	−0.06667	−0.08660	0.60622	−0.08660	−0.69282
	t_6	2.0000	0.00000	1.0000	0.00000	0.00000	−0.00000	0.00000

Making use of the linearlity of simplex tableau, the new linear programming problem may be dealt with as a problem of dual type[9] which is represented by adding the following part $[A]$ to the tableau of Table 2:

$$\underset{3\times7}{[\boldsymbol{A}]}=\underset{3\times7}{[\boldsymbol{E}]}-\underset{3\times5}{[\boldsymbol{F}]}\underset{5\times7}{[\boldsymbol{H}]}$$

where

Table 3

Basic variables	Nonbasic variables						
	s_0	λ_8	λ_{10}	λ_3	λ_1	λ_7	λ_4
P_0	0.63597	0.13333	0.10264	0.13333	0.06667	0.13333	0.06667
m_7	2.0000	−0.00000	0.00000	0.00000	0.00000	1.0000	−0.00000
m_6	1.8619	−0.40000	0.46188	0.60000	0.30000	0.10000	−0.20000
m_5	1.4101	−0.33333	−0.25660	0.66667	−0.16666	0.16667	0.33333
m_2	1.3640	−0.13333	−0.10264	−0.13333	0.93334	−0.13333	−0.06667
m_4	2.0000	0.00000	0.00000	0.00000	−0.00000	0.00000	1.0000
λ_{11}	−0.08532	−0.03578	−0.02754	−1.03575	0.25042	−0.03578	−0.28619
m_8	2.0000	1.0000	0.00000	0.00000	−0.00000	0.00000	0.00000
λ_{12}	−0.37142	0.34476	−1.39810	−0.51715	−0.25857	−0.08619	0.17238
λ_{13}	−0.49999	−1.0000	−1.0000	−0.00000	0.00000	−0.00000	−0.00000
m_3	2.0000	−0.00000	0.00000	1.0000	0.00000	0.00000	−0.00000
t_3	1.4131	0.08660	0.06667	0.08660	−0.60622	0.08660	0.69282
m_1	2.0000	−0.00000	−0.00000	−0.00000	1.0000	−0.00000	−0.00000
λ_9	0.58693	−0.08660	−0.06667	−0.08660	0.60622	−0.08660	−0.69282
t_6	2.0000	0.00000	1.0000	0.00000	0.00000	0.00000	0.00000

Table 4

Basic variables	Nonbasic variables						
	s_0	λ_8	λ_{13}	λ_{11}	λ_1	λ_7	λ_4
P_0	0.57544	0.02963	0.04955	0.06436	0.09890	0.12873	0.02983
m_7	2.0000	−0.00000	0.00000	0.00000	0.00000	1.0000	−0.00000
m_6	1.5895	−0.86665	0.22296	0.28964	0.44506	0.07928	−0.36578
m_5	1.4923	−0.08204	−0.13716	0.32182	−0.00549	0.14364	0.14913
m_2	1.4246	−0.02963	−0.04955	−0.06436	0.90110	−0.12873	−0.02983
m_4	2.0000	−0.00000	0.00000	0.00000	−0.00000	0.00000	1.0000
λ_3	0.06908	0.00795	0.01329	−0.48273	−0.24176	0.03454	0.27630
m_8	2.0000	1.0000	0.00000	0.00000	−0.00000	0.00000	0.00000
λ_{12}	0.36333	1.74695	−0.69215	−0.24964	−0.38360	−0.06833	0.31526
λ_{13}	0.24999	0.49999	−0.25000	0.00000	−0.00000	0.00000	0.00000
m_3	1.9309	−0.00795	−0.01329	−0.48273	0.24176	−0.03454	−0.27630
t_3	1.3738	0.01925	0.03218	0.04180	−0.58528	0.08361	0.66889
m_1	2.0000	0.00000	−0.00000	−0.00000	1.0000	−0.00000	−0.00000
λ_9	0.62624	−0.01925	−0.03218	−0.04180	0.58528	−0.08361	−0.66889
t_5	1.5000	−0.99999	0.50000	−0.00000	0.00000	−0.00000	−0.00000

$[E]$ is the matrix of the coefficients of (25) and the corresponding variables are in the set of the nonbasic ones of the tableau of Table 2. The coefficients corresponding to slack variables are taken as zero, i.e.,

$$[E]=\begin{bmatrix} 2.4985 & 0.0 & 0.0\cdots\cdots \\ 3.2334 & 0.0 & 0.0\cdots\cdots \\ 3.5000 & 0.0 & 0.0\cdots\cdots \end{bmatrix},$$

Table 5

Step i	P_{0i}	$\{m_j\}^i$	$\{t_k\}^i$	$\{\delta_j\}^i$
1	0.63597	2.0000	1.4131	0.0000
		1.3640	2.0000	−0.8675
		2.0000		1.0000
		2.0000		0.0000
		1.4101		−0.6612
		1.8619		0.74287
		2.0000		0.0000
		2.0000		1.0000
2	0.575441	2.0000	1.3738	0.0000
		1.4246	1.5000	−0.8197
		1.9309		0.0063
		2.0000		0.0000
		1.4923		−0.6179
		1.5895		−0.4025
		2.0000		0.0000
		2.0000		0.2500
3	0.567589	2.0000	1.3687	0.0000
		1.4324	1.7500	−0.8130
		1.9296		0.0001
		2.0000		0.0000
		1.5106		−0.6033
		1.8042		0.2093
		2.0000		0.0000
		1.7500		0.1250
4	0.561795	2.0000	1.3649	0.0000
		1.4382	1.5833	−0.8080
		1.9311		0.0001
		2.0000		0.0000
		1.5266		−0.5895
		1.6948		−0.1770
		2.0000		0.0000
		1.8333		0.0346
5	0.559735	2.0000	1.3636	0.0000
		1.4403	1.6527	−0.8061
		1.9316		0.0001
		2.0000		0.0000
		1.5323		−0.5845
		1.7549		−0.0040
		2.0000		0.0000
		1.7639		0.0097
6	0.559368	2.0000	1.3633	0.0000
		1.4406	1.6119	−0.8059
		1.9317		0.0001
		2.0000		0.0000
		1.5333		−0.5836
		1.7247		−0.1004
		2.0000		0.0000
		1.7925		0.0025

Underlines show the elements which do not satisfy the condition $\delta^* \leq 0.005$.

$[F]$ is the matrix of the coefficients of (25) and the corresponding variables are in the set of the basic ones of the tableau of Table 2, i.e.,

$$[F]=\begin{bmatrix} 1.0000 & 0.0 & 0.0 & 0.4131 & 0.0 \\ 0.0 & 0.8619 & 0.0 & 0.0 & 1.0000 \\ 0.0 & 0.0 & 1.0000 & 0.0 & 1.0000 \end{bmatrix},$$

$[H]$ is the matrix consisting of the rows of Table 2 and the basic variables belonging to those rows correspond to the variables of (25), i.e.,

$$[H]=\begin{bmatrix} 2.0000 & -0.0000 & 0.0000 & 1.0000\cdots\cdots \\ 1.8619 & -0.4000 & 0.4619 & 0.6000\cdots\cdots \\ 2.0000 & 1.0000 & 0.0000 & 0.0000\cdots\cdots \\ 1.4131 & 0.0866 & 0.0667 & 0.0866\cdots\cdots \\ 2.0000 & 0.0000 & 1.0000 & 0.0000\cdots\cdots \end{bmatrix}.$$

By adding such the part the number of rows in the tableau increases. The more the iteration of the calculation proceeds, the larger the number of rows becomes. By introducing the new constraints (25) the number of slack variables becomes large but one of actual variables remains constant. Since the values of slack variables are not needed in the final results, the size of the tableau is able to remain constant by substituting the additional part $[A]$ into such the rows that the basic variables belonging to those rows are slack ones as shown by the underlines in Table 3. Then the optimal solution at the second step can be calculated from the tableau of Table 3 by use of the dual simplex method. The optimal solution and the corresponding $\{\delta_j\}^2$ are shown in Table 4 and the second row of Table 5. The condition $\delta^*\leq 0.005$ is not fulfilled now. Hence the new constraints must be added concerning the elements of the third and the eight rows in $\{\delta_j\}^2$, and the optimal solution $\{m_j\}^3$, $\{t_k\}^3$ at the third step is calculated in the similar manner. Proceeding such the iteration, it is found that the condition $\delta^*\leq 0.005$ is fulfilled at the sixth step. The information on the convergency of solution is furnished in Table 5.

6. SOME RESULTS FROM THE NUMERICAL EXAMPLES

6.1 Square grillage girders

In order to compare the numerical solution obtained by the described method with the known analytical one the problem of square grillage girder with members of the same cross section has been treated under a uniform load as shown in Fig. 5. For the same problem P. G. Hodge[10] gives the analytical solution by making use of an inscribed octagonal yield curve for the circular yield curve. The numerical solution evaluated by the linear programming problems for the same approximate yield curve has been proved to agree with precisely P. G. Hodge's one. While for the circular yield curve J. Heymann[11] gives the limit load $4aP/M_0=2.32$ in the case $\beta=1$ and for the same case the method in this paper gives $4aP/M_0=2.3238$ under the condition $\delta^*\leq 0.005$. The results for the various values of β are indicated graphically in Fig. 6.

6.2 Other grillage girders

The numerical results of the other grillage girders with simply supported ends, subjected to the various loads as shown in Fig. 7, are presented. The numerical results for the grillage girder with three main girders and three cross

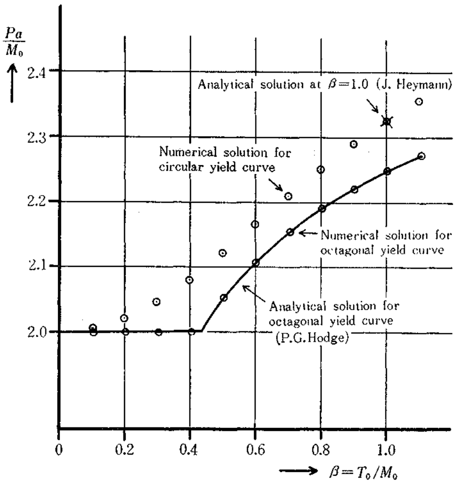

Fig. 6 Comparison of the numerical solution obtained by the method in this paper and P. G. Hodge's analytical one for the uniformly loaded square grillage girder of the uniform cross section as shown in Fig. 5.

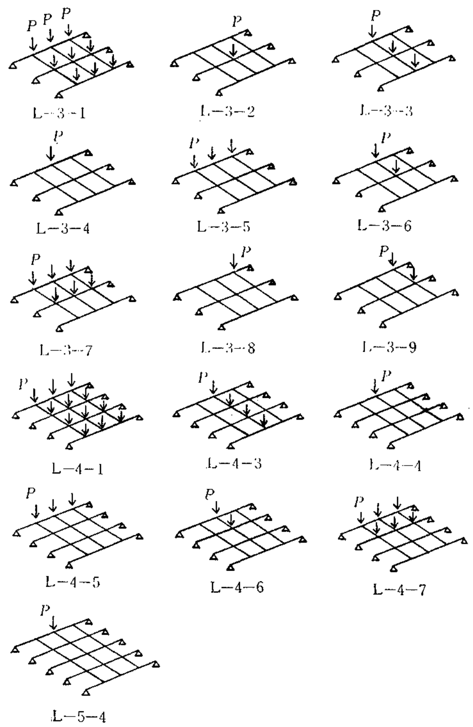

Fig. 7 Loading conditions for the grillage girders where P shows the intensity of the point load.

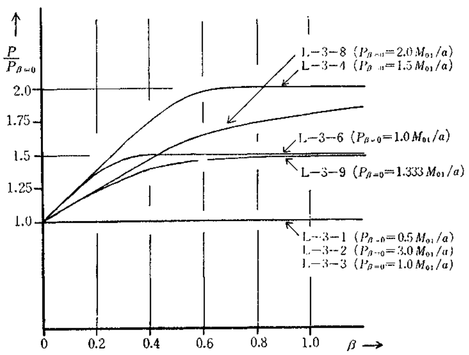

Fig. 8 Limit loads for the grillage girder with three main girders and three cross ones, where $\beta=\beta_1=\beta_2$ and $\gamma=1.0$

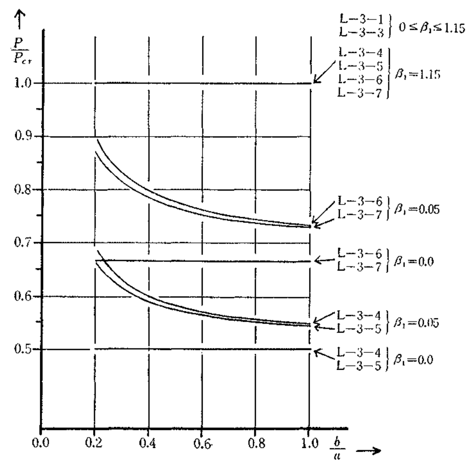

Fig. 9 Limit loads of the grillage girders with three main girders and three cross ones, where $\beta_2=\beta_1$ and $\gamma=1.0$.

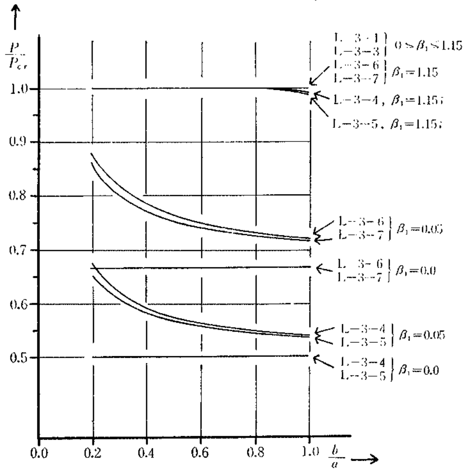

Fig. 10 Limit loads of the grillage girders with three main girders and three cross ones, where $\beta_2=\beta_1$ and $\gamma=0.5$.

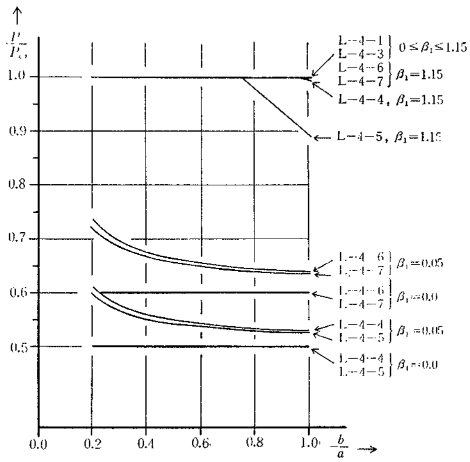

Fig. 11 Limit loads of the grillage girders with four main girders and three four cross ones, where $\beta_2=\beta_1$ and $\gamma=1.0$.

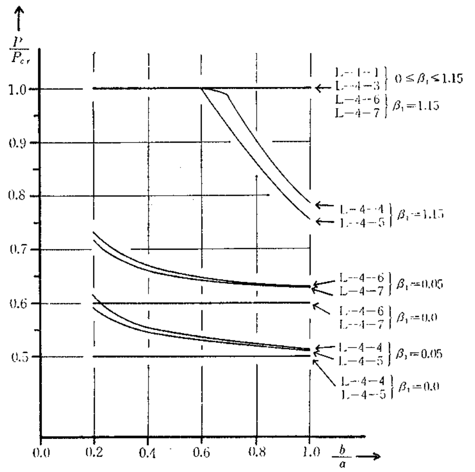

Fig. 12 Limit loads of the grillage girders with four main girders and three cross ones, where $\beta_2=\beta_1$ and $\gamma=0.5$.

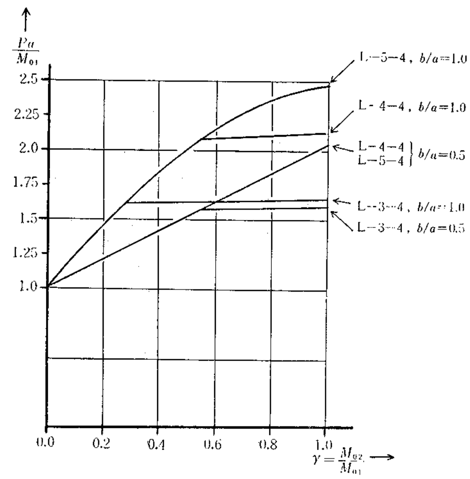

Fig. 13 Influence of γ for the limit loads of grillage girders, where $\beta_1=\beta_2=0.05$.

ones of the uniform cross section are shown in Fig. 8. From Fig. 9 to Fig. 12, the relations between the values of the limit load and the spacing of main girders are shown for the grillage girders with three cross girders and three or more main ones having such the typical cross sections that the ratios of the fully plastic torque to the fully plastic moment are 0, 0.05 and 1.15, where 0.05 shows the mean value of H-sections and 1.15 represents the value for an idealized box section with thin web plates and thick flange ones. And in Fig. 13 how the sectional size of the cross girder influences the values of limit load under various loadings is investigated. In these examples each main girder and each cross one are assumed to be of the same cross sections respectively and arranged with the

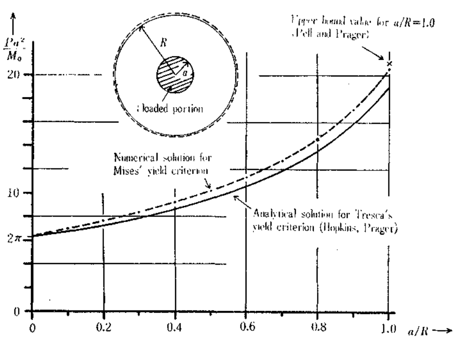

Fig. 14 Limit loads for partially loaded circular plate with simply supported edge.

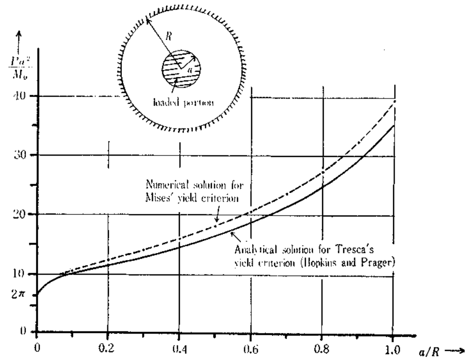

Fig. 15 Limit loads for partially loaded circular plate with clamped edge.

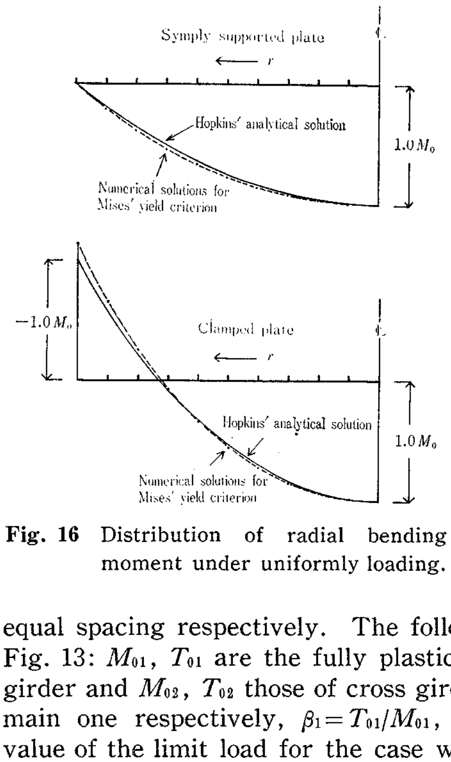

Fig. 16 Distribution of radial bending moment under uniformly loading.

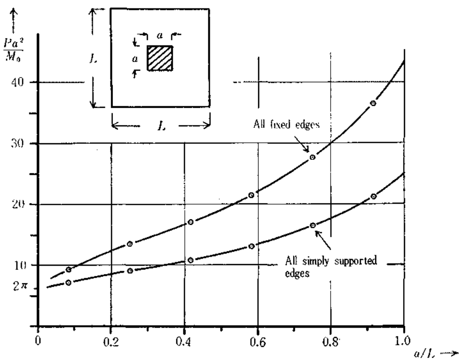

Fig. 17 Limit loads of square plate under partially loading (Number of mesh: 7×7 in a quadrant and $\delta^*\leq0.01$).

equal spacing respectively. The following nomenclatures are used in Fig. 8 to Fig. 13: M_{01}, T_{01} are the fully plastic moment and the fully plastic torque of main girder and M_{02}, T_{02} those of cross girder, a, b are the spacing of cross girder and main one respectively, $\beta_1=T_{01}/M_{01}$, $\beta_2=T_{02}/M_{02}$, $\gamma=M_{02}/M_{01}$, and P_{cr} shows the value of the limit load for the case when $T_{01}=M_{02}=\infty$, namely, each main girder is collapsed by the same mode as a simple beam with span length $4a$.

In general the torsional resistance of I-type or H-type girder is very small and may be neglected in practical design. However, the above numerical examples imply that the neglect of the torsional resistance may give considerably conservative results in the limit design of the structures with such girders. While, Fig. 13 shows that to enlarge the sectional size of cross girder doesn't always result to increase the values of the limit load. Thus to find the optimal size of cross girder requires a special technical aspect in practical design.

6.3 Circular plates

H. G. Hopkins and W. Prager[12] give analytical solutions for the axially symmetric problems of circular plate obeying Tresca's yield criterion. The same problems under Mises' yield criterion have been studied where the spacing of mesh in the finite difference approximation is taken as 1/10 of the radius. The comparison of the numerical solutions and the analytical ones is given in Fig. 14 ~Fig. 16.

6.4 Rectangular plates

The calculated values of the limit load for the partially loaded square plates and the uniformly loaded rectangular plates are shown in Fig. 17 and Table 6. In Table 7 the relations between the spacing of mesh in the finite difference approximation and the obtained limit loads for the uniformly loaded square plates are investigated and at the same time the evaluated values of the limit loads are compared with the lower and upper bound values given by P. G. Hodge and T. Belytschko[6]. From a result of this it may be said that for the simply supported edges even the rather coarse mesh gives a sufficiently reilable limit load but for the clamped edges the close mesh is required to find the reliable limit load.

Table 6 Values of limit load for uniformly loaded rectangular plates (PL_y^2/M_0)

L_x/L_y	1.1	1.25	1.5	2.0	3.0
Simply supported plates	22.910	20.527	17.859	15.008	12.640
Clamped plates	39.486	35.385	30.976	26.374	22.728

Number of mesh: 10×8, $\delta^* \leq 1/100$ and L_x, L_y: lengths of the plate side in X and Y-directions.

Table 7 Values of limit load for uniformly loaded square plate (PL^2/M_0)

Number of mesh	Simply supported plate	Clamped plate
4× 4	24.954	41.477
6× 6	25.018	42.891
8× 8	25.090	43.164
10×10	25.094	43.328
12×12	25.091	43.470
Lower bounded*	24.864	42.864
Upper bounded*	26.544	49.248

$\delta^* \leq 1/100$, L: length of the plate side.

* The values evaluated by P. G. Hodge and T. Belytschko[6].

7. CONCLUSIONS

This paper dealts with the problems of the evaluation of the limit load for grillage girders, circular plates and rectangular ones. The method of analysis is based on the idea that the determination of limit load can be reduced to a nonlinear programming problem, especially to a convex programming one which can be solved by the cutting plane method developed by J. E. Kelley. The reliability of this method is confirmed by comparing the evaluated values in the various numerical examples with the known analytical solutions. Especially, for the grillage girder problems it is possible to obtain the precise limit load if the sufficient number of the iteration is performed. For the plate problems, to find the precise limit loads becomes more difficult by the finite difference approximation of the equilibrium equations, but in a practical point of view the limit loads with enough precision may be obtained by using moderate mesh in the finite difference calculation.

Since the method of calculation described in this paper is based essentially on the lower bound theorem of limit analysis, it may be applied to the problems of anisotropic plates, arbitrarily shaped plates and more complicate structures. Hence it may be expected that the numerical method described herein would become an effective weapon for the evaluation of the limit load in the various structures. Lastly, it is reported that the running time (including the time of print out) of the digital computer amounts to 10∼60 seconds in a grillage girder problem, about 30 seconds in a circular plate and 2∼8 minutes in a rectangular plate. The computer FACOM-270-30 of Osaka City University has been used for the computation.

NOTATIONS

Λ	:	panel length of a grillage girder
L	:	reference length of a grillage girder or a rectangular plate
λ	:	L/Λ
R	:	radius of a circular plate
X, Y	:	rectangular coordinates for plates
L_x, L_y	:	lengths of the sides of a rectangular plate
a, b	:	spacing of the cross girder and the main one of a grillage girder, respectively
Δ	:	spacing of mesh in the finite difference approximation for a circular plate problem
h	:	Δ/R
Δ_x, Δ_y	:	spacing of mesh in the finite difference approximation for a rectangular plate problem
α	:	Δ_x/Δ_y
P	:	concentrated load in a grillage girder or intensity of a distributed load in a circular plate or a rectangular one
P_0, p_0	:	reference loads
M	:	bending moment of a grillage girder or bending moment of a circular plate in the radial direction
T	:	twisting moment of a grillage girder
N	:	bending moment of a circular plate in the circumferential direction
M_x, M_y, M_{xy}	:	bending and twisting moments of a rectangular plate
M_0	:	fully plastic moment of a girder or fully plastic bending moment per unit length of a plate
T_0	:	fully plastic torque of a girder or fully plastic twisting mo-

		ment per unit length of a plate
M_b	:	fully plastic moment of a reference girder
m', t', n'	:	non-dimensional moments: $m'=M/M_0$ and $t'=T/T_0$ for grillage girders, $m'=M/M_0$ and $n'=N/M_0$ for circular plates, and $m'=M_x/M_0$, $n'=M_y/M_0$ and $t'=M_{xy}/T_0$ for rectangular plates
m, t, n	:	non-negative variables for the problems of mathematical programming: $m=m'+1$ and $t=t'+1$ for grillage girder problems, $m=m'+2/\sqrt{3}$ and $n=n'+2/\sqrt{3}$ for circular plate problems, and $m=m'+2/\sqrt{3}$, $n=n'+2/\sqrt{3}$ and $t=t'+1$ for rectangular plate problems
β	:	T_0/M_0
μ	:	M_0/M_b
$\{m_j\}, \{t_j\}, \{n_j\}$	:	column matrices consisting of the variables m, t and n
$\{m_j\}^i, \{t_j\}^i, \{n_j\}^i$	:	optimal solution of the linear programming problem at the ith iteration
$\{I\}$	:	column matrix whose elements are unity
$\{0\}$	:	zero vector
$\{\delta_j\}^i$	:	a column matrix necessary to estimate the error in the calculated limit load at the ith iteration
δ^*	:	the maximum element of $\{\delta_j\}^i$
$\lambda_1, \lambda_2, \cdots$	:	slack variables for a linear programming problem
$\mu_1, \mu_2, \cdots$	:	artificial variables for a linear programming problem

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