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A NOTE ON THE DIFFUSION DUE TO LINEARIZED RANDOM WAVES

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SUMMARY

The coefficient of eddy diffusion is obtained for the randomized Lagrangian representation of short crested waves. It is proved that random waves have no diffusivity. The result mentioned above is essential limitation of the second order, stationary stochastic theory.

INTRODUCTION

The ocean development is in progress in many aspects. In accordance with the increase of human activities in the ocean, the preservation of environment becomes one of critical subjects. Mixing problems are concerned with the protection of human environment from contamination in connection with the ocean outfalls of sewage or the disposal of radioactive substances.

The objective of this study is to express the coefficient of eddy diffusion in terms of wave spectra. The randomized Lagrangian expression of short crested waves would be able to describe the mixing process of floating materials at the surface of the ocean.

Although no physically meaningful result has not yet been obtained, the writer considers that the publication of this report may be useful from the following standpoint:

1) Dr. M. Hino discussed the same subject by different method of approach in the 14th Japanese Conference on Hydraulics, February, 1970, and he personally expressed his desire to publish my report.
2) There may be potential existence of investigators interested in the subject.

THEORY

For simplicity one-dimensional problem is considered. Consider the movement of a tracer particle which has a fluctuating velocity $v(t)$. $v(t)$ is supposed to be a mean zero, second order, covariance stationary random function.

Suppose that a particle is put into this fluctutaing velocity field at $y=0$ at time zero. The position of the tracer particle after time t is

$$y(t)=\int_0^t v(\xi)d\xi \,. \tag{1}$$

Consequently the following two expressions are derived:

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$$y^2(t)=\int_0^t\int_0^t v(\xi)v(\zeta)d\xi d\zeta\,, \tag{2}$$

$$E[y^2(t)]=\int_0^t\int_0^t E[v(\xi)v(\zeta)]d\xi d\zeta\,, \tag{3}$$

where $E[y^2(t)]$ represents the expected value of $y^2(t)$.

According to the assumption that covariance is stationary, only time interval between ξ and ζ is an important factor in the correlation coefficient. Setting

$$\tau=\zeta-\xi\,, \tag{4}$$

Lagrangian correlation coefficient is given by

$$\psi(\tau)=\frac{E[v(t)v(t+\tau)]}{E[v^2]}\,, \tag{5}$$

where $E[v^2]$ is called the intensity of turbulence in the theory of turbulence and is a constant by the assumption of covariance stationarity.

Following the G. I. Taylor's consideration, one obtains the coefficient of eddy diffusion, D, for infinitely large value of t as below:

$$D=E[v^2]\cdot T\,, \tag{6}$$

where

$$T=\int_0^\infty \psi(\tau)d\tau\,, \tag{7}$$

is called the Lagrangian time scale.

The Lagrangian equations are stated in the following paragraphs. Let α, β, and δ be the x, y, and z coordinates of a particle of fluid. In the Lagrangian system of equations, a solution to the equations consists of finding the positions x, y, and z of all the particles in the fluid as a function of time and the initial positions of the particles, α, β, and δ. The Lagrangian equations are given by Eq. (8) where subscripts denote partial differentiation.

$$\left.\begin{aligned} &x_{tt}x_\alpha+y_{tt}y_\alpha+(z_{tt}+g)z_\alpha+p_\alpha/\rho=0\\ &x_{tt}x_\beta+y_{tt}y_\beta+(z_{tt}+g)z_\beta+p_\beta/\rho=0\\ &x_{tt}x_\delta+y_{tt}y_\delta+(z_{tt}+g)z_\delta+p_\delta/\rho=0 \end{aligned}\right\} \tag{8}$$

The equation of continuity is most conveniently expressed by

$$\frac{d}{dt}\left[\frac{\partial(x, y, z)}{\partial(\alpha, \beta, \delta)}\right]=0\,. \tag{9}$$

Such solutions need not be irrotational, but, if a function, $F(\alpha, \beta, \delta, t)$, can be found such that

$$dF=(x_tx_\alpha+y_ty_\alpha+z_tz_\alpha)d\alpha+(x_tx_\beta+y_ty_\beta+z_tz_\beta)d\beta+(x_tx_\delta+y_ty_\delta+z_tz_\delta)d\delta \tag{10}$$

is a perfect differential, there is no vorticity.

A zero order solution to these equations is given by

$$x=\alpha\,, \qquad y=\beta\,, \qquad z=\delta\,, \qquad p=p_0-g\rho\delta\,, \tag{11}$$

in which all fluid particles are at rest in hydrostatic equilibrium under the force of gravity. Here δ is set to be positive in the upward direction.

We expand about the zero order solution in terms of a small parameter, ε, as in Eq. (12) in which F_0 is a constant. Here we think of ε as equal to ak, but the first order terms must have the dimensions of a length, and so $a=\varepsilon/k$ is used in the various solutions that are obtained. The parameter ε never appears explicitly.

$$\left.\begin{aligned} &x=\alpha+\varepsilon x_1+\varepsilon^2 x_2\,,\\ &y=\beta+\varepsilon y_1+\varepsilon^2 y_2\,,\\ &z=\delta+\varepsilon z_1+\varepsilon^2 z_2\,,\\ &p=p_0-g\rho\delta+\varepsilon p_1+\varepsilon^2 p_2\,,\\ &F=F_0+\varepsilon F_1+\varepsilon^2 F_2\,. \end{aligned}\right\} \tag{12}$$

A solution to Eq. (12), applicable to waves in deep water, when obtained up to the first order solution, yields Eq. (13).

$$\left.\begin{aligned} &x=\alpha+\varepsilon x_1=\alpha-\frac{ak_1}{k}e^{k\delta}\sin(k_1\alpha+k_2\beta-\omega t)\,,\\ &y=\beta+\varepsilon y_1=\beta-\frac{ak_2}{k}e^{k\delta}\sin(k_1\alpha+k_2\beta-\omega t)\,,\\ &z=\delta+\varepsilon z_1=\delta+ae^{k\delta}\cos(k_1\alpha+k_2\beta-\omega t)\,,\\ &p=p_0-\rho g\delta+\varepsilon p_1=p_0+\rho g\delta\,,\\ &F_1=\frac{a\omega}{k}e^{k\delta}\sin(k_1\alpha+k_2\beta-\omega t)\,. \end{aligned}\right\} \tag{13}$$

In order to impose the condition that $p(\alpha, \beta, \delta)=p_0$ at $\delta=0$, it is required that

$$\omega^2/g=k\,. \tag{14}$$

The equation of continuity imposes the condition that

$$k^2=k_1^2+k_2^2\,. \tag{15}$$

The first order solution has no vorticity to first order as F_1 has been found. An alternate form for the solution given by Eq. (13) is Eq. (16).

$$\left.\begin{aligned} &x=\alpha-a\cos\theta\cdot e^{\omega^2\delta/g}\sin\left(\frac{\omega^2}{g}(\alpha\cos\theta+\beta\sin\theta)-\omega t\right),\\ &y=\beta-a\sin\theta\cdot e^{\omega^2\delta/g}\sin\left(\frac{\omega^2}{g}(\alpha\cos\theta+\beta\sin\theta)-\omega t\right),\\ &z=\delta+ae^{\omega^2\delta/g}\cos\left(\frac{\omega^2}{g}(\alpha\cos\theta+\beta\sin\theta)-\omega t\right). \end{aligned}\right\} \tag{16}$$

The angle θ is the direction toward which the wave is traveling.

If it is assumed that each particle has a displacement from its rest position, α_0, β_0, and δ_0, that is described by a stationary Gaussian vector process with specified coherency relationships, a solution to Eq. (8) that is correct to first order is given by Eq. (17).

$$\left.\begin{aligned} x&=\alpha-\int_0^\infty\int_{-\pi}^{\pi} e^{\omega^2\delta/g}\cos\theta\cdot\sin\left(\frac{\omega^2}{g}(\alpha\cos\theta+\beta\sin\theta)-\omega t+\epsilon(\omega,\theta)\right)\sqrt{2S(\omega,\theta)d\omega d\theta}\ ,\\ y&=\beta-\int_0^\infty\int_{-\pi}^{\pi} e^{\omega^2\delta/g}\sin\theta\cdot\sin\left(\frac{\omega^2}{g}(\alpha\cos\theta+\beta\sin\theta)-\omega t+\epsilon(\omega,\theta)\right)\sqrt{2S(\omega,\theta)d\omega d\theta}\ ,\\ z&=\delta+\int_0^\infty\int_{-\pi}^{\pi} e^{\omega^2\delta/g}\cos\left(\frac{\omega^2}{g}(\alpha\cos\theta+\beta\sin\theta)-\omega t+\epsilon(\omega,\theta)\right)\sqrt{2S(\omega,\theta)d\omega d\theta}\ . \end{aligned}\right\} \tag{17}$$

In Eq. (17) $S(\omega,\theta)$ is the resolution of the variance of the particle motions into frequency and direction and $\epsilon(\omega,\theta)$ is the random phase lag expressed by Gaussian vector process.

CALCULATION OF THE COEFFICIENT OF EDDY DIFFUSION

Let us consider the simplest example. Suppose we put a floating tracer particle at $(\alpha,\beta,\delta)=(0,0,0)$ at time zero. Let us consider we have uniform flow with velocity U in x direction and intensity of fluctuating velocity in x direction is small compared with U. We are mainly concerned with the contribution of random waves to the expected lateral position of the floating particle, where y is taken in the direction parpendicular to x at water surface.

Because the floating particle is chosen, movement in z direction is out of inquiry in this report. y component of the velocity vector is derived from Eq. (17).

$$v=y_t=\int_0^\infty\int_{-\pi}^{\pi}\omega\sin\theta\cdot\cos(\omega t-\epsilon(\omega,\theta))\sqrt{2S(\omega,\theta)d\theta d\omega} \tag{18}$$

for $\alpha=\beta=\delta=0$.

Let covariance of the velocity component be $c(\tau)$.

$$c(\tau)=E[v(t)\cdot v(t+\tau)]=\int_0^\infty\int_{-\pi}^{\pi}\omega^2\sin^2\theta\cdot S(\omega,\theta)\cos\omega\tau d\theta d\omega\ . \tag{19}$$

Through the definition of Eqs. (5) and (7), the Lagrangian time scale can be expressed by,

$$T=\int_0^\infty\frac{c(\tau)}{c(0)}d\tau=\frac{1}{c(0)}\int_0^\infty c(\tau)d\tau\ . \tag{20}$$

Because the integrand of Eq. (19) is an even function, Eq. (19) can be rewritten as,

$$\begin{aligned} c(\tau)&=\frac{1}{2}\int_{-\infty}^\infty\int_{-\pi}^{\pi}\omega^2\sin^2\theta\cdot S(\omega,\theta)(\cos\omega\tau+i\sin\omega\tau)d\theta d\omega\\ &=\int_{-\infty}^\infty\frac{\omega^2}{2}\left[\int_{-\pi}^{\pi}\sin^2\theta\cdot S(\omega,\theta)d\theta\right]e^{i\omega\tau}d\omega\ . \end{aligned} \tag{21}$$

This is the expression of the inverse Fourier transform. Therefore, the Fourier pair of Eq. (21) takes the form as follows:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} c(\tau)e^{-i\omega\tau}d\tau=\frac{\omega^2}{2}\int_{-\pi}^{\pi} \sin^2\theta\cdot S(\omega, \theta)d\theta \; . \tag{22}$$

Setting $\omega=0$ in Eq. (22), we obtain the integral of $c(\tau)$.

$$\frac{1}{2\pi}\int_{0}^{\infty} c(\tau)d\tau=0 \; . \tag{23}$$

According to Eqs. (6), (20), and (23), the coefficient of eddy diffusion due to randomized short crested waves comes to be,

$$D=0 \; . \tag{24}$$

DISCUSSION

The result given by Eq. (24) shows that the linearized random waves have no diffusivity. Due to the basic condition of the covariance stationary, second order stochastic process, the variance of the position of a tracer particle is proved to be constant. On the other hand, in turbulence theory only the velocity fluctuation is bounded by statistical assumptions. Because the position of a tracer particle has no restriction, G. I. Taylor successfully described the diffusion in the turbulent flow.

The writer suspects that the covariance stationary, second order stochastic theory of waves does not afford a powerful tool for the analysis of diffusion. The second order approximation does not work either, because the covariance stationarity is also applied to the second order approximation. Non-stationary theory might give the idea on diffusion.

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