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PROBABILISTIC DISPLACEMENT TIME HISTORY OF A SINGLE DEGREE OF FREEDOM SYSTEM

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I. INTRODUCTION

Research on structural systems with stochastic or probabilistic parameters has been studied by many authors. Mathematicians initiated the research in stochastic differential equations[1),4)]. This interest has led to many powerful methodological tools for engineers primarily in the field of random vibration. Interest was focused on dynamical responses such as natural frequency or displacement of structures with uncertain parameters. Owing to the randomness that exists in material properties and construction, the actual values of the parameters cannot be determined exactly. The responses under this situation will be imprecisely known.

Free vibration of elastic beams was studied by W. E. Boyce and B. E. Goodwin, in which geometric parameters such as moment of inertia and cross sectional area of the beam are assumed to be stochastic[2)]. Vibration characteristics of beam-columns and plates were investigated by author[3)]. He obtained the generalized stochastic equations of mean and variance for natural frequency of the systems when all the geometric as well as material properties are probabilistic or stochastic functions. Samuels applied theory of probability to stochastic equations whose parameters are gaussianly distributed random variables which vary with time[5)].

This concept of the probabilistic parameters was applied to one degree of freedom systems by W. Tang[6)], in which probability distribution functions for amplitude, frequency and phase angle of the system are obtained for normally distributed input parameters.

This investigation attempts to extend W. Tang's work and develop primarily the time dependent displacement functions in the form of mean and variance for vibration problems of a mass, spring and dashpot model with one degree of freedom systems shown in Fig. 1. Spring and damping coefficients are assumed to have small random perturbations. Due to this assumption, the displacement can be expressed in the series of power of the small parameters where the higher terms may be neglected. The general expression for the displacement will

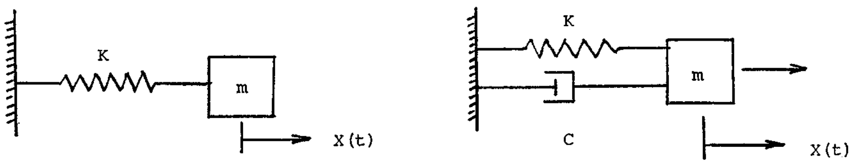

Fig. 1 Mass-Spring-Damping Systems

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be applied to the case where the excitation force is also random. Finally, numerical examples are given when the system is subjected to an impulsive excitation.

II. FREE VIBRATION

Let us first discuss response quantities of the frequency, amplitude and phase angle of the model shown in Fig. 1. The governing equation of the undamped free vibration is given by[7]

$$m\frac{d^2x}{dt^2}+kx=0 \tag{1}$$

and the solution is

$$X(t)=X_0\cos\omega t+\frac{\dot{X}_0}{\omega}\sin\omega t \tag{2}$$

or

$$X(t)=A\cos(\omega t-B) \tag{3}$$

where $X(t)$ is a displacement and m, k and ω are the mass, spring coefficient and natural frequency respectively. ω is given by the relation $\omega^2=k/m$. X_0 and $\dot{X}_0$ are the initial displacement and velocity respectively. A represents the amptitude, whereas B stands for the phase angle.

Let us assume that K is a random variable given by

$$K=K_0+\varepsilon_k \tag{4}$$

where K_0 is a deterministic constant and ε_k is a small random variable with zero mean. Thus, there is almost zero possibility that ε_k will exceed K_0.

Frequency ω is expressed by using eq. (4)

$$\omega=\sqrt{\frac{K}{m}}=\sqrt{\frac{K_0}{m}\left(1+\frac{\varepsilon_k}{k_0}\right)}\cong\sqrt{\frac{K_0}{m}}\left(1+\frac{\varepsilon_k}{2k_0}\right) \tag{5}$$

Hence, if ε_k is normally distributed with zero mean and small deviation $\sigma_{\varepsilon k}$, ω is also normal with mean

$$\mu_\omega=\sqrt{\frac{K_0}{m}} \tag{6}$$

and deviation

$$\sigma_\omega=-\frac{1}{2\sqrt{mk_0}}\sigma_{\varepsilon k} \tag{7}$$

Amplitude A is from eqs. (2) and (3) together with eq. (4)

$$A=\sqrt{X_0^2+\frac{\dot{X}_0^2}{\omega^2}}=\sqrt{X_0^2+\frac{\dot{X}_0^2m}{K_0+\varepsilon_k}}\cong\sqrt{X_0^2+\frac{\dot{X}_0^2m}{K_0}}\left\{1-\frac{\dot{X}_0^2m}{2K_0(X_0^2K_0+\dot{X}_0^2m)}\varepsilon_k\right\} \tag{8}$$

Amplitude A is also normal with mean

$$\mu_A=\sqrt{X_0^2+\frac{\dot{X}_0^2 m}{K_0}} \tag{9}$$

and deviation

$$\sigma_A=\frac{\dot{X}_0^2 m}{2K_0^2\sqrt{X_0^2+\frac{\dot{X}_0^2 m}{K_0}}}\sigma_{\varepsilon k} \tag{10}$$

Phase angle B is given by

$$B=\tan^{-1}\frac{\dot{X}_0}{\omega X_0} \tag{11}$$

Since B is a function of frequency ω, we can obtain the probability distribution function of B, $P_B(B)$ by knowing the probability distribution function of ω, $P(\omega)$ as follows:

$$P_B(B)=\frac{P_\omega(\omega)}{\left|\frac{dB}{d\omega}\right|}=\frac{\dot{X}_0}{X_0\sin^2\beta}P_\omega\left(\frac{\dot{X}_0}{X_0}\frac{1}{\tan\beta}\right) \tag{12}$$

We note that probability distribution functions for ω, A and B are time independent and depend only on the deviation of ε_k.

III. DISPLACEMENT OF FREE VIBRATION

Let us assume the solution of eq. (1) in the expanded form of a parameter ε_k

$$X(t)=\sum_{i=1}^{n}X_i(t)\varepsilon_k^i=X_0(t)+X_1(t)\varepsilon_k+X_2(t)\varepsilon_k^2+\cdots \tag{13}$$

Substitution of eqs. (13) and (4) in eq. (1) leads to the following couple of differential equations after equating the coefficients of power of ε_k up to the linear terms of ε_k and by neglecting the higher ones.

$$m\ddot{X}_0+K_0X_0=0 \tag{14}$$

$$m\ddot{X}_1+X_0+K_0X_1=0 \tag{15}$$

Initial conditions are from eq. (13) together with the relationship $X(0)=X_0$ and $\dot{X}(0)=\dot{X}_0$.

$$X_0(0)=X_0, \qquad \dot{X}_0(0)=\dot{X}_0 \tag{16}$$

and

$$X_1(0)=\dot{X}_1(0)=0 \tag{17}$$

Solving eq. (14) with initial conditions (16), we have

$$X_0(t)=X_0\cos\sqrt{\frac{k_0}{m}}\,t+\frac{\dot{X}_0}{\sqrt{\frac{k_0}{m}}}\sin\sqrt{\frac{k_0}{m}}\,t \tag{18}$$

or by using eq. (6)

$$X_0(t)=X_0\cos\mu_\omega t+\frac{\dot{X}_0}{\mu_\omega}\sin\mu_\omega t \tag{19}$$

Eq. (15) now becomes with eq. (19)

$$m\ddot{X}_1+K_0X_1=-X_0\cos\mu_\omega t-\frac{\dot{X}_0}{\mu\omega}\sin\mu_\omega t \tag{20}$$

Solving this, we obtain

$$X_1(t)=\left(-\frac{\dot{X}_0}{2m\mu_\omega^3}-\frac{X_0}{2m\mu_\omega}t\right)\sin\mu_\omega t+\frac{\dot{X}_0}{2m\mu_\omega^2}t\cos\mu_\omega t \tag{21}$$

If we take terms up to linear ones in eq. (13), we finally obtain the displacement $x(t)$ in the form of

$$X(t)\cong X_0(t)+X_1(t)\varepsilon_k \tag{22}$$

where $X_0(t)$ and $X_1(t)$ are given by eqs. (19) and (21) respectively. Taking the expected value of $x(t)$ from eq. (22), assuming zero mean of ε_k, we have

$$E[x(t)]\cong X_0(t) \tag{23}$$

Where $E[\ \ ]$ represents the expected value of ensemble. Squaring eq. (22) and introducing the expected value, we have the mean square value of $x(t)$ as follows.

$$E[x^2(t)]\cong X_0^2(t)+X_1^2(t)E[\varepsilon_k^2] \tag{24}$$

From eqs. (23) and (24), we obtain the variance of $x(t)$, var $[x(t)]$.

$$\text{var}\,[x(t)]=E[x^2(t)]-E[x(t)]^2\cong X_1^2(t)E[\varepsilon_k^2] \tag{25}$$

If we use the notations $\sigma^2_{x(t)}$ and σ_ε^2 for variances of $x(t)$ and ε_k respectively, we may write eq. (25) as follows.

$$\sigma^2_{x(t)}\cong X_1^2(t)\sigma^2_{\varepsilon k} \tag{26}$$

Standard deviation $\sigma_{x(t)}$ is given by

$$\sigma_{x(t)}\cong|X_1(t)|\sigma_{\varepsilon k} \tag{27}$$

It turns out that the mean of displacement $E[x(t)]$ is identical to the solution of the system with spring coefficient K_0 which is the mean of K. On the other hand, standard deviation of the displacement is a trigonometric function and varies with time t (eqs. (21) and (27)). It is interesting to know that the mean and standard deviation of the displacement varies with time, while those of the amplitude are time independent... eqs. (9) and (10).

IV. FORCED VIBRATION

The forced vibration of the system given in Fig. 1 is investigated. Damping of viscousity is considered and is shown as a dashpot. The governing equation is

$$m\frac{d^2x}{dt^2}+C\frac{dx}{dt}+KX=f(t) \tag{28}$$

where C is a damping coefficient and $f(t)$ is a excitation force acting on the mass m.

Initially the system is at rest and hence, we have for initial conditions

$$X(0)=\dot{X}(0)=0 \tag{29}$$

The following assumptions are made on the spring and damping coefficients.

(1) K and C are random variables and can be put in the forms

$$K=K_0+\varepsilon_k \tag{30}$$

$$C=C_0+\varepsilon_c \tag{31}$$

where K_0 and C_0 are deterministic and ε_k and ε_c are small random variables with zero mean. Hence, the mean of K or C is K_0 or C_0 respectively.

(2) Correlation exists between K and C. In other words, ε_k and ε_c are statistically dependent with correlation coefficient ρ_{kc}. Introducing eqs. (30) and (31) into eq. (28), we have

$$\ddot{X}+\frac{C_0+\varepsilon_c}{m}\dot{X}+\frac{K_0+\varepsilon_k}{m}=\frac{f(t)}{m} \tag{32}$$

Assuming

$$X(t)=\sum_{i,j=1}^{\infty} X_{ij}(t)\varepsilon_k{}^i\varepsilon_c{}^j=X_{00}(t)+X_{10}(t)\varepsilon_k+X_{01}(t)\varepsilon_c+\cdots \tag{33}$$

we substitute eq. (33) in eq. (32). Equating the coefficients of the expanded series of $\varepsilon_k{}^i\varepsilon_c{}^j$ in the resulted equation up to linear terms, we will have the following successive differential equations.

$$\ddot{X}_{00}(t)+\frac{C_0}{m}\dot{X}_{00}(t)+\frac{K_0}{m}X_{00}(t)=\frac{1}{m}f(t) \tag{34}$$

$$\ddot{X}_{10}(t)+\frac{C_0}{m}\dot{X}_{10}(t)+\frac{K_0}{m}X_{10}(t)=-\frac{1}{m}X_{00}(t) \tag{35}$$

$$\ddot{X}_{01}(t)+\frac{C_0}{m}\dot{X}_{01}(t)+\frac{K_0}{m}X_{01}(t)=-\frac{1}{m}\dot{X}_{00}(t) \tag{36}$$

The corresponding initial conditions are similarily obtained from eqs. (29) and (33)

$$X_{00}(0)=X_{10}(0)=X_{01}(0)=0 \tag{37}$$

$$\dot{X}_{00}(0)=\dot{X}_{10}(0)=\dot{X}_{10}(0)=0 \tag{38}$$

If we put eqs. (34) to (36) in the matrix form, we have

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\ddot{X}_{00}\\ \ddot{X}_{10}\\ \ddot{X}_{01}\end{pmatrix}+\begin{pmatrix}\frac{C_0}{m}&0&0\\0&\frac{C_0}{m}&0\\ \frac{1}{m}&0&\frac{C_0}{m}\end{pmatrix}\begin{pmatrix}\dot{X}_{00}\\ \dot{X}_{10}\\ \dot{X}_{01}\end{pmatrix}+\begin{pmatrix}\frac{K_0}{m}&0&0\\ \frac{1}{m}&\frac{K_0}{m}&0\\0&0&\frac{K_0}{m}\end{pmatrix}\begin{pmatrix}X_{00}\\X_{10}\\X_{01}\end{pmatrix}=\begin{pmatrix}\frac{1}{m}f(t)\\0\\0\end{pmatrix} \tag{39}$$

and initial conditions are:

$$\begin{Bmatrix} X_{00}(0) \\ X_{10}(0) \\ X_{01}(0) \end{Bmatrix} = \begin{Bmatrix} \dot{X}_{00}(0) \\ \dot{X}_{10}(0) \\ \dot{X}_{01}(0) \end{Bmatrix} = 0 \tag{40}$$

Equation (39) is conceivable to be an equivalent differential equation to govern three degree of freedom systems for X_{00}, X_{10}, and X_{01}. It is easy to see that in general if m uncertain parameters are invelved in a system with n degree of freedom, the problem can be converted to $n(m+1)$ degree of freedom system by linearizing perturbed random parameter effect. Then, modal analysis is a well known powerful method for high degree of freedom systems.

Solving eq. (39) with eq. (40), we will have

$$X_{00}(t)=\int_0^t A(t-\tau_1)f(\tau_1)d\tau_1 \tag{41}$$

$$X_{10}(t)=-\int_0^t A(t-\tau_2)\int_0^t A(\tau_2-\tau_1)f(\tau_1)d\tau_1\, d\tau_2 \tag{42}$$

and

$$X_{01}(t)=-\int_0^t A(t-\tau_3)\int_0^{\tau_3}\frac{\partial}{\partial\tau_3}A(\tau_3-\tau_1)f(\tau_1)d\tau_1\, d\tau_2 \tag{43}$$

where we assumed the light damping $C_0<2\sqrt{K_0 m}$ and

$$A(t)=\beta e^{-\alpha t}\sin\lambda t \tag{44}$$

$$\alpha=\frac{C_0}{m} \tag{45}$$

$$\beta=\frac{1}{m\sqrt{\mu_\omega{}^2-\alpha^2}} \tag{46}$$

$$\lambda=\sqrt{\mu_\omega{}^2-\alpha^2} \tag{47}$$

Putting back eqs. (41) to (43) into eq. (33), we obtain the expression for $X(t)$.

$$X(t)\cong X_{00}(t)+X_{10}(t)\varepsilon_k+X_{01}(t)\varepsilon_c \tag{48}$$

where $X_{00}(t)$, $X_{10}(t)$ and $X_{01}(t)$ are given by eqs. (41), (42), and (43) respectively. If $f(t)$ is a deterministic function, the mean of $X(t)$ is given by

$$E[X(t)]\cong X_{00}(t) \tag{49}$$

The variance of $X(t)$ is given by

$$\sigma^2_{x(t)}=X^2_{10}\sigma^2_{\varepsilon k}+X^2_{01}\sigma^2_{\varepsilon c}+2X_{10}X_{01}\rho_{kc}\sigma_{\varepsilon k}\sigma_{\varepsilon c} \tag{50}$$

where $\sigma^2_{\varepsilon k}$ and $\sigma^2_{\varepsilon c}$ are variances of ε_k and ε_c respectively. For random function $f(t)$, we will have

$$E[X(t)]\cong E[X_{00}(t)] \tag{51}$$

and

$$\sigma_{x(t)}^2 \cong E[X_{10}^2]\sigma_{\varepsilon k}^2 + E[X_{01}^2]\sigma_{\varepsilon c}^2 + 2E[X_{10}X_{01}]\rho_{kc}\sigma_{\varepsilon k}\sigma_{\varepsilon c} \tag{52}$$

where we assumed that $f(t)$ and $\{\varepsilon_k, \varepsilon_c\}$ and independent. We note that in eq. (52) $E[X_{10}^2]$, $E[X_{01}^2]$ and $E[X_{10}X_{01}]$ are functions of the autocorrelation of the excitation force $f(t)$ and are given by

$$E[X_{10}^2(t)] = \int_0^t\int_0^t\int_0^{u_2}\int_0^{t_2} A(t-\tau_2)A(\tau_2-\tau_1)A(t-u_2)A(u_2-u_1)E[f(\tau_1)f(u_1)]d\tau_1\,du_1\,d\tau_2\,du_2 \tag{53}$$

$$E[X_{01}^2(t)] = \int_0^t\int_0^t\int_0^{u_2}\int_0^{\tau_2} A(t-\tau_2)\frac{\partial}{\partial\tau_2}A(\tau_2-\tau_1)A(t-u_2)\frac{\partial}{\partial U_2}A(u_2-u_1) \times E[f(\tau_1)f(u_1)]d\tau_1\,du_1\,d\tau_2\,du_2 \tag{54}$$

$$E[X_{10}(t)X_{01}(t)u] = \int_0^t\int_0^t\int_0^{u_2}\int_0^{\sigma_2} A(t-\tau_2)A(\tau_2-\tau_1)A(t-u_2)\frac{\partial}{\partial U_2}A(u_2-u_1) \times E[f(\tau_1)f(u_1)]d\tau_1\,du_1\,d\tau_2\,du_2 \tag{55}$$

V. NUMERICAL EXAMPLES

Free vibration with initial displacement and velocity, and forced vibration of a system with damping, which is subjected to an impulsive force at time zero are demonstrated. We supporse that the data on the mass, spring and damping are given as follows:

$$m = 0.13\ \text{lb}\cdot\text{sec}^2/\text{in} \qquad (W = 50\ \text{lb})$$

$$K_0 = 100\ \text{lb/in}$$

$$C_0 = 0.2\ \text{lb}\cdot\text{sec/in}$$

Standard deviations $\sigma_{\varepsilon k}$ and $\sigma_{\varepsilon c}$ are 1 lb/in and 0.002 lb·sec/in respectively. The mean of the frequency, μ_ω is calculated by $\mu_\omega = \sqrt{K_0/m}$ and is equal to 27.7 rad./sec and hence the mean cycle per second is 4.41 cycle/sec. For the free undamped vibration, we use for initial conditions

$$X_0 = 0.2\ \text{in}$$

$$\dot{X}_0 = 12\ \text{in/sec}$$

For the forced vibration, the dirac delta type of impulsive force with unity magnitude is imposed at $t=0$. The correlation coefficient ρ_{kc} between ε_k and ε_c is assumed to be 0.8.

The mean and standard deviation of the displacement of the single degree of freedom systems are calculated by computer with each time interval 0.01 seconds both in the free and forced vibrations. Figure 2 shows in the mean and standard deviation of the displacement of the free vibration for the initial part of time history. The mean displacement history is identical to the one obtained from the corresponding deterministic system with spring K_0 and damping C_0. However, this fact is valid only if approximation is limited to first order term.

Table 1 Displacement Time History

(sec) Time	Free Vibration		Impulsive Excitation	
	mean	deviation	mean	deviation
0	0.200	0	0.000	0
0.01	0.311	0.912×10^{-4}	0.075	0.981×10^{-5}
0.02	0.398	0.412×10^{-3}	0.144	0.769×10^{-4}
0.03	0.455	0.100×10^{-2}	0.200	0.250×10^{-3}
0.04	0.477	0.186×10^{-2}	0.241	0.560×10^{-3}
0.05	0.463	0.294×10^{-2}	0.263	0.101×10^{-2}
0.06	0.413	0.415×10^{-2}	0.264	0.158×10^{-2}
0.07	0.332	0.536×10^{-2}	0.246	0.222×10^{-2}
0.08	0.226	0.640×10^{-2}	0.209	0.286×10^{-2}
0.09	0.102	0.714×10^{-2}	0.157	0.342×10^{-2}
0.10	−0.029	0.740×10^{-2}	0.094	0.383×10^{-2}

Unit: inch

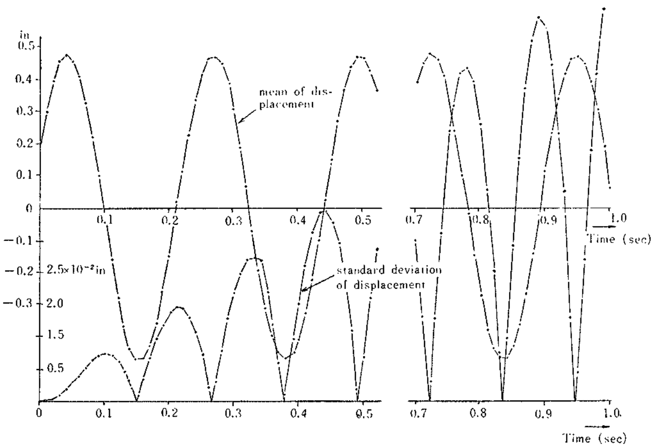

Fig. 2 Displacement Time History in Free Vibration.

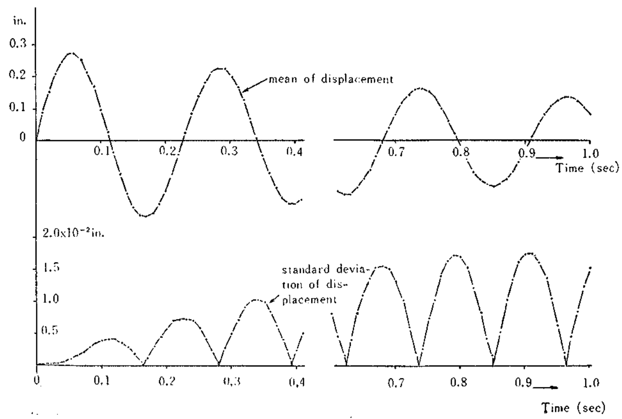

Fig. 3 Displacement Time History for Impulsive Excitation.

It is noteworthly that the deviation from the mean is a periodic function of time, whose maximum value in each period occurs at the same time with zero mean of the displacement. This maximum deviation increases with time lapse. The errors due to the linear approximation may increase with time, but are very small for the initial part of time. Figure 3 indicates that the mean displacement will gradually damp out, while the deviation from the mean will reach to the stationary periodicity, which depends on the damping C_0 and on the correlation between ε_k and ε_c.

VI. CONCLUSIONS

Series approximation by perturbation method for the single degree of freedom systems with probabilistic spring and damping are used in this analysis. Displacement time history is obtained in the form of mean and standard deviation both for free and forced vibrations. It is concluded that small imperfection in the system parameters will result in the appreciable deviation of the displacement. As is suggested in the main part of this paper, the author intends to extend this analysis to multi degree of freedom systems in imperfection.

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Appendix I—REFERENCES

1) Bharucha-Reid, A. T.: "Probabilistic Methods in Applied Mathematics", Vol. 1, Academic Press, 1968.
2) Boyce, W. E. and Goodwin, B. E.: "Random Transverse Vibrations of Elastic Beams", SIAM Journal 12, 1964, pp. 613-629.
3) Hoshiya, M.: "Dynamic and Eigenvalue Analysis of Stochastic Structural Systems", Stanford University, CE. Tech. Rt. 107.
4) Keller, J. B.: "Wave Propagation in Random Media", Proc. Symp. Appl. Math., New York, 1960, 13, pp. 227-246.
5) Samuels, J. C.: "Theory of Stochastic Linear Systems with Gaussian Parameter Variations", Jour. of Accoust. Soc. of Ame., Vol. 33, 1961.
6) Tang, W. H. C.: "Dynamic Response of Single Degree of Freedom Systems with Probabilistic Springs and Damping", Stanford Univ., 1968.
7) Timoshenko, S.: "Vibration Problems in Engineering", D. Van Nostrand, 1955.

Appendix II—*NOTATION*

The following symbols are used in this paper:

A = amplitude
$A(t)$ = $\beta e^{-\alpha t} \sin \lambda t$
B = phase angle
C = coefficient of viscous damping
C_0 = deterministic coefficient of viscous damping corresponding to C
$E[\ ]$ = expected value of ensemble
$f(t)$ = excitation force
K = spring coefficient
K_0 = deterministic spring coefficient corresponding to K
m = mass
$P_B(B)$ = probability distribution function of B
$P_\omega(\omega)$ = probability distribution function of ω
$\mathrm{Var}[X(t)]$ = variance of $x(t)$
$X, X(t)$ = displacement
$X_i(t)$ = coefficient of ε_k^i
$X_{ij}(t)$ = coefficient of $\varepsilon_k^i \varepsilon_c^j$
X_0 = initial displacement
$\dot{X}_0$ = initial velocity
α = $\dfrac{C_0}{m}$
β = $\dfrac{1}{m\sqrt{\mu_\omega^2 - \alpha^2}}$
λ = $\sqrt{\mu_\omega^2 - \alpha^2}$
ε_c = small random variable of C
ε_k = small random variable of K
μ_A = mean of A
μ_ω = mean of natural frequency ω
ρ_{kc} = correlation coefficient of ε_k and ε_c
σ_A = standard deviation of A
$\sigma_{X(t)}$ = standard deviation of $X(t)$
$\sigma_{\varepsilon k}$ = standard deviation of ε_k
σ_ω = standard deviation of ω
ω = natural frequency

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