THE STABLE LONGITUDINAL PROFILES OF RIVER BEDS

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SYNOPSIS

All present river bed equilibrium theories presuppose rectangular sections. This practice will bring satisfactory results in the case where the cross section of the river bed is nearly horizontal like that of the upper reaches of a sand-bank dam, but is not accurate in the case of a complicated cross section or of the cross section of a compound channel often seen at a repaired river. In this paper, therefore, the analysis is made of equilibrium longitudinal sections of river channel with consideration for the cross sections. As the cross sections in the dynamic equilibrium state still remain unknown, examples of the cross sections of compound channels and parabola have been taken for cross sections. Two kinds of solutions under varied equilibrium conditions have been worked out by using the Manning's roughness coefficient n and the equivalent roughness ks for the friction term. Also studies have been made on theoretically unsolved factors and application problems. By the method of this paper it will become possible to obtain solutions for general stable cross sections too in similar ways.

1. INTRODUCTION

In executing river improvement work such as building dams, beds and spur dikes or changing the river widths and channels, it is important to foresee the various changes expected to take place in the river beds after the improvements. There is now a remarkable tendency that while the volume of the gravel artificially gathered from most of the Japanese rivers to meet the demand from the construction industry is rapidly increasing, that of the gravel supplied to the rivers from their upper reaches is decreasing because of the progress of dam construction and anti-erosion work at the catchment areas. This factor causes the lowering of river beds especially those in the vicinity of cities and affects the normal functions of the rivers by making it difficult to obtain service water and damaging river conservation facilities and other structures. In order that the functions of rivers may be properly maintained under this condition, it is necessary to predetermine stabilized river shapes for the future by foreseeing the possible changes of the river beds.

For equilibrium river bed theory, the one known as the "regime theory" suggested by Kennedy in 1895 was the first. Later this theory was developed by Lacey in 1929, but it was not based on sufficient study about its physical grounds and lacked universality for application. In Japan, Mononobe and Aki formed a theory based on the notion of critical tractive force and studied practical methods in 1951 out of necessity to draw out coherent river improvement programs. In recent years, keeping pace with the progress of researches on sediment problems, there have been many improved studies based on the theory of sediment transport, such as by Sato (1957), Kikkawa and Sone (1953), Sugio (1957)[1], Yano and Daido (1958), Tsuchiya (1958 and 1962)[2], and Masuda and Komura (1960)[3,4]. Some of these recent theories of sediment deal with the static equilibrium state and others with the dynamic equilibrium state. The former is the case where the river bed materials constituting the runway do not move, that is, the tractive force is below the critical and the latter is the case where the quantity of sediment does not change longitudinally and is in a state of equilibrium.

These studies are made on the basis of fundamental equations of non-uniform flow and carry out analysis by the use of the sediment transport function or the critical tractive force function, but all of them presuppose rectangular sections. Where the cross section of the river bed is nearly horizontal as in the case of the upper reaches of a sand-bank dam or where the section is single as in the case of an artificial channel, these theories are fully applicable and can lead to satisfactory results. However, in many rivers the cross sections are complicated and it is not advisable in respect of accuracy to discuss the flow velocity and depth in terms of mean values, and therefore in such cases a consideration of the characteristics of the cross sections is highly desirable.

As an attempt for considering such cross sections

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of rivers, a study is made in this paper about the case where general cross sections (including double sections and quadruple cross sections) are considered as divided into several sections and also about the case of parabola, both on supposition that the water level and energy gradient do not approximately change to the cross sectional direction. This study has been made under various equilibrium conditions in the light of the actual situations of rivers. For the resistance rule, the solution has been reached by the Manning's Formula and by the formula based on the equivalent roughness k_s.

Concerning stable cross sections, there are a few examples of studies on static equilibrium, but dynamic equilibrium remains still unsolved as it is not so simple because of the problem of secondary flow and others. Furthermore, there has not yet been known any rule on the distribution of the average diameter of particles of the river bed materials in the cross sections.

This also makes it difficult to discuss the profiles of cross sections of river beds. There are many rivers having considerablly large width. As the stable cross sections in the static equilibrium theory give the gradients of side walls, they do not have any large influence as a whole in rivers having large width compared with the depth. Since most of the Japanese rivers are provided with sea walls, the problem of static cross sections is not important and therefore shall not be discussed in this paper.

When such equilibrium theories are put to actual applications for rivers, there are still a lot of questions to be answered. Rivers do not always have constant discharge. The characteristics of river bed materials, such as rules on the changes of longitudinal profiles like average particle diameter and particle size distribution are not certain. Also it is extremely difficult to foreknow the future changes of river bed materials and the production quantities of sediment. Such complicated unknown factors are involved in the equilibrium theories. The problem of whether or not there exist equilibrium river beds still have much to be discussed. Whether rivers approach equilibrium dynamically or statically is also one of the most important problems in the application of the theories. Here studies are attempted on factors which have large influence on the application of the theories and the values of calculation of equilibrium longitudinal profiles for studying the equilibrium beds of actual rivers.

2. BASIC EQUATIONS AND CALCULATION MODEL

To analyze the problem theoletically, the following seven equations are considered: (1) The equation of motion, (2) the equation of continuity for flow, (3) the sediment transport formula, (4) the equation of continuity for sediment transport, (5) the equation of critical tractive force, (6) the geometrical equation, and (7) the law of resistance to flow.

For minute discussions of river equilibrium shapes, there are many other equations required. They are (1) the equation for giving the volume and particle size distribution of the sediment flowing to the area under consideration in an equilibrium state, (2) the equation for giving the longitudinal distribution of the representative particle diameters of river bed materials in equilibrium, (3) the function of the channel width in equilibrium, (4) the function for giving the stable cross section, (5) the function of plane characteristics such as meandering, and (6) the equations related to the volume of water such as flow rate, hydrograph and flow rate frequency. These are very important functions, but at present they are either unknown or artificially controlled, and therefore shall be substituted by given values in the present analysis.

Consequently, as far as this paper is concerned, the theoretical solution of the problem of river equilibrium longitudinal profile may be done by the simultaneous use of necessary equations taken out according to the dynamic or static equilibrium case.

First, there are equations employing Manning's formula, Chezy's formula, and equivalent roughness k_s as the law of resistance to flow. All these equations are not much different from each other, and any of them may be used without causing fundamental difference. For instance, when the average flow velocity at each point is expressed by u, Manning's coefficient of roughness n is given by

$$n=\frac{h^{1/6}}{\sqrt{g}}\frac{1}{u/u_*} \cdots\cdots (1)$$

And the relation between the coefficient of roughness and the equivalent roughness k_s is expressed by the following semi-theoretical equation for flow velocity distribution through u/u_*:

$$\frac{u}{u_*}=8.5-\frac{1}{\kappa}+\frac{2.3}{\kappa}\log_{10}\frac{h}{k_s} \cdots\cdots (2)$$

where

h=water depth, g=gravity acceleration, $u_*=\sqrt{ghI}$ is the friction velocity, I=energy gradient, κ= Kárman's constant. κ is the function of the concentration of sediment, and in the case of pure water, $\kappa=0.4$. The equivalent roughness, though not without problems, may be expressed generally by

$$\frac{k_s}{d_m}=K\left[\frac{u_*^2}{\{(\rho_s/\rho)-1\}gd_m}\right]^m \quad \cdots\cdots\cdots\cdots\cdots(3)$$

where
K and m are constants, d_m=the average particle diameter of river bed materials, ρ_s=the density of river bed material, ρ=the density of water. The values of the constants K and m, as long as the tractive force is small and sand ripples are not yet produced, are approximately $K=0.5\sim4$ and $m=0$, but when sand is produced to the extent of causing ripples or dunes, they[6] are approximately $K=10^3$, $m=2$. When the river bed is in transition and forming antidunes, the value of k_s/d_m is small, but is quantitatively not well known. In the stage where sand waves are produced, it is not adequate to make k_s dimensionless with the average particle diameter of sand and gravel d_m, and therefore the expression (3) involves some questions to be solved in the future. In this respect the equation (3) should be improved on. Since the equation (2) is not convenient, it is sometimes substituted by the approximate equation in indicative form as follows:

$$\frac{u}{u_*}=E\left(\frac{h}{k_s}\right)^q \quad \cdots\cdots\cdots\cdots\cdots(4)$$

where
E and q=constants. According to Manning's equation, $q=1/6$, $E=k_s^q/(n\sqrt{g})$. E may be regarded as approximately constant when h/k_s is within the range of $1\sim10^3$; and according to Einstein and Barbarossa, $E=7.76$; when Strikler's formula ($n=dm^{1/6}/21.2$) is used, $E=6.74$; when the writer's formula ($n=0.04\ k_s^{1/6}$) is used, $E=8.0$, and the value is always approximately 8.

In the case where the influence of the cross section of channel is taken into account, the equation for the motion of steady non-uniform flow, if the axis x is set in the direction of the flow, is, in the stationary state, as follows:

$$\frac{dH}{dx}+\frac{Q^2}{2g}\frac{d}{dx}\left(\frac{D}{A^2}\right)+I=0 \quad \cdots\cdots\cdots\cdots\cdots(5)$$

where
H=water level, Q=discharge, A=sectional area, D=the corrected coefficient of the flow velocity distribution by the influence of cross section. The equation (5) is based on the presumption that (1) the flow may be regarded approximately as one dimensional, (2) therefore the water level and energy gradient are constant in the cross sectional direction, and (3) the corrected coefficient α of the flow velocity distribution in the perpendicular direction is constant throughout the entire sectional area.

Under these conditions, D of the equation (5) may be calculated from

$$D=\frac{\alpha}{A}\int_{-B/2}^{B/2}\left(\frac{u}{U}\right)^3 hdy \quad \cdots\cdots\cdots\cdots\cdots(6)$$

where
B=channel width, $U=Q/A$ is the average flow velocity of the cross section, y=the distance from the center of channel (See Fig. 1). In this case, there are Manning's equation and an equation based on the equivalent roughness k_s available for obtaining the flow velocity distribution. The former is

$$D=\alpha\frac{A^2\int_{-B/2}^{B/2}\frac{h^3}{n^3}dy}{\left[\int_{-B/2}^{B/2}\frac{h^{5/3}}{n}dy\right]^3} \quad \cdots\cdots\cdots\cdots\cdots(7)$$

The equation (7) was given by Ida[7]. The latter, with u of the equation (4) substituted into the equation (6) and employing the equation (3) with regard to k_s, is

$$D=\alpha\frac{A^2\int_{-B/2}^{B/2}d_m^{3(m-1)q}h^{\frac{5+6q-6mq}{2}}dy}{\left[\int_{-B/2}^{B/2}d_m^{(m-1)q}h^{\frac{3+2q-2mq}{2}}dy\right]^3} \quad \cdots\cdots\cdots\cdots\cdots(8)$$

where
E, K, q, ρ_s, and m are constants. As to each divided channel in the cross section, when the coefficient of roughness is known the equation (7) may be used, while it is not known the equation (8) may be used.

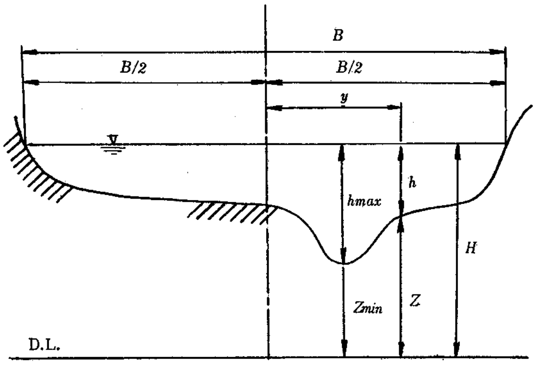

Fig. 1 Definition Sketch for Cross Section of River Channel

Of the energy gradient I of the equation (5), no other consideration than the loss of energy by friction shall be made. This may be obtained from condition of continuity $Q=\int uhdy$ by Manning's equation or the equation based on the equivalent roughness k_s, considering it as quasi-uniform flow. When the variation of the channel width is large it is necessary to adopt the condition of non-uniform flow or to separately consider the energy loss by sudden widening or narrowing. When the channel width is gradually changing, I may be given by the Manning's equation as

$$I=I_0\left\{\frac{\int_{-B/2}^{B/2}\frac{h^{5/3}}{n}dy}{\int_{-B_0/2}^{B_0/2}\frac{h_0^{5/3}}{n_0}dy}\right\}^{-2} \quad\cdots\cdots(9)$$

and by the equations (4) and (3) as

$$I=I_0\left\{\frac{\int_{-B/2}^{B/2}d_m^{(m-1)q}h^{\frac{3+2q-2mq}{2}}dy}{\int_{-B_0/2}^{B_0/2}d_{m0}^{(m-1)q}h_0^{\frac{3+2q-2mq}{2}}dy}\right\}^{\frac{-2}{1-2mq}} \quad\cdots\cdots(10)$$

where

$1-2mq\neq0$. The suffix o as used here shows the value at the reference point. In case $1-2mq=0$, the flow velocity u becomes unrelated to the gradient I and the solution cannot be made.

The gradient at reference point I_0 is given by Manning's equation

$$I_0=Q^2\left[\int_{-B_0/2}^{B_0/2}\frac{h_0^{5/3}}{n_0}dy\right]^{-2} \quad\cdots\cdots(11)$$

and by the equation based on k_s

$$I_0=\left[\frac{Q}{g^{1/2}EK^{-q}\{(\rho_s/\rho)-1\}^{mq}}\right]^{\frac{2}{1-2mq}}\times\left[\int_{-B_0/2}^{B_0/2}d_{m0}^{(m-1)q}h_0^{\frac{3+2q-2mq}{2}}dy\right]^{\frac{-2}{1-2mq}} \quad\cdots\cdots(12)$$

In this case, a point near uniform flow where the variation of the river bed is small should be selected for the reference point, and the calculation should be made by the water depth h_0 complying with the volume of sediment flowing to the area under consideration. h_0 will be calculated later.

In the equation (5), the condition of continuity $Q=AU$ is included.

Although there are various formulae for the quantity of sediment transport so far announced, the following may be generally adopted:

$$\frac{q_T}{u_*d_m}=a_s\left[\frac{u_*^2-u_{*c}^2}{\{(\rho_s/\rho)-1\}gd_m}\right]^P \quad\cdots\cdots(13)$$

where

$q_T=q_S+q_B$, which is the total quantity of sediment transport per unit of width, q_S and q_B are the suspended load and bed load respectively per unit of width, u_{*c}=the critical friction velocity corresponding to the critical tractive force, A_S and P=constants. Where the tractive force is sufficiently large and $u_*\gg u_{*c}$, the equation (13) becomes approximately

$$q_T=a_s[\{(\rho_s/\rho)-1\}g]^{-P}d_m^{1-P}u_*^{1+2P} \quad\cdots\cdots(14)$$

In the equation (14), according to Brown's formula for giving the total of suspended load and bed load, $P=2$, $a_s=10$; and according to Sato, Kikkawa and Ashida's formula, $P=1$, $a_s=0.623\,(40\,n)^{-3.5}$, $n\leq 0.025$ (m-sec unit), $a_s=0.623$ and $n\geq0.025$ (if $F(\tau_c/\tau_0)=1$).

In the dynamic equilibrium state, the condition of continuity of sediment transport, if the total quantity of sediment transport for each section is expressed by Q_T, is

$$Q_T=\int_{-B/2}^{B/2}q_Tdy=\text{const} \quad\cdots\cdots(15)$$

Needless to say, in the static equilibrium state, $Q_T=0$.

As to the formula for critical tractive force, there are many experimental ones but few theoretical ones. There are only three of them, namely White's, Kurihara's and Iwagaki's. The most commonly adopted Iwagaki's formula may be generally expressed as

$$u_{*c}^2=\frac{\tau_c}{\rho}=a_c\{(\rho_s/\rho)-1\}g\cdot d_m \quad\cdots\cdots(16)$$

where

τ_c=the critical tractive force, a_c=the constant determined by the value of u_*d_m/ν (See Fig. 2), ν=the coefficient of dynamic viscosity.

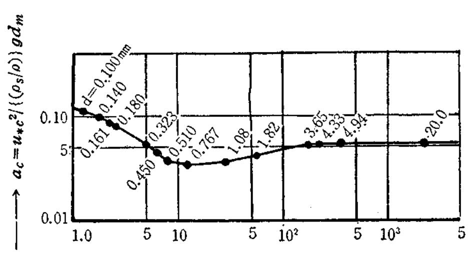

Fig. 2 $a_c-u_cd_m/\nu$

The equation for geometric conditions consist of the relations between the water level, river bed height, and river bed gradient i as follows:

$$\left.\begin{aligned}&H=z+h,\quad z_{\min}=H-h_{\max}\\&i=-\frac{dz}{dx}\end{aligned}\right\} \quad\cdots\cdots(17)$$

The equilibrium river bed longitudinal profile may be obtained by simultaneous use of the above fundamental equations. In this case, there are two expressions, one by substituting the equations (7) and (9) into the equation (5), the other by substituting the equations (8) and (10) into the equation (5). Also there are two ways of solution. One is by differentiating the first and second terms of the equation (5), and the other by integrating them. In either case, the relation with the water depth determined by the conditions of equilibrium is of course used. While the former requires the previous substitution of the relation with water depth before conducting the differentiation, the latter permits the previous integration before substituting the relation with the water depth for obtaining the equilibrium river bed. If integrated, the equation (5) becomes as follows:

$$z=z_0+h_0-h-\frac{Q^2}{2g}\left(\frac{D}{A^2}-\frac{D_0}{A_0^2}\right)-\int_{x_0}^{x} Idx \quad \cdots\cdots(18)$$

where

suffix o is the value where the reference point $x=x_0$. As the third term in the righthand member of the equation (18) shows the case where x is set in the direction of lower reaches from the upper reaches, it is necessary to reverse the sign to plus in the case of coordinates setting x from the lower to upper reaches.

The remaining work necessary for obtaining the equilibrium river bed is to determine the water depth h from the conditions of each equilibrium. This point shall be stated later in this paper under proper head.

3. RIVER BED EQUILIBRIUM LONGITUDINAL SECTIONS FOR CROSS SECTIONS OF DEFORMED CHANNELS

The cross sections of rivers are sometimes highly complicated. The dynamic cross sections are still unsolved, and shall not be discussed here. The present study has been made on the presumption that the law of cross sections may be given. The equation (5) is expressing cross sections continuously. If approximated to step-shaped sections of the discrete type, this does not lose the universality in principle. Most of improved rivers have double sectional channels. In that case, the calculation is possible by dividing the sections in two or three. Quadruple sectional channels may be treated in the similar way.

Suppose the cross section of a river is divided into n' rectangular sections and n or k_s in each section is a constant and also the shear strength on the boundary of each section may be approximately disregarded. Then the following relations are possible:

$$B=B_1+B_2+\cdots\cdots=\sum_{i=1}^{n'} B_i \quad \cdots\cdots(19)$$

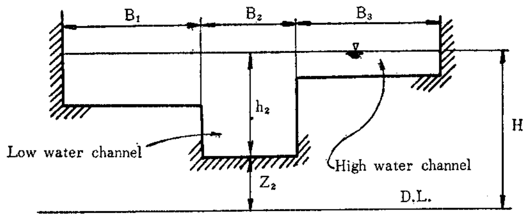

Fig. 3 Definition Sketch for Cross Section of Compound Channel

$$A=B_1h_1+B_2h_2+\cdots\cdots=\sum_{i=1}^{n'} B_ih_i \quad \cdots\cdots(20)$$

$$Q_T=Q_{T_1}+Q_{T_2}+\cdots\cdots=\sum_{i=1}^{n'} Q_{Ti} \quad \cdots\cdots(21)$$

where

B_i is a given quantity. As the equilibrium conditions for channels having such section, the following shall be handled:

(1) Q_{T_1}=constant, Q_{T_2}=const.,…… Q_{Ti}=const., ……

(2) $Q_T=\sum_{i=1}^{n'} Q_{Ti}$=const., ($Q_{Ti}$ is disregarded)

(3) Low water channel and in dynamic equilibrium, critical tractive force at high water channel (static equilibrium)

(4) Low water channel and in dynamic equilibrium, no change at high water channel (stationary, given quantity)

(5) Critical tractive force at low water channel, no change at high water channel (stationary, given quantity).

(1) is a case where the cross sectional division of the channel does not change longitudinally and each divided channel is in independent dynamic equilibrium. (2) is a case where the division of the section does change longitudinally but the flow of sand is continuous and the entire quantity of the sediment transport of the cross section may be considered as constant, for instance, the case where a double section continuously changes into a single section. (3), (4) and (5) are cases of double sectional channels. In many rivers the beds of high water channels are considerablly stable. In case a bed of a high water channel has, for example, facilities built on it or plants growing on it, it is a case mentioned in (4) or (5).

Next, the water depth h at equilibrium state is obtained with regard to each of these cases. Depending upon the conditions of continuity for the sediment transport and the conditions of the river bed in each of the cases (1)-(5), the results will be various. As mentioned above, two expressions are considered for the law of resistance, so one or the other will be used for each case. The analysis of each case shall be made later in detail.

In order to obtain the equilibrium river bed height z, it is necessary first to determine the reference point. A place where the change in river bed hardly occurs and not away from the uniform flow should be selected for it. Next, I_0 and h_{0i} shall be obtained from the quantity of sediment transport Q_T at the time of equilibrium of the area under consideration and the coefficient of roughness.

Next, n_i, d_{mi} and B_i shall be estimated. Next, after the water height h at the time of equilibrium for each case is obtained by the method to be stated later in detail, D and I shall be calculated by the equations (7) and (8) respectively in case Manning's formula is used or by the equations (8) and (10) respectively in case the equation based on the equivalent roughness k_s is used. Next, the river bed height z is obtained by substituting these values into the equation (18).

(1) **Where Q_{Ti}=constant**

In the case where, in each of the rectangular channels divided into n', the sediment transit is longitudinally continuous and the movement of sediment transit between each channel can be disregarded,

$$Q_{Ti}=a_s\cdot\{(\rho_s/\rho)-1\}^{-p}\cdot g^{1/2}\times I^{\frac{1+2p}{2}}\cdot d_{mi}^{1-p}\cdot h_i^{\frac{1+2p}{2}}\cdot B_i \quad\cdots\cdots(22)$$

according to the equations (14) and (15). As the quantity of sediment transit at each divided channel is independently constant,

$$\left(\frac{h_i}{h_{0i}}\right)=\left(\frac{d_{m\cdot i}}{d_{m\cdot 0\cdot i}}\right)^{-\frac{2(1-p)}{1+2p}}\left(\frac{B_i}{B_{0i}}\right)^{-\frac{2}{1+2p}}\left(\frac{I}{I_0}\right)^{-1}\quad\cdots\cdots(23)$$

from the dondition Q_{Ti}=constant as given in the equation (22) when a_s, ρ_s and P do not change longitudinally, where o is the value at each reference point. If the energy gradient I is given by the equation (9) based on Manning's formula and by the equation (10) based on the equivalent roughness, the equation (23) becomes

$$\left(\frac{h_i}{h_{0i}}\right)=\left(\frac{d_{mi}}{d_{m0i}}\right)^{-\frac{2(1-p)}{1+2p}}\left(\frac{B_i}{B_{0i}}\right)^{-\frac{2}{1+2p}}\times\left\{\frac{\sum_{i=1}^{n'} n_i^{-1}h_i^{5/3}B_i}{\sum_{i=1}^{n'} n_{0i}^{-1}h_{0i}^{5/3}B_{0i}}\right\}^2\quad\cdots\cdots(24)$$

and

$$\left(\frac{h_i}{h_{0i}}\right)=\left(\frac{d_{mi}}{d_{m0i}}\right)^{-\frac{2(1-p)}{1+2p}}\left(\frac{B_i}{B_{0i}}\right)^{-\frac{2}{1+2p}}\times\left\{\frac{\sum_{i=1}^{n'} d_{mi}^{(m-1)q}h_i^{\frac{3+2q-2mq}{2}}B_i}{\sum_{i=1}^{n'} d_{m0i}^{(m-1)q}h_{0i}^{\frac{3+2q-2mq}{2}}B_{0i}}\right\}^{\frac{2}{1-2mq}}\quad\cdots\cdots(25)$$

respectively. The number of h_i is n' and that of the equations is also n', so in case the n' is not so large, h_i can be obtained by the method of trial and error.

The double sectional channel of a river consists of a low water channel and a high water channel, and the latter is usually divided into two parts though they are often nearly equal in height. The equilibrium longitudinal profile of such a channel may be treated approximately as $n'=2$. If n' is set as 2 in the equations (24) and (25), respectively,

$$h_1=M_1^{-3/7}\left[\frac{B_1}{n_1}+\frac{B_2}{n_2}\left(\frac{M_2}{M_1}\right)^{5/3}\right]^{-6/7}\quad\cdots\cdots(26)$$

and

$$h_2=M_2^{-3/7}\left[\frac{B_1}{n_1}\left(\frac{M_1}{M_2}\right)^{5/3}+\frac{B_2}{n_2}\right]^{-6/7}\quad\cdots\cdots(27)$$

where

$$M_1=\frac{h_{01}\left(\frac{d_{m1}}{d_{m01}}\right)^{-\frac{2(1-p)}{1+2p}}\left(\frac{B_1}{B_{01}}\right)^{-\frac{2}{1+2p}}}{(n_{01}^{-1}h_{01}^{5/3}B_{01}+n_{02}^{-1}h_{02}^{5/3}B_{02})^2}\quad\cdots\cdots(28)$$

$$M_2=\frac{h_{02}\left(\frac{d_{m2}}{d_{m02}}\right)^{-\frac{2(1-p)}{1+2p}}\left(\frac{B_2}{B_{02}}\right)^{-\frac{2}{1+2p}}}{(n_{01}^{-1}h_0^{5/3}B_{01}+n_{02}^{-1}h_{02}^{5/3}B_{02})^2}\quad\cdots\cdots(29)$$

and

$$h_1=N_1^{-\frac{1-2mq}{2(1+q)}}\left[d_{m1}^{(m-1)q}B_1+d_{m2}^{(m-1)q}B_2\left(\frac{N_2}{N_1}\right)^{\frac{3+2q-2mq}{2}}\right]^{-\frac{1}{1+q}}\quad\cdots(30)$$

$$h_2=N_2^{-\frac{1-2mq}{2(1+q)}}\left[d_{m1}^{(m-1)q}B_1\left(\frac{N_1}{N_2}\right)^{\frac{3+2q-2mq}{2}}+d_{m2}^{(m-1)q}B_2\right]^{-\frac{1}{1+q}}\quad\cdots\cdots(31)$$

where

$$N_1=\frac{h_{01}\left(\frac{d_{m1}}{d_{m01}}\right)^{-\frac{2(1-p)}{1+2p}}}{(d_{m01}^{(m-1)q}h_{01}^{\frac{3+2q-2mq}{2}}B_{01}}\times\frac{\left(\frac{B_1}{B_{01}}\right)^{-\frac{2}{1+2p}}}{+d_{m02}^{(m-1)q}h_{02}^{\frac{3+2q-2mq}{2}}B_{02})^{\frac{2}{1-2mq}}}\quad\cdots\cdots(32)$$

$$N_2=\frac{h_{02}\left(\frac{d_{m2}}{d_{m02}}\right)^{-\frac{2(1-p)}{1+2p}}}{(d_{m01}^{(m-1)q}h_{01}^{\frac{3+2q-2mq}{2}}B_{01}}\times\frac{\left(\frac{B_2}{B_{02}}\right)^{-\frac{2}{1+2p}}}{+d_{m02}^{(m-1)q}h_{02}^{\frac{3+2q-2mq}{2}}B_{02})^{\frac{2}{1-2mq}}}\quad\cdots\cdots(33)$$

Needless to say, where $n'=1$, the above equations may apply also in the case of a rectangular section.

(2) **Where $Q_T=\sum_{i=1}^{n'}Q_{Ti}$=constant ($Q_{Ti}$ is disregarded)**

This is the case where, irrespective of the quantity of sediment transit of each divided channel, the quantity of sediment transit of the entire channel is longitudinally constant and stable. Therefore, according to the equation (14),

$$\left\{\frac{\sum_{i=1} d_{mi}^{1-p}h_i^{\frac{1+2p}{2}}B_i}{\sum_{j=1} d_{m0j}^{1-p}h_{0j}^{\frac{1+2p}{2}}B_{0j}}\right\}^{\frac{2}{1+2p}}\left(\frac{I}{I_0}\right)=1\quad\cdots\cdots(34)$$

Likewise, if the equations (9) and (10) are adopted with regard to (I/I_0), the equation (34) becomes,

$$\frac{\sum_{i=1} d_{mi}^{1-p} h_i^{\frac{1+2p}{2}} B_i}{\sum_{j=1} d_{m0j}^{1-p} h_{0j}^{\frac{1+2p}{2}} B_{0j}} = \left(\frac{\sum_{i=1} n_i^{-1} h_i^{5/3} B_i}{\sum_{j=1} n_{0j}^{-1} h_{0j}^{5/3} B_{0j}} \right)^{1+2p} \quad \cdots\cdots(35)$$

and

$$\frac{\sum_{i=1} d_{mi}^{1-p} h_i^{\frac{1+2p}{2}} B_i}{\sum_{j=1} d_{m0j}^{1-p} h_i^{\frac{1+2p}{2}} B_i} = \left(\frac{\sum_{i=1} d_{mi}^{(m-1)q} h_i^{\frac{3+2q-2mq}{2}} B_i}{\sum_{j=1} d_{m0j}^{(m-1)q} h_{0j}^{\frac{3+2q-2mq}{2}} B_{0j}} \right)^{\frac{1+2p}{1-2mq}} \quad \cdots\cdots(36)$$

respectively. These remain unsolved without knowing their relation with h_i. Conversely, if the conditions of the equation (35) or (36) is satisfied, each water depth h_i in the cross section may be determined in any way desired, but it should be determined in the light of the actuality. These may apply to cases where a single section (or rectangular profile) is changed into a double section or the reverse.

(3) Low water channel and in dynamic equilibrium, static equilibrium at high water channel

To the quantities of low water channel and high water channel, L and H are attached respectively. The water depth h_L of low water channel is expressed in the equation (23) with the attached i changed to L. From the equation (16) for expressing the critical tractive force the water depth h_H of the high water channel is given by

$$\left(\frac{h_H}{h_{0H}}\right) = \left(\frac{a_{cH}}{a_{c0H}}\right)\left(\frac{d_{mH}}{d_{m0H}}\right)\left(\frac{I}{I_0}\right)^{-1} \quad \cdots\cdots(37)$$

To obtain the energy gradient I, in case Manning's formula or the equivalent roughness k_s is used, either the equation (9) or (10) may be used. Both h_L and h_H may be obtained by simultaneously employing each of the equations. In case Manning's formula of the equation (9) is adopted, the equations for obtaining I are equal to the equations (26) and (27) with the attached 1 and 2 replaced by L and H respectively. The equation for M_L is equal to the equation (28) with the attached 1 and 2 replaced by L and H respectively. But that for M_H given below has no reference to the equation (29) :

$$M_H = \frac{h_{0H}\left(\frac{a_{cH}}{a_{c0H}}\right)\left(\frac{d_{mH}}{d_{m0H}}\right)}{(n_{0L}^{-1} h_{0L}^{5/3} B_{0L} + n_{0H}^{-1} h_{0H}^{5/3} B_{0H})^2} \quad \cdots\cdots(38)$$

In case the equivalent roughness of the equation (10) is adopted, the equations for obtaining I are equal to the equations (30) and (31) with the attached 1 and 2 replaced by L and H. The equation for N_L is equal to the equation (32) with the attached 1 and 2 replaced by L and H respectively. But that for N_H given below has no reference to the equation (33) :

$$N_H = \frac{h_{0H}\left(\frac{a_{cH}}{a_{c0H}}\right)}{(d_{m0L}^{(m-1)q} h_{0L}^{\frac{3+2q-2mq}{2}} B_{0L}} \times \frac{\left(\frac{d_{mH}}{d_{m0H}}\right)}{+ d_{m0H}^{(m-1)q} h_{0H}^{\frac{3+2q-2mq}{2}} B_{0H})^{\frac{2}{1-2mq}}} \quad \cdots\cdots(39)$$

(4) Low water channel and in dynamic equilibrium, stationary at high water channel

At rivers of double sectional channels it is not unusual that their beds have plants growing, fields cultivated or facilities and structures built on them. In such a case, the change of river bed at high water level is small, so it is desirable to analyze the height of the bed of the high water channel as given value and fixed. When the height of the bed of the channel is expressed by Z_H, the water depth of the high water channel $h_H = H - Z_H$. Therefore, when the Manning's formula or the equivalent roughness k_s is used, from the equation (24) or (25) respectively

$$h_L = M_L\left[\frac{B_L}{n_L} h_L^{5/3} + \frac{B_H}{n_H}(H-Z_H)^{5/3}\right]^2, \quad (H-Z_H>0) \quad \cdots\cdots(40)$$

or

$$h_L = N_L[d_{mL}^{(m-1)q} h_L^{\frac{3+2q-2mq}{2}} B_L + d_{mH}^{(m-1)q}(H-Z_H)^{\frac{3+2q-2mq}{2}} B_H]^{\frac{2}{1-2mq}}, \quad (H-Z_H>0) \quad \cdots\cdots(41)$$

where M_L and N_L are equal to the equations (28) and (32) with the attached 1 and 2 replaced by L and H respectively. To obtain the water depth h_L from the equation (40) or (41) (Be careful in taking proper value, because there may be three solutions at least), h_L is first trial calculated on the presumptive H, and then by using the value thus obtained H is obtained from the equation (5) or (18) respectively. This trial calculation shall be repeated until at last the value obtained and the presumptive H agree with each other.

(5) Low water channel and in static equilibrium, stationary at high water channel

The state of critical tractive force at low water channel is expressed by the equation equal to the equation (37) with the attached H replaced by L. In case Manning's formula or the equivalent roughness k_s is adopted for I, from the equation (24) or (25) respectively

$$h_L=\frac{h_{0L}\left(\frac{a_{cL}}{a_{c0L}}\right)\left(\frac{d_{mL}}{d_{m0L}}\right)}{[n_{0L}^{-1}h_{0L}^{5/3}B_{0L}+n_{0H}^{-1}h_{0H}^{5/3}B_{0H}]^2}\times\left[\frac{B_L}{n_L}h_L^{5/3}+\frac{B_H}{n_H}(H-Z_H)^{5/3}\right]^2. \quad (H-Z_H>0) \quad \cdots\cdots(42)$$

$$h_L=\frac{h_{0L}\left(\frac{a_{cL}}{a_{c0L}}\right)}{[d_{m0L}^{(m-1)q}h_{0L}^{\frac{3+2q-2mq}{2}}B_{0L}}\times\frac{\left(\frac{d_{mL}}{d_{m0L}}\right)}{+d_{m0H}^{(m-1)q}h_{0H}^{\frac{3+2q-2mq}{2}}B_{0H}]^{\frac{2}{1-2mq}}}\times[d_{mL}^{(m-1)q}h_L^{\frac{3+2q-2mq}{2}}B_L+d_{mH}^{(m-1)q}\times(H-Z_H)^{\frac{3+2q-2mq}{2}}B_H]^{\frac{2}{1-2mq}}, \quad (H-Z_H>0) \quad \cdots\cdots(43)$$

To obtain the water depth h_L from the equation (42) or (43) (Be careful in taking proper value, because there may be three solutions at least), h_L is trial calculated on the presumptive H, and by using the value thus obtained H is obtained from the equation (5) or (18). This trial calculation shall be repeated until the value obtained and the presumptive H agree with each other.

4. DYNAMIC EQUILIBRIUM AT PARABOLA CHANNELS

There are rivers whose cross sections can be regarded approximately as parabola and are stable. In most of the cases, the average particle diameter of their river bed materials are not distributed extremely in the crosss sectional direction. This paper shall treat of a case where the water surface width B is gradually changing and on the supposition that the coefficient of roughness n, equivalent roughness k_s, and average particle diameter are constant on the cross section. The cross section of the parabola is given by

$$h=h_{\max}\left(1-\frac{4}{B^2}y^2\right) \quad \cdots\cdots(44)$$

In this case, the cross sectional area of the channel A is

$$A=2\int_0^{B/2}h_{\max}\left(1-\frac{4}{B^2}y^2\right)dy=\frac{2}{3}Bh_{\max} \quad (45)$$

which may be determined only by the channel width B and the maximum water depth $h_{\max}$. As in the case of the rectangular profile, therefore, this analysis is rather easy.

The corrected coefficient D for the flow velocity distribution by the influence of the cross section is expressed by the equation (7) or (8) when Manning's formula or the equivalent roughness k_s is used respectively. In the case of parabola channels, they are

$$D=\alpha\cdot\frac{A^2\left(2h_{\max}^3\frac{8}{35}B\right)}{(2h_{\max}^{5/3}\cdot BX_1)^3}=\frac{1}{4}\cdot\frac{4}{9}\cdot\frac{8}{35}\cdot\frac{\alpha}{X_1^3}=1.089\,\alpha \quad \cdots\cdots(46)$$

and

$$D=\frac{\alpha}{9}\frac{X_3}{X_2^3}=(\text{constant}) \quad \cdots\cdots(47)$$

respectively, where

$$\left.\begin{aligned}X_i&=\frac{1}{4}\int_0^1\frac{\eta^{t_i}}{\sqrt{1-\eta}}d\eta=\frac{1}{4}\frac{\Gamma(t_i+1)\,\Gamma(0.5)}{\Gamma(t_i+1.5)}\\ t_1&=\frac{5}{3},\ t_2=\frac{3+2q-2mq}{2},\ t_3=\frac{5+6q-6mq}{2}\end{aligned}\right\} \quad \cdots\cdots(48)$$

Especially, $x_1 \doteqdot \frac{1.145}{4} \doteqdot \frac{2}{7}$. In the equation (46), $D=1.089\,\alpha$, which is a value not much larger than 1. The calculated value of the equation (47) is nearly the same. Therefore, the second term of the equation (5) is

$$\frac{DQ^2}{2g}\frac{d}{dx}\left(\frac{1}{A^2}\right)=-\frac{9D}{4g}\frac{Q^2}{B^3h_{\max}^3}\times\left(h_{\max}\frac{dB}{dx}+B\frac{dh_{\max}}{dx}\right) \quad (49)$$

As the first term is $H=z_{\min}+h_{\max}$,

$$\frac{dH}{dx}=\frac{dz_{\min}}{dx}+\frac{dh_{\max}}{dx}=-i_{\min}+\frac{dh_{\max}}{dx} \quad \cdots(50)$$

where $i_{\min}$ is the river bed gradient at the deepest place. The gradient I is expressed by the equation (9) or (10) when Manning's formula or the equivalent roughness k_s is used respectively. In the case of parabola channels, they are

$$I=I_0\left(\frac{n}{n_0}\right)^2\left(\frac{h_{\max}}{h_{0\max}}\right)^{-10/3}\left(\frac{B}{B_0}\right)^{-2} \quad \cdots\cdots(51)$$

and

$$I=I_0\left(\frac{d_m}{d_{m0}}\right)^{\frac{-2(m-1)q}{1-2mq}}\left(\frac{h_{\max}}{h_{0\max}}\right)^{\frac{-3-2q+2mq}{1-2mq}}\times\left(\frac{B}{B_0}\right)^{\frac{-2}{1-2mq}} \quad \cdots\cdots(52)$$

respectively. Next, the condition of continuity of sediment transport is used for determining $h_{\max}$. Based on the equation (14) and from the condition $Q_T=2\int_0^{B/2}q_Tdy=\text{constant}$

$$\left(\frac{h_{\max}}{h_{0\max}}\right)=\left(\frac{d_m}{d_{m0}}\right)^{\frac{-2(1-p)}{1+2p}}\left(\frac{B}{B_0}\right)^{-\frac{2}{1+2p}}\left(\frac{I}{I_0}\right)^{-1} \quad \cdots\cdots(53)$$

By using the eqution (51) or (52) for I, the equation (53) is

$$\left(\frac{h_{\max}}{h_{0\max}}\right)=\left(\frac{d_m}{d_{m0}}\right)^{\frac{6(1-p)}{7(1+2p)}}\left(\frac{B}{B_0}\right)^{\frac{-12p}{7(1+2p)}}\left(\frac{n}{n_0}\right)^{6/7} \quad \cdots\cdots(54)$$

or

$$\left(\frac{h_{\max}}{h_{0\max}}\right)=\left(\frac{d_m}{d_{m0}}\right)^{\frac{1-p+q+2pq-3mq}{(1+2p)(1+q)}}\times\left(\frac{B}{B_0}\right)^{\frac{-2mq-2p}{(1+2p)(1+q)}} \quad \cdots\cdots(55)$$

respectively. If differentiated by x, the equations (54) and (55) become

$$\frac{dh_{\max}}{dx}=h_{0\max}\left(\frac{d_m}{d_{m0}}\right)^{a}\left(\frac{n}{n_0}\right)^{6/7}\left(\frac{B}{B_0}\right)^{b}\times\left[\frac{a}{d_m}\frac{dd_m}{dx}+\frac{6}{7n}\frac{dn}{dx}+\frac{b}{B}\frac{dB}{dx}\right] \quad \cdots\cdots(56)$$

and

$$\frac{dh_{\max}}{dx}=h_{0\max}\left(\frac{d_m}{d_{m0}}\right)^{a'}\left(\frac{B}{B_0}\right)^{b'}\times\left[\frac{a'}{d_m}\frac{dd_m}{dx}+\frac{b'}{B}\frac{dB}{dx}\right] \quad \cdots\cdots(57)$$

respectively. The I in the equations (51) and (52), according to the equations (54) and (55), is

$$I=I_0\left(\frac{n}{n_0}\right)^{-6/7}\left(\frac{d_m}{d_{m0}}\right)^{c}\left(\frac{B}{B_0}\right)^{d} \quad \cdots\cdots(58)$$

and

$$I=I_0\left(\frac{d_m}{d_{m0}}\right)^{c'}\left(\frac{B}{B_0}\right)^{d'} \quad \cdots\cdots(59)$$

respectively, where

$$\left(\begin{array}{ll} a=\dfrac{6(1-p)}{7(1+2p)}, & b=\dfrac{-12p}{7(1+2p)} \\ c=\dfrac{-20(1-p)}{7(1+2p)}, & d=\dfrac{-14+12p}{7(1+2p)} \\ a'=\dfrac{1-p+q+2pq-3mq}{(1+2p)(1+q)}, & b'=\dfrac{-2mq-2p}{(1+2p)(1+q)} \\ c'=\dfrac{3mq-3q+3p-3}{(1+2p)(1+q)}, & d'=\dfrac{2mq-2q+2p-2}{(1+2p)(1+q)} \end{array}\right) \quad \cdots\cdots(60)$$

Therefore, the solution is obtained by substituting into the fundamental equation for non-uniform flow (5) the equations (54), (56), and (58) in the case of Manning's formula and (55), (57), and (59) in the case of the equivalent roughness k_s. However, the equations (49) and (50) are used in this case. The solution in the former case is

$$i_{\min}=I_0\left(\frac{n}{n_0}\right)^{-6/7}\left(\frac{d_m}{d_{m0}}\right)^{c}\left(\frac{B}{B_0}\right)^{d}+h_{0\max}\left(\frac{d_m}{d_{m0}}\right)^{a}\times\left(\frac{n}{n_0}\right)^{6/7}\left(\frac{B}{B_0}\right)^{b}\left[\frac{a}{d_m}\frac{dd_m}{dx}+\frac{6}{7n}\frac{dn}{dx}+\frac{b}{B}\frac{dB}{dx}\right]-\frac{9D}{4}\frac{Q^2}{gB^2}\frac{1}{h_{0\max}{}^2}\left(\frac{d_m}{d_{m0}}\right)^{-2a}\times\left(\frac{n}{n_0}\right)^{-12/7}\left(\frac{B}{B_0}\right)^{-2b}\left[\frac{a}{d_m}\frac{dd_m}{dx}+\frac{6}{7n}\times\frac{dn}{dx}+\frac{b+1}{B}\frac{dB}{dx}\right] \quad \cdots\cdots(61)$$

and the solution in the latter case is

$$i_{\min}=I_0\left(\frac{d_m}{d_{m0}}\right)^{c'}\left(\frac{B}{B_0}\right)^{d'}+h_{0\max}\left(\frac{d_m}{d_{m0}}\right)^{a'}\left(\frac{B}{B_0}\right)^{b}\times\left[\frac{a'}{d_m}\frac{dd_m}{dx}+\frac{b'}{B}\frac{dB}{dx}\right]-\frac{9D}{4}\frac{Q^2}{gB^2}\frac{1}{h_0{}^2{}_{\max}}\times\left(\frac{d_m}{d_{m0}}\right)^{-2a'}\left(\frac{B}{B_0}\right)^{-2b'}\left[\frac{a'}{d_m}\frac{d_m}{dx}+\frac{b'+1}{B}\frac{dB}{dx}\right] \quad \cdots\cdots(62)$$

The D in the equation (46) is used for the equation (61), and the D in the equation (47) for the equation (62). The solutions by the equations (61) and (62) are quite resembling those in the case of rectangular profiles in form. If the $9D/4$ in the third term of the righthand member is taken for instance, it agrees with the equation for rectangular profiles with $h_{\max}$ as the water depth.

When calculating the height of a river bed successively by dividing it into minute sections, the differences ΔZ_η in the height at the divided sections ΔX_η in the case of the equations (61) and (62) are

$$\Delta Z_\eta=-I_0\left(\frac{n}{n_0}\right)_m^{-6/7}\left(\frac{d_m}{d_{m0}}\right)_m^{c}\left(\frac{B}{B_0}\right)_m^{d}\Delta X_\eta-h_{0\max}\times\left(\frac{d_m}{d_{m0}}\right)_m^{a}\left(\frac{n}{n_0}\right)_m^{6/7}\left(\frac{B}{B_0}\right)_m^{b}\left[a\left(\frac{d_{m0}}{d_m}\right)_m\left(\frac{\Delta d_m}{d_{m0}}\right)+\frac{6}{7}\left(\frac{n_0}{n}\right)_m\left(\frac{\Delta n}{n_0}\right)+b\left(\frac{B_0}{B}\right)_m\left(\frac{\Delta B}{B_0}\right)\right]+\frac{9D}{4}\frac{Q^2}{gB^2}\frac{1}{h_{0\max}{}^2}\left(\frac{d_m}{d_{m0}}\right)_m^{-2a}\left(\frac{n}{n_0}\right)_m^{-12/7}\times\left(\frac{B}{B_0}\right)_m^{-2b}\left[a\left(\frac{d_{m0}}{d_m}\right)_m\left(\frac{\Delta d_m}{d_{m0}}\right)+\frac{6}{7}\left(\frac{n_0}{n}\right)_m\times\frac{\Delta n}{n_0}+(b+1)\left(\frac{B_0}{B}\right)_m\left(\frac{\Delta B}{B_0}\right)\right] \quad \cdots\cdots(63)$$

and

$$\Delta Z_\eta=-I_0\left(\frac{d_m}{d_{m0}}\right)_m^{c'}\left(\frac{B}{B_0}\right)_m^{d'}\Delta X_\eta-h_{0\max}\left(\frac{d_m}{d_{m0}}\right)_m^{a'}\times\left(\frac{B}{B_0}\right)_m^{b'}\left[a'\left(\frac{d_{m0}}{d_m}\right)_m\left(\frac{\Delta d_m}{d_{m0}}\right)+b'\left(\frac{B_0}{B}\right)\times\left(\frac{\Delta B}{B_0}\right)\right]+\frac{9D}{4}\frac{Q^2}{gB^2}\frac{1}{h_{0\max}{}^2}\left(\frac{d_m}{d_{m0}}\right)_m^{-2a'}\times\left(\frac{B}{B_0}\right)_m^{-2b'}\left[a'\left(\frac{d_{m0}}{d_m}\right)_m\left(\frac{\Delta d_m}{d_{m0}}\right)+(b'+1)\left(\frac{B_0}{B}\right)\left(\frac{\Delta B}{B_0}\right)\right] \quad \cdots\cdots(64)$$

respectively. The height of the river bed Z_η at point η divided into n' is

$$Z_\eta=Z_0+\sum_{\eta=1}^{u}\Delta Z_\eta \quad \cdots\cdots(65)$$

where the attached m=the mean value at the section ΔX_η, and Δd_m, Δn, and ΔB are differences respectively of d_m, n, and B at both ends at intervals of ΔX_η.

5. PROBLEMS CONCERNING APPLICATION OF THE THEORY

The theory of equilibrium at present is a very imperfect thing. As referred to in Chapter 2 above, the coefficient of roughness at the time of equilibrium, quantity of sediment transport supplied, the longitudinal distribution of the average particle diameter of the river bed materials, and river width B_i should be estimated by some other methods and treated as given values. These are determined based on the findings of age changing and by referring to the present various values of rivers. The same can be said of the selection of reference points. It is hardly possible to consider the three dimensional characters of rivers. In addition, there are considerable number of artificial factors involved. Under the present circumstance there is still much room for discussions on the existance itself of equilibrium river beds.

However, in studying the subject on data collected in a matter of several decades at the longest, the presumption of the existance of equilibrium river beds will be permitted as a practical means. In that case, there are two kinds of conceptions, that is, dynamic equilibrium and static equilibrium. Then which is better suited to the actual state of rivers? Before this question can be answered it is necessary to study as to how river beds change their shapes. Generally, rivers having river bed materials of mixed particle diameters. If such rivers are flooded, sediment will move but when the water subsides gravel having grain diameters larger than those corresponding to the critical tractive force stops moving and accumulates at each point. As a result, the longitudinal distribution of the grain diameters of gravel forms the distribution of many heaps having peaks at different positions classified by diameter. If these do not change even in floods, the theory of static equilibrium comes to have large significance. However, in order to suppose such a model it is also necessary to consider the tendency of the river bed materials toward pulverization. In other words, if the change is seen in the order of very long years, either the river beds rise or the river extention becomes longer. If sediment of particle diameters which should stop controlled by the critical tractive force is transported to the lower reaches as the diameters become smaller, and the quantity is in balance with that of the sediment of transport supplied from the upper reaches, the river bed is in the state of equilibrium. The details of the phenomenon of sediment to become finer and finer is almost unknown, but this conception is compatible to that of dynamic equilibrium. Statically, a river bed changes to become near the dynamic equilibrium river bed meeting the flow rate, and produces heaps by the critical tractive force determined by the river bed shape obtained at flood seasons.

The change of the river bed at a river approaching equilibrium is considerably slow. At such a river, the present values have very important meanings if the estimation of the equilibrium is made in the order of several decades. Although the equilibrium river bed may be considered to have a double-structural characteristics combining static and dynamic equilibrium as mentioned above, it is not impossible to judge which of the two is playing a superior part in the present values. For instance, at areas on the upper reaches of sand-bank dam, in the case where the distribution of particle diameters are closely related to the change of river width, or at a short space of area where the particle diameters suddenly change, it is difficult to readily apply the theory of dynamic equilibrium. On the other hand, at the lower reaches of a river where there is no large influence of the quantity of sediment of transport from tributaries, it is easy to maintain a dynamically stable condition because the influence of the quantity of sediment supplied may be eliminated by the adjusting action of the riverway. In this respect, which theory is superior to apply for accounting the actual states is roughly classified in Table 1.

Table 1 Conditions for Raising the Applicability of Each Theory

Theory	Place	River Width	Particle Dia.	Hydrograph
Static Equil.	1) Upper reaches of sand-bank dam 2) Lower reaches of sand-bank dam 3) Upper reaches (where sediment seldom collapses) 4) Near deltas at river-mouths	1) Large change in river width	1) Where large 2) Where it changes with river width 3) Where it suddenly changes	1) Water subsides quickly
Dynamic Equil.	1) Middle and lower reaches (small influence by tributaries)	1) Where river width scarcely changes	1) Where the longitudinal change of particle diameters is small 2) Particle diameters are small 3) Particle diameters are nearly uniform	1) Where water subsides slowly

When each formula is applied, the flow rate and representative particle diameters of river bed materials are predominant for static equilibrium and the quantity of sediment supplied and the coefficient of roughness are predominant for dynamic equilibrium. These items have big influence on the calculated results. They must be investigated carefully.

As for the rivers in Japan, embankments and rivetments are well performed in the middle and down reaches of the rivers, and the so-called natural rivers are remained only in small rivers or in the upper reaches of the rivers. In such rivers that the training works have been made, or planned, there are many cases that the shapes of cross-sections are compound or double-compound. The equilibrium theories of rectangular sections can also be apllied to the compound channels when discharges are small, but when the water levels go up and some discharges flow in high water channels, the exact results can not be expected. Moreover in compound channels, in general, the bed materials and the coefficients of roughness are different between low and high water channels, and therefore the equilibrium theory of compound channels is desired to be applied.

Now, the stable conditions of rivers may not be explained by such a simple theory as static or dynamic equilibrium theory, but the five kinds of stable conditions concerning low and high water channels are analysed in this paper. There are many complicated rivers, for example, the rivers in which the low and the high water channels are difficult to be distinguished, and the meandering rivers in which the high water channels are not continuous longitudinally.

As for the formulas of sediment transport, the functions are not necessarily coincident between the low and high water channels. It may be necessary for bed load and suspended load to be considered independently. But the problems above mentioned, are to be analysed for special purposes. And it had better that the simple examples of rivers are investigated first, based on the theories of five kinds of conditions.

But there are still other problems about the applicability of the theories to the simple rivers. The first one is a doubt about the existence of equilibrium beds, and the second one is that the rules how to give the conditions are not yet determined.

These problems are common to any other equilibrium theories of river beds. The estimation of the distribution of grain size of the river bed material in future, and the estimation of the supplied bed material load in equilibrium state are very difficult, and the influence of the picking of river sand and gravel is not always negligible. But there is not any other method than the one based on the data of present rivers, in order to estimate even the equilibrium state of the rivers. The profiles of river beds in near future can be well estimated by the present data of rivers, because the changes of river beds, in general, are not so rapid, though the local changes are not so slow.

Before the calculations of equilibrium river beds are made, the followings must be investigated

a. As for the determination of the hight of equilibrium river bed of a reference point, enough data of long period must be used

b. The values of I_0 and h_0 at the reference point act the most important roles in the investigation. These two values should be determined by the supplied bed matireal load and coefficient of roughness (or equivalent roughness), in case of dynamic equilibrium state, and by the mean diameter of bed material and coefficient of roughness (or k_s), in case of static equilibrium state.

The gradient at the reference point I_0 is given by Manning's equation or by the equation based on equivalent roughness, as equation (11) or equation (12) respectively, in which h_0 is not yet unknown.

The water depth at the reference point h_{0i} (h_{0L}, h_{0H}) is given by friction velocity $u_{*0i}=\sqrt{gh_{0i}I_0}$ and Manning's equation, using the condition $I_0=$const. i.e., $h_{0H}=(u_{*0H}/u_{*0L})^2 h_{0L}$, as

$$h_{0L}=\left\{\frac{\sqrt{g}\,Q}{\dfrac{u_{*0L}\,B_{0L}}{n_{0L}}+\left(\dfrac{u_{*0H}}{u_{*0L}}\right)^{7/3}\dfrac{u_{*0H}\,B_{0H}}{n_{0H}}}\right\}^{6/7} \qquad \cdots\cdots(66)$$

$$h_{0H}=\left(\frac{u_{*0H}}{u_{*0L}}\right)^2 \times h_{0L}=\left\{\frac{\sqrt{g}\,Q}{\left(\dfrac{u_{*0L}}{u_{*0H}}\right)^{7/3}\dfrac{u_{*0L}\,B_{0L}}{n_{0L}}+\dfrac{u_{*0H}\,B_{0H}}{n_{0H}}}\right\}^{6/7} \qquad \cdots\cdots(67)$$

and by the equation base on k_s, as

$$h_{0L}=\left\{\frac{Q\Big/\Big\{EK^{-q}}{d_{m0L}{}^{(m-1)q}\,u_{*0L}{}^{1-2mq}\,B_{0L}+d_{m0H}{}^{(m-1)q}}\times\frac{\left(\dfrac{\rho_s}{\rho}-1\right)^{mq} g^{mq}\Big\}}{u_{*0H}{}^{1-2mq}\left(\dfrac{u_{*0H}}{u_{*0L}}\right)^{2(1+q)}B_{0H}}\right\}^{\frac{1}{Hq}} \qquad \cdots\cdots(68)$$

$$h_{0H}=\left[\frac{Q\Big/\Big\{EK^{-q}}{d_{m0L}{}^{(m-1)q}\,u_{*0L}{}^{1-2mq}\left(\dfrac{u_{*0L}}{u_{*0H}}\right)^{2(1+q)}B_{0L}}\times\frac{\left(\dfrac{\rho_s}{\rho}-1\right)^{mq}g^{mq}\Big\}}{+d_{m0H}{}^{(m-1)q}u_{*0H}{}^{1-2mq}\,B_{0H}}\right]^{\frac{1}{Hq}} \quad \cdots\cdots(69)$$

respectively, where u_{*0L} and u_{*0H} are given by equation (14) or equation (16), acording to the kind of equilibrium condition of chapter 3. In case of condition (1), u_{*0L} and u_{*0H} are given by,

$$u_{*0i}=\left[\frac{Q_{Ti}}{a_s\{(\rho_s/\rho)-1\}^{-p}g^{-p}d_{m0i}{}^{1-p}B_{0i}}\right]^{\frac{1}{1+2p}} \quad \cdots\cdots(70)$$

and in case of condition (3), u_{*0L} is given by equation (70) and u_{*0H} is given by equation (16). In case of condition (2), (4) and (5), h_{0H} should be considered to be a given value. In these cases, h_{0L} can be calculated by the following equations, for example by try and error method, instead of equation (66) and equation (68) respectively,

$$h_{0L}{}^{7/6}=\frac{\sqrt{g}\,n_{0L}Q}{B_{0L}\,u_{*0L}}-\frac{n_{0L}\,B_{0H}\,h_{0H}{}^{5/3}}{n_{0H}\,B_{0L}\,h_{0L}{}^{1/2}} \quad \cdots\cdots(71)$$

$$h_{0L}{}^{1+q}=\frac{Q/[EK^{-q}\{(\rho_s/\rho)-1\}^{mq}g^{mq}}{d_{m0L}{}^{(m-1)q}B_{0L}\,u_{*0L}{}^{1-2mq}}-\left(\frac{d_{m0H}}{d_{m0L}}\right)^{(m-1)q}\left(\frac{B_{0H}}{B_{0L}}\right)\frac{h_{0H}{}^{(3-2mq+2q)/2}}{h_{0L}{}^{(1-2mq)/2}} \quad \cdots\cdots(72)$$

where u_{*0L} is given by equation (70) in case of condition (4), and by equation (16) in case of condition (5). In case of condition (2), u_{*0L} must be determined for the total bed load $Q_T=Q_{TL}+Q_{TH}$ to be adjusted correctly, by repeat calculations.

c. The distributions of the grain size of river bed materials must be obtained both in low and high water channels. Bed materials must be taken from the beds which are considered to be the beds when the dominant discharge was flowing.

d. The coefficient of roughness (or k_s) is so important that it must be determined based on the enough data both in low and high water channels.

e. As for the supplied bed material load, it is considered to be better to adopt the mean value of calculated values at many points of regular intervals along the river including the upper part of the section under consideration. But if there is a stable section, the bed material load of the section can be adopted.

The sediment functions are not perfectly reliable, so that the investigations by measurements of bed load and suspended load are desired.

f. The problem as to the dominant discharge is very difficult. Calculation should be done assuming some kinds of discharges which had better to be investiged as to the return periods, the charactors of flood waves, the catchment areas, and the charactors of bed materials of the rivers.

g. The supplied bed material load is restrained by a weir or a dam. This is an another boundary condition instead of supplied bed material load.

h. The problems are found very complicated when there are a large range of the change of water level at the end of the section under consideration, such as the estuaries or the rivers that flow into big rivers. It is also necessary to pay attention to estuary sand bars and salt wedges.

6. CONCLUSION

With regard to the deformed sectional channels and parbola channels, equations for giving equilibrium longitudinal profiles are obtained. They presuppose that (1) the flow may be regarded approximately as one dimensional, (2) therefore, the water level and energy gradient are constant in the cross sectional direction, and (3) the corrected coefficient for the distribution of flow velocity in the parpendicular direction is constant throughout the entire section.

The formulae for equilibrium longitudinal profiles are obtained for both cases where Manning's formula is used and where the formula based on the equivalent roughness k_s is used. It will be well to adopt the former when Manning's coefficient of roughness is known and the latter when it is not known.

As to the equilibrium shape of deformed sectional channels, study has been made on a total of 5 conditions including those of continuity of the quantity of sediment of transport and those of river beds. Namely, (1) the quantity of the sediment of transport at each divided channel is constant, (2) the quantity of the sediment of transport at each divided channel is not considered but that for the entire section is constant, (3) dynamic equilibrium at low water channel, static equilibrium at high water channels, (4) dynamic equilibrium at low water channels, stable at high water channels, and (5) static equilibrium at low water channels, stable at high water channels.

The above equilibrium river bed theory lacks important functions such as the quantity of the sediment of supplied, distribution of particle size, and longitudinal change of the distribution of particle size in equilibrium state, and therefore have many defects in their application. Some study is made about the applica ility of the theory.

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Appendix—Notation

The following symbols have been adopted for use in this paper

A : Cross-sectional area
a, a' : defined by Eq. (60)
a_c : Constant
a_s : Constant
B : Surface width of channel
ΔB : $=(B)x=x_\eta+\Delta x_\eta-(B)x=x_\eta$
b, b' : Defined by Eq. (60)
c, c' : Defined by Eq. (60)
D : Constant defined by Eq. (6)
d, d' : Defined by Eq. (60)
d_m : Mean diameter of bed materials
Δd_m : $=(d_m)x=x_\eta+\Delta x_\eta-(d_m)x=x_\eta$
E : Constant
g : Acceleration of gravity
H : Water surface elevation measured from an arbitrary datum
h : Depth
$h_{\max}$: Maximum depth
I : Slope of the energy gradient
i : Slope of the bed gradient
$i_{\min}$: Slope of the bed gradient of minimum elevation
K : Constant
k_s : Equivalent roughness
M_1 : Defined by Eq. (28)
M_2 : Defined by Eq. (29)
M_L : Given by the equation equal to the equation (28) with the attached 1 and 2 replaced by L and H respectively.
M_H : Defined by Eq. (38)
m : Constant
N_1 : Defined by Eq. (32)
N_2 : Defined by Eq. (33)
N_L : Given by the equation equal to the equation (32) with the attached 1 and 2 replaced by L and H respectively.
N_H : Defined by Eq. (39)
n : Manning resistance coefficient
n' : Number
Δn : $=(n)x=x_\eta+\Delta x_\eta-(n)x=x_\eta$
p : Constant
Q : Discharge
Q_T : Total sediment transport
q : Constant
$q_T=q_S+q_B$: Total sediment transport per unit width
q_S : Suspended load per unit width
q_B : Bed load per unit width
t_i : Defined by Eq. (48)
U : Mean velocity
u : Local verocity
u_* : Local friction velocity
u_{*c} : Local critical friction velocity
X_i : Defined by Eq. (48)
x : Coordinate along the channel
x_η : Distance at $x=\eta$
Δx_η : Distance between sections
y : Distance from the center line of the cross-section
Z : Bed elevation measured from an arbitrary datum
$Z_{\min}$: Minimum bed elevation
ΔZ_η : $=(Z)x=x_\eta+\Delta x_\eta-(Z)x=x_\eta$
α : Constant in velocity distribution relationship
η : Distance
κ : Kárman's constant
ν : Kinematic viscosity
ρ : Density of water
ρ_s : Density of bed materials
τ_c : Critical shear force
Attached 0 : Shows the value at the reference point
Attached L, H : Show the values of low water channel and high water channel respectively
Attached m : Shows the average of the distance between 2 sections
Attached i, j : Show the values of divided channels

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