MECHANICS OF GRANULAR MATERIAL COMPOSED OF PARTICLES OF VARIOUS SIZES

By *Takeo Mogami**

1. Introduction

In the author's paper on the mechanics of granular material in 1965[1], the concept of the volume element was introduced on which his theory of mechanics was based.

As shown in the figure 1, a surface S which passes through the centres of particles surrounding a void was considered. In the space enclosed by this surface S, there exists a void of volume V_v and a half of volumes of particles which belong to this particle group, this volume being V_S, hence the void ratio of this volume element surrounded by the surface S is

$$e=\frac{V_v}{V_S} \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Total volume of the material can be covered by such volume elements without overlapping. Among these volume elements, there might exist some which have equal void ratio. And the maximum and minimum void ratios can be found among them. they are designated as e_M, $e_{M'}$.

We consider an adequate interval (e_1, e_n) which includes e_M and $e_{M'}$ and we divide this interval (e_1, e_n) into n-1 sub-intervals as shown in the figure 2, that is,

$$e_1, e_2, \cdots\cdots, e_n$$

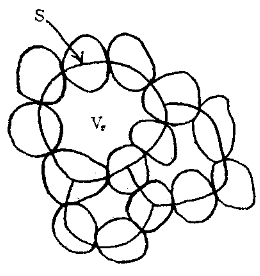

Fig. 1

Fig. 2

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There might exist volume elements of which void ratios are in the sub-intervals.

$$e_i-\frac{1}{2(n-1)}(e_n-e_1),\ e_i+\frac{1}{2(n-1)}(e_n-e_1),$$
$$i=1, 2, \cdots\cdots, n$$

The total volume of particles in such volume elements is denoted by N_i, the volume is expressed by the number of particles, that is, the number N_i is given as the ratio of the total volume of particles to the volume of one particle, the volume of each particle being assumed equal.

And the total volume of such volume elements is denoted by V_i.

Then we get a matrix which shows the structure of the material,

$$\begin{pmatrix} e_1 & e_2 & \cdots\cdots & e_n \\ N_1 & N_2 & \cdots\cdots & N_n \\ V_1 & V_2 & \cdots\cdots & V_n \end{pmatrix} \cdots\cdots\cdots\cdots (2)$$

Above explained is theauthor's basic concept of volume element, but he found out that this concept contains some ambiguity and it should be improved.

The points of improvement are as follows. The void ratio of the volume element depends on the choise of the surface S, because any definite rule for selection of the surface is not given. When the surface S for one volume element is chosen to include more volume of particles in this volume element, the volume of particles in the next element would be less, hence it seems that no trouble would take place when the total volume is considered, however, this would not be verified so easily.

The author's concept of volume element was thought out basing on the two dimensional model of the assembrage as shown in the figure 1, hence the above definition of the volume element was quite easily understandable. However, for three-dimensional model, the conditions are not so simple, for example, a void would not be surrounded by such group of particles as shown in the figure 1. When a surface S which passes through centres of particles surrounding the void, is drawn, this surface would go not only through particles but also through void. By such reasons, an improvement is proposed in this paper, this would not be a complete one, but it must be a better one.

2. Cubic volume element

When we construct three sets of plane in the material which are perpendicular with each other, the whole body is divided into many volumes of equal size, a^3.

The volume of each cell, volume element, is small compared with the total volume, but it should be taken as large as such that each contains sufficient number of particle, because we shall have to apply the Stirling's formula to particles in this volume. A set of void ratios $e_1, e_2, \cdots\cdots, e_m$ can be obtained when the void ratio of each volume element is calculated. Among them some are equal and others are different. An interval (E_1, E_n) which contains $e_1, e_2, \cdots\cdots, e_m$ is divided into sub-intervals $E_1, E_2, \cdots\cdots, E_n$ of width δ. In the interval $\left(E_i-\frac{\delta}{2}, E_i+\frac{\delta}{2}\right)$ some of e_s, $s=1,2,\cdots\cdots,m$, are contained.

If the side length or the direction of the above difined cubic volume element is varied continuously, the set $e_1, e_2, \cdots\cdots, e_m$ would vary continuously in accordance with the change of the volume element. By such procedure, an infinite sets of $e_1, e_2, \cdots\cdots, e_m$ would be obtained. If we consider these infinite number of such set, we would be able to find out the void ratio

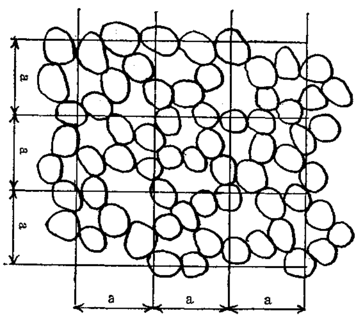

Fig. 3

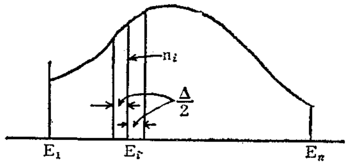

Fig. 4

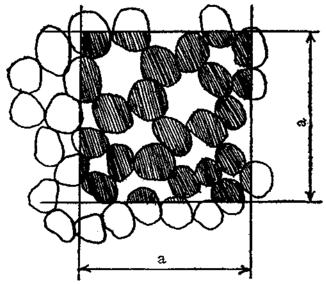

Fig. 5

included in the interval $\left(E_i-\frac{\delta}{2}, E_i+\frac{\delta}{2}\right)$, however small the value δ would be. And the number n_i of void ratios which are contained in the interval $\left(E_i-\frac{\delta}{2}, E_i+\frac{\delta}{2}\right)$ would be continuous with respect to E_i. To get better understanding, the following explanation will be added.

We divide the interval (E_1, E_n) into n sub-intervals and the width of the sub-intervals is denoted by $\varDelta$. The number of e-s; n_i which are contained in the sub-interval

$$\left(E_i-\frac{\varDelta}{2}, E_i+\frac{\varDelta}{2}\right) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

is the height of the curve at the point E_i shown in the figure 4.

In one volume element

$$a^3=V_S+V_v \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

,where V_S and V_v are volumes of particles and that of void respectively, and

$$e=\frac{V_v}{V_S}=\frac{a^3}{V_S}-1 \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

From (4) and (5), we get

$$\varDelta V_S+\varDelta V_v=0,\ \varDelta e=-\frac{a^3}{V_S{}^2}\varDelta V_S \quad\cdots\cdots(6)$$

Now, in some volume element, the void ratio e_i satisfies the following condition.

$$E_K-\frac{\varDelta}{2}<e_i<E_K+\frac{\varDelta}{2} \quad\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

When the volume element is changed in the manner as described, the change in void ratio is given by the equations (6), and moreover if the condition

$$E_K-\frac{\Delta}{2}<e_i\pm\Delta e<E_K+\frac{\Delta}{2} \quad \cdots\cdots\cdots\cdots(8)$$

is satisfied, the relationship between Δe and Δ should be

$$-\Delta<\pm\Delta e<\Delta \quad \cdots\cdots\cdots\cdots(9)$$

,that is, $$\left|a^3\frac{\Delta V_S}{V_S}\right|<\Delta \quad \cdots\cdots\cdots\cdots(10)$$

This condition can be satisfied if the change of volume element is made adequately. Hence, the condition

$$|\Delta n_i|<\varepsilon \quad \cdots\cdots\cdots\cdots(11)$$

can be satisfied.

When the volume of the volume element is v, the volume of particles which are contained in the interval (3) is

$$n_iv\frac{1}{1+E_i} \quad \cdots\cdots\cdots\cdots(12)$$

Total volume of the volume elements of which void ratios are in the interval (3) is n_iv. If the volume expressed by (12) is counted in number of particles, this is

$$n_i\frac{v}{v_0}\frac{1}{1+E_i} \quad \cdots\cdots\cdots\cdots(13)$$

,where the volume of one particle is v_0.

Hence we have,

$$V_i=n_iv$$

$$N_i=\frac{n_iv}{v_0(1+E_i)} \quad \cdots\cdots\cdots\cdots(14)$$

Therefore, as a matrix which represents the characteristics of a granular material, we have

$$\begin{pmatrix} E_1, & E_2,\cdots\cdots\cdots & E_n \\ V_1, & V_2,\cdots\cdots\cdots & V_n \\ N_1, & N_2,\cdots\cdots\cdots & N_n \end{pmatrix} \quad \cdots\cdots\cdots\cdots(15)$$

This matrix has the same form as the matrix (2), hence the theory of the mechanics of granular material can be developed along the same line as described in the former paper[1].

3. The quantity which expresses the characteristics of a granular material composed of particles of various sizes.

The material is composed of N_1 particles, each particle has the volume v_1; N_2 particles, each of which has the volume v_2;……, and N_m particles each of which has the volume v_m.

The voidratio and the total volume of the material as a whole are e and V respectively.

A matrix which expresses the characteristics of the material is obtained when we consider following the descriptions given in the former paragraph,

$$\begin{pmatrix} e_1, & e_2, & \cdots\cdots\cdots, & e_n \\ N_1^*, & N_2^*\cdots\cdots\cdots, & & N_n^* \\ V_1^*, & V_2^*\cdots\cdots\cdots, & & V_n^* \end{pmatrix} \quad \cdots\cdots\cdots\cdots(16)$$

In the above matrix, the volume V_i^* is the total volume of the volume elements which has the void ratio e_i. In this volume V_i^*, there are contained N_{1i} particles of volume v_2……, and N_{mi} particles of volume v_m.

Hence we have

$$N_i^*=N_{1i}+N_{2i}+\cdots\cdots+N_{mi} \quad \cdots\cdots\cdots\cdots(17)$$

and

$$V_i^*=(N_{1i}v_1+N_{2i}v_2+\cdots\cdots+N_{mi}v_m)(1+e_i) \quad \cdots\cdots\cdots\cdots(18)$$

As shown in the previous paper, the number of partitioning of volume V into volume V_1^*, V_2^*, V_3^*,……, V_n^* is

$$Z_0={}_{V+n-1}C_n \quad \cdots\cdots\cdots\cdots(19)$$

Let us put

$$V_{1i}=N_{1i}v_1,\ V_{2i}=N_{2i}v_2,\cdots\cdots,\ V_{mi}=N_{mi}v_m \quad \cdots\cdots\cdots\cdots(20)$$

Putting N_{1i} particles of volume v_1 into the volume V_i^*, is the same as taking the volume V_{1i} out of the volume V_i^*, so that the number of ways of such procedure is

$${}_{V_i^*}C_{V_{1i}}={}_{V_i^*}C_{V_i^*-V_{1i}} \quad \cdots\cdots\cdots\cdots(21)$$

The number of ways of taking out the volume V_{2i} from the remaining volume $V_i^*-V_{1i}$ is

$${}_{V_i^*-V_{1i}}C_{V_{2i}}={}_{V_i^*-V_{1i}}C_{V_i^*-V_{1i}-V_{2i}} \cdots\cdots(22)$$

Proceeding as above, the number of ways of putting N_{1i} particles of volume v_1, N_{2i} particles of volume v_2,……,N_m particles of volume v_m into the volume V_i^* is obtained as

$$Z_i={}_{V_i^*}C_{V_i^*-V_{1i}}\cdot{}_{V_i^*-V_{1i}}C_{V_i^*-V_{1i}-V_{2i}}\cdots\cdots$$
$$=\frac{V_i^*!}{V_{1i}!\,V_{2i}!\cdots V_{mi}!\,V_{vi}!} \quad \cdots\cdots\cdots\cdots(23)$$

The number of states of granular material characterized by the matrix (16) is, with the aid of the equations (19) and (23)

$$Z=Z_0\prod_{i=1}^{n}Z_i \quad \cdots\cdots\cdots\cdots(24)$$

Applying the Stirling's formula

$$m!\sim m\log m,\ m\gg 0 \quad \cdots\cdots\cdots\cdots(25)$$

We have

$$\log Z_0\sim(V+n-1)\log(V+n-1)$$
$$-n\log n-(V-1)\log(V-1)\cdots\cdots(26)$$

On the other hand, the total volume V is expressed by the equation,

$$V=(1+e)\sum_{i=1}^{n}(N_{1i}v_1+N_{2i}v_2+\cdots\cdots+N_{mi}v_m) \quad \cdots\cdots(27)$$

As shown in the former paper written by the present author in 1943[2], the most probable distribution of $N_{1i}, N_{2i}\cdots\cdots, N_{mi}$ is given when the ratio $N_{1i}:N_{2i}:\cdots\cdots:N_{mi}$ is equal to the ratio $N_1:N_2:\cdots\cdots:N_m$, where

$$N_1=\sum_i N_{1i},\ N_2=\sum_i N_{2i},\cdots\cdots,N_m=\sum_i N_{mi} \quad \cdots\cdots(28)$$

Hence we can put

$$N_{1i}=\frac{N_1}{n}+\lambda_{1i},\ N_{2i}=\frac{N_2}{n}+\lambda_{2i},\cdots\cdots,$$

$$N_{mi}=\frac{N_m}{n}+\lambda_{mi} \quad \cdots\cdots(29)$$

The quantities λ_{ji} are very small and from the equation (28) we have

$$\sum_i \lambda_{ji}=0 \quad \cdots\cdots(30)$$

Concerning void ratios, we can have the followings,

$$e_i=\bar{e}+\epsilon_i,\ \bar{e}=\frac{1}{n}\sum_i e_i,\ \sum_i \epsilon_i=0 \quad \cdots\cdots(31)$$

The following assumptions can be made, because the quantities ϵ_i, λ_{ji} are small.

$$\sum_i \epsilon_i\lambda_{ji}=0,\ \sum_i \lambda_{ji}\lambda_{ki}=0,\ (j\neq k) \quad \cdots\cdots(32)$$

From the equation (29), we have

$$N_{ji}=\frac{N_j}{n}+\lambda_{ji}=\frac{N_j}{n}\left(1+\frac{n}{N_j}\lambda_{ji}\right) \quad \cdots(33)$$

Hence we get

$$V_{ji}=\frac{N_jv_j}{n}\left(1+\frac{n}{N_j}\lambda_{ji}\right)$$

$$=\frac{\kappa_j}{n}N_1v_1\left(1+\frac{n}{N_j}\lambda_{ji}\right) \quad \cdots\cdots(34)$$

,where

$$\kappa_j=\frac{N_jv_j}{N_1v_1} \quad \cdots\cdots(35)$$

Applying these expressions to (18), we obtain

$$V_i^*=(1+e_i)\sum_j V_{ji}$$

$$=(1+e)\left(1+\frac{1}{1+e}\epsilon_i\right)\frac{N_1v_1}{n}\left(\sum_j \kappa_j+\sum_j\frac{n\,\kappa_j}{N_j}\lambda_{ji}\right)$$

$$\fallingdotseq\frac{N_1v_1K_1(1+e)}{n}\left(1+\frac{1}{1+e}\epsilon_i+\sum_i\frac{n\,\kappa_j}{N_jK_1}\lambda_{ji}\right) \quad \cdots\cdots(36)$$

,where

$$K_1=\sum \kappa_j \quad \cdots\cdots(37)$$

and

$$V_{vi}=V_i-\sum_j V_{ji}$$

$$=\frac{N_1v_1K_1e}{n}\left(1+\frac{1}{e}\epsilon_i+\sum_j\frac{n\,\kappa_j}{N_jK_1}\lambda_{ji}\right) \quad (38)$$

Hence, we have

$$\log Z_i\sim\log V_i^*!-\sum_j\log V_{ji}!-\sum\log V_{vi}!$$

$$\sim\frac{N_1v_1K_1(1+e)}{n}\left[1+\frac{1}{1+e}\epsilon_i+\sum_j\frac{n\kappa_j}{N_jK_1}\lambda_{ji}\right]$$

$$\times\left[\log\frac{N_1v_1K_1(1+e)}{n}+\frac{1}{1+e}\epsilon_i+\sum_j\frac{n\kappa_j}{N_jK_1}\lambda_{ji}\right.$$

$$\left.-\frac{1}{2}\frac{1}{(1+e)^2}\epsilon_i-\frac{1}{2}\sum_j\frac{n^2\kappa_j^2}{N_j^2K_1^2}\lambda_{ji}^2\cdots\right]$$

$$-\sum_j\frac{N_1v_1}{n}\kappa_j\left(1+\frac{n}{N_j}\lambda_{ji}\right)\left[\log\frac{N_1v_1}{n}\right.$$

$$\left.+\log\kappa_j+\frac{n}{N_j}\lambda_{ji}-\frac{1}{2}\frac{n^2}{N_j^2}\lambda_{ji}^2\cdots\right]$$

$$-\frac{N_1v_1K_1e}{n}\left[1+\frac{1}{e}\epsilon_i+\sum_j\frac{n\,\kappa_j}{N_jK_1}\lambda_{ji}\right]$$

$$\times\left[\log\frac{N_1v_1K_1e}{n}+\frac{1}{e}\epsilon_i+\sum_j\frac{n\,\kappa_j}{N_jK_1}\lambda_{ji}\right.$$

$$\left.-\frac{1}{2}\frac{1}{e^2}\epsilon_i^2-\frac{1}{2}\sum_j\frac{n^2\kappa_j^2}{N_j^2K_1^2}\lambda_{ji}^2\cdots\right] \quad \cdots\cdots(39)$$

Therefore we get

$$\sum_j\log Z_i\sim\frac{N_1v_1K_1(1+e)}{n}\left[n\log\frac{N_1v_1K_1(1+e)}{n}\right.$$

$$\left.+\frac{1}{2}\frac{1}{(1+e)^2}\sum_i\epsilon_i^2+\frac{1}{2}\sum_j\frac{n^2\kappa_j^2}{N_j^2K_1^2}\sum_i\lambda_{ji}^2\right]$$

$$-\frac{N_1v_1}{n}\left[n\sum_j\kappa_j\log\frac{N_1v_1}{n}+n\sum_j\kappa_j\log\kappa_j\right.$$

$$\left.+\sum_j\frac{n^2\kappa_j}{2N_j^2}\sum_j\lambda^2{}_{ji}\right]-\frac{N_1v_1K_1e}{n}$$

$$\times\left[n\log\frac{N_1v_1K_1e}{n}+\frac{1}{2}\frac{1}{e^2}\sum_i\epsilon_i^2\right.$$

$$\left.+\frac{1}{2}\sum_j\frac{n^2\kappa_j^2}{N_j^2K_1^2}\sum_i\lambda^2{}_{ji}\right]$$

$$=N_1v_1K_1[(1+e)\log(1+e)-e\log e]$$

$$-N_1v_1K_1\log\frac{N_1v_1}{n}-N_1v_1\sum_j\kappa_j\log\kappa_j$$

$$-\frac{N_1v_1K_1}{2}\frac{s}{e(1+e)}$$

$$-\frac{nN_1v_1}{2}\sum_j\frac{\kappa_j}{N_j^2}\left(1-\frac{\kappa_j}{K_1}\right)\sum_i\lambda^2{}_{ji}\cdots \quad (40)$$

,where

$$s=\frac{1}{n}\sum\epsilon_i^2 \quad \cdots\cdots(41)$$

From the equation (26), we get

$$\log Z_0\sim\{N_1v_1K_1(1+e)+n-1\}$$

$$\times\log\{N_1v_1K_1(1+e)+n-1\}-n\log n$$

$$-\{N_1v_1K_1(1+e)-1\}\log\{N_1v_1K_1(1+e)-1\}$$

$$\sim\frac{n}{N_1v_1K_1(1+e)}$$

$$\times \log \{N_1 v_1 K_1 (1+e)\} + n \quad \cdots\cdots\cdots\cdots (42)$$

Hence we have

$$\log Z \sim A + \frac{n}{N_1 v_1 K_1 (1+e)} \log \{N_1 v_1 K_1 \times (1+e)\} + N_1 v_1 K_1 [(1+e)\log(1+e) - e\log e] - \frac{N_1 v_1 K_1}{2} \frac{s}{e(1+e)} - \frac{nN_1 v_1}{2} \sum_j \frac{\kappa_j}{N_j^2}\left(1-\frac{\kappa_j}{K_1}\right) \sum_i \lambda^2{}_{ji} \quad (43)$$

,where A is a quantity which is independent of e and s.

The quantity Z is the number of states, so that we can put

$$S = K \log Z \quad \cdots\cdots\cdots\cdots (44)$$

and S can be considered as an entropy of the system.

Since $\sum_j \lambda^2{}_{ji}$ and their coefficients in the equation (43) are positive, $\log Z$ is maximum when $\lambda_{ji}=0$, therefore the maximum of $\log Z$ when the values of e and s are constants, is

$$\log Z = \frac{n}{N_1 v_1 K_1 (1+e)} \log \{N_1 v_1 K_1 (1+e)\} + N_1 v_1 K_1 [(1+e)\log(1+e) - e\log e] - \frac{N_1 v_1 K_1 s}{2e(1+e)} \quad \cdots\cdots (45)$$

The first term of this equation is negligibly small compaired with others, therefore if this first term is neglected we obtain

$$\log Z \sim N_1 v_1 K_1 [(1+e)\log(1+e) - e\log e] - \frac{N_1 v_1 K_1 s}{2e(1+e)} \quad \cdots\cdots\cdots\cdots (46)$$

This equation has the same form as the log Z obtained in case when the granular material is composed of particles of equal size.

Therefore we can conclude that the relationship between the void ratio and the angle of internal friction of the granular material has the same form even if this material is composed of particles of various size as that in the case when the material is composed of equal size[1].

Reference

1) T. Mogami, A statistical approach to the mechanics of granular material, Soil and Foundation, Vol. V, No. 2, 1965 or A Statistical Theory of Mechanics of Granular Materials, Journ. Faculty of Engineering, University of Tokyo, Ser. (B), Vol. 28, No. 2, 1965

2) T. Mogami, A mathematical study on the distribution of particle size in sand (in Japanese) Journ.Japan Soc. of Civil Engineers, Vol. 29, No. 8, 1943

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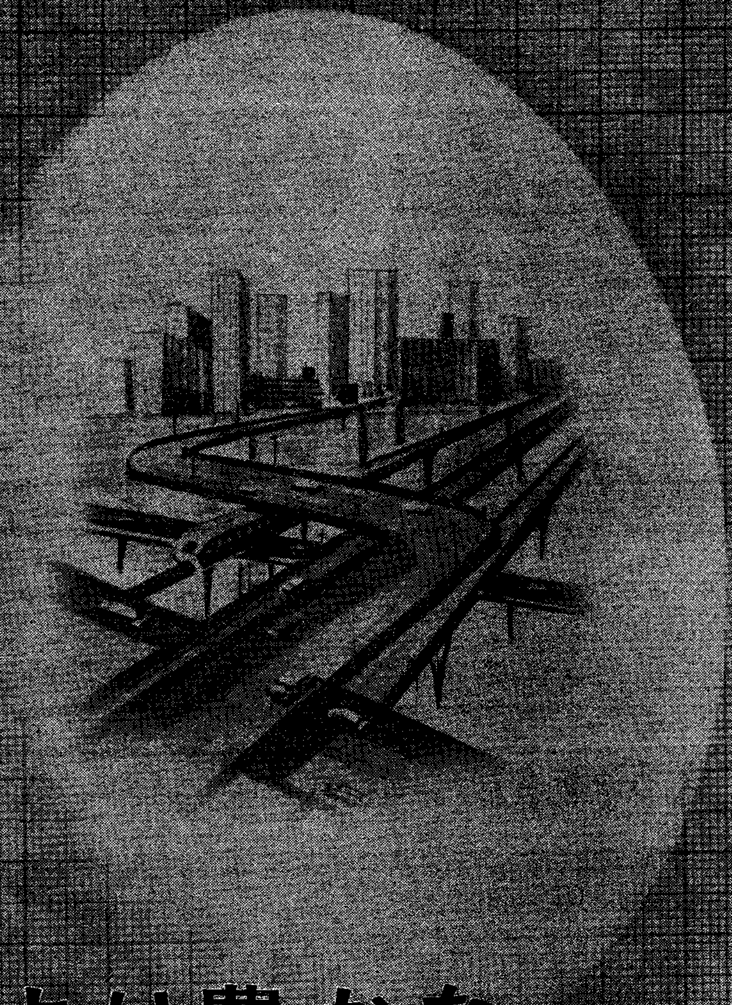

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