EXPERIMENTAL STUDIES ON THE STRENGTE OF RECTANGULAR REINFORCED AND PRESTRESSED CONCRETE BEAMS UNDER COMBINED FLEXURE AND TORSION

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1. OBJECTIVE OF INVESTIGATION

Experimental or theoretical studies on the resisting properties of plain or reinforced concrete members to torsion and to combined bending and torsion have been carried out since about 1900 or 1 950, respectively. The existing test data show that the failure of beam under combined stresses is very complicated phenomenon and not yet well-defined. There remain many problems unsolved yet, for instance, informations of influences of the followings described below are little available;

repeated loading, variation of cross section, ratio of bending moment to torsion, strength of concrete, properties of reinforcement and prestressing etc.

In this study totally eighty rectangular concrete beams were tested statically to investigate the following three items: (1) Effects of horizontal bar or stirrup on the elastic-plastic properties and on the ultimate strength of beam. (2) Effects of prestressing on the ultimate strength of plain or reinforced beam. (3) Calculation of the ultimate strength of beams by assuming the failure mechanism under the combinedloading.

2. DESCRIPTION OF SPECIMENS AND TEST METHOD

Test specimens

All beams are of 10×20 cm section and 150 cm long as shown in Fig. 1. Round bar of dia. 6 mm is used as stirrup in the beams with three kinds of spacing of 5, 10 and 15 cm. The concrete cover over the stirrup is 2 cm. As the longitudinal reinforcement four round bars of dia. 13 mm or 16 mm are used in type I and

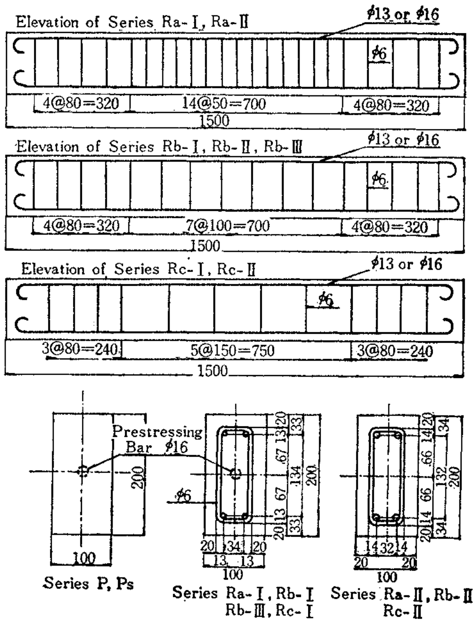

Fig. 1 Elevation and Cross Section of Specimen.

III, or type II specimens, also as shown in Fig. 1.

In series Ps and Rb-III all beams are post-tensioned with one prestressing bar of dia. 16 mm, passing through the center of the cross-section. The uniform prestress remaining effective in the beam at the time of testing is estimated as 75 kg/cm². Other characteristics of the beams are shown in Table 1 (a).

For each series twelve standard cylinders of 30 ϕ 15 cm were prepared, half of which were for compressive strength test and other half for splitting test, and six standard beam specimens of 15×15×53 cm were cast for flexural strength test of the concrete used.

To examine specifically the behavior of beam under large ratio of bending to torsional moment ($M/T=6$), the additional tests as shown in Table 1 (b) were carried out. All beam specimens were cast in wooden molds. The concrete used except in the additional tests had a nominal composition of 336 kg cement, 750 kg sand, 1 094 kg gravel with maximum size 25 mm, and a

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Table 1 (a) Characteristics of the beams

Series	Spacing of stirrup	Horizontal bar		Steel ratio horizontal bar	Effective prestress	Compressive strength of concrete	Tensile strength of concrete	Number of specimens tested				
No.	s	Dia. ϕ (mm)		$p=p'$	σ_{ps}	σ_{cu}	σ_{tu}	M/T (Ratio of bending moment to torque)				
	(cm)	Top	Bottom	(%)	(kg/cm²)	(kg/cm²)	(kg/cm²)	0	0.41	1.00	2.41	∞
P	—	—	—	—	—	322	22.0	2	2	2	2	—
P_s	—	—	—	—	75	402	31.7	2	2	2	2	2
Ra-I	5	2ϕ13	2ϕ13	1.6	—	268	20.5	2	2	2	2	—
Ra-II	5	2ϕ16	2ϕ16	2.4	—	396	27.2	2	2	2	2	—
Rb-I	10	2ϕ13	2ϕ13	1.6	—	294	24.3	2	2	2	2	2
Rb-II	10	2ϕ16	2ϕ16	2.4	—	363	28.5	2	2	2	2	2
Rb-III	10	2ϕ13	2ϕ13	1.6	75	480	32.4	2	2	2	2	2
Rc-I	15	2ϕ13	2ϕ13	1.6	—	341	27.0	2	2	2	2	—
Rc-II	15	2ϕ16	2ϕ16	2.4	—	410	34.4	2	2	2	2	—

Table 1 (b)

Series No. / M/T	Number of specimens tested						σ_{cu}=438 kg/cm²
	Ra-I	Ra-II	Rb-I	Rb-II	Rc-I	Rc-II	
6.00	1	1	1	1	1	1	σ_{tu}= 32.1 kg/cm²
∞	1	1	1	1	—	—	

Table 2

	Dia. ϕ (mm)	σ_t (kg/cm²) (failure)	σ_y (kg/cm²) (yield)	Elongation (%)
Round bar	ϕ16	4 890	3 449	27.7
	ϕ13	4 500	3 400	34.1
	ϕ 6	3 010	2 153	14.4
P.C. bar	ϕ16	12 300	11 250	9.4

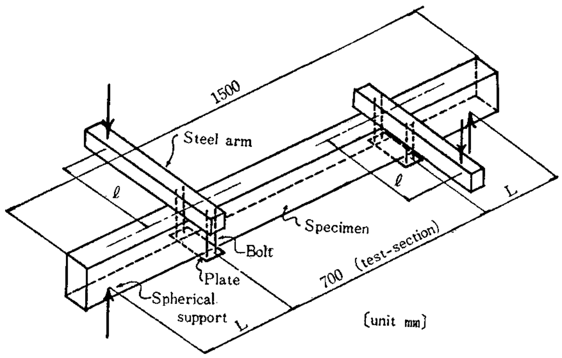


Fig. 2 Test set-up.

slump of 7.5±1 cm. All spcimens were exposed to the air in the room after removal of mold. The strength of concrete at ages of beam test (78～105 days) varied fairly much from series to series as shown in Table 1 (a), and the higher strength of concrete as shown in Table 1 (b) was used in the additional test where the beams were tested at earlier age of 20 days. Properties of the reinforcing steel and the prestressing steel bar are given in Table 2. It is noted that the steel used for stirrup has rather lower tensile strength comparing with other two kinds of steel used for horizontal bar.

Test set-up and procedure

Various ratios of bending moment to torque could be achived by the testing device as shown in Fig. 2. The load was applied by the steel arms to the specimen supported at both ends. In this case the ratio M/T was held constant in the test section nominally 70 cm long and was equal to the ratio L/l indicated in Fig. 2.

To avoid the influence of stress concentrations at the junction of the loading arms and the beam, the rubber pads of 2.5 mm thick were inserted there. Two rollers were used at the beam supports also to avoid the support restriction. Precise measurements made on two selected specimens of Rb-III Series using twelve electric resistant strain gages showed that this testing device gave the required load condition quite satisfactorily. (photo. 1)

To measure the deflection at the mid-span and the rotation of beam, both electric resistant strain gages and dial gages of 1/100 mm scale were used as illustrated in Fig. 3.

Load was applied with speed of about 50～100 kg/min to the specimens in various increment and the load was held constant while the

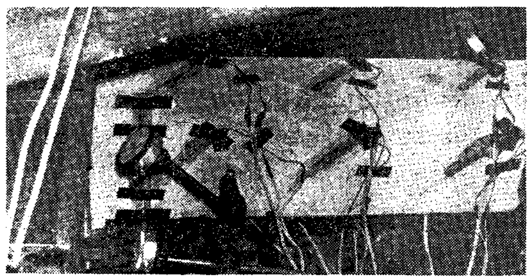


Photo 1 Gage Location

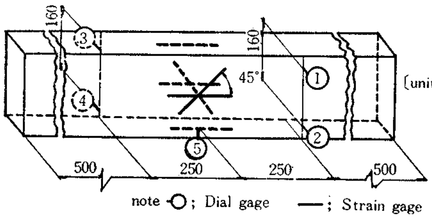


Fig. 3 Location of gases.

observations were made during about 1 min. The load was returned to near zero when the strain reading of electric resistant gage began to show a non-linear relation with load, and then the load was again increased until the beam failed.

3. TEST RESULTS

Principal test data

Table 3 and Fig. 4 (a)-(j) illustrate the ultimate strengths of all the beams in pure or in combined bending and torsion.

Mode of Failure

(1) P Series (plain concrete beams)

In each of these beams the ultimate failure occurred immediately after the formation of the first crack. The strains measured by electric resistant gages usually began to flow when the load approached the failure of beam, clearly indicating the time of beam failure. But in some beams the brittle failure occurred quite independently of the strain readings.

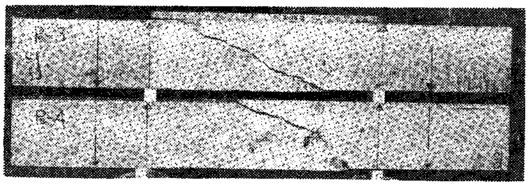


Photo 2 Inclination of Cracks (front view $M/T=0.41$)

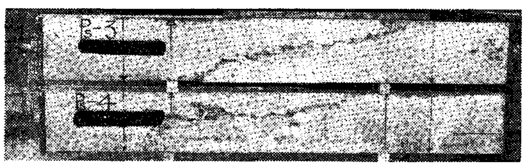


Photo 3 (back view, $M/T=0.41$)

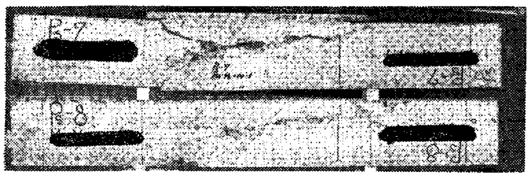


Photo 4 ($M/T=2.41$)

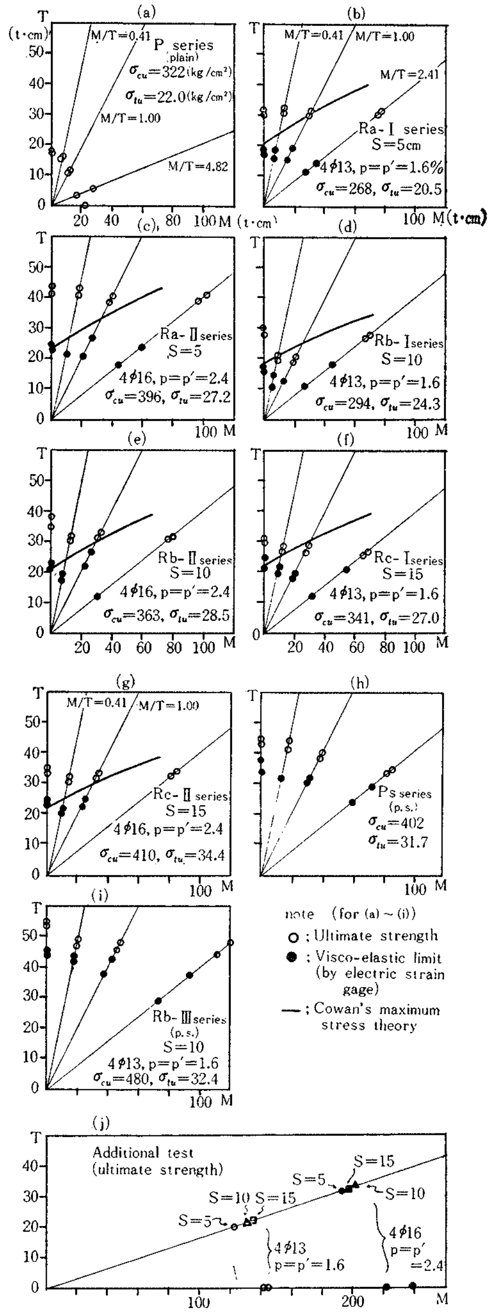


Fig. 4 Ultimate strength and Visco-elastic limit.

(2) Ps Series (prestressed beams)

The ratio of the effective prestress to the ultimate compressive strength of concrete was about 20%, so the mode of failure was not so brittle as explained anywhere, probably because of the influence of the beam-holding rigs and the loading steel arms. Except under pure bending the initial crack could not be detected with naked eye until the beam failed. The

Table 3 Results of Test

Series No.	Beam No.	M/T	Visco-elastic limit (t·cm) T_B	Visco-elastic limit (t·cm) M_B	Ultimate strength (t·cm) T_u	Ultimate strength (t·cm) M_u
P (plain)	1	0	15.5	—	18.0	—
	2	0	15.0	—	18.5	—
	3	0.41	13.0	5.4	16.3	6.8
	4	0.41	13.8	5.7	15.9	6.6
	5	1.00	9.7	9.7	10.9	10.9
	6	1.00	13.8	13.8	11.1	11.1
	7	4.82	—	—	5.2	25.2
	8	4.82	—	—	3.4	16.8
Ra-I (s=5)	1	0	17.5	—	30.3	—
	2	0	18.5	—	31.8	—
	3	0.41	16.5	6.8	31.2	12.9
	4	0.41	18.0	7.5	32.7	13.5
	5	1.00	15.7	15.7	31.5	31.5
	6	1.00	19.0	19.0	31.1	31.1
	7	2.41	14.0	33.8	31.5	76.2
	8	2.41	11.2	27.0	30.7	74.3
Ra-II (s=5)	1	0	25.5	—	44.2	—
	2	0	24.0	—	42.3	—
	3	0.41	21.5	8.9	42.3	17.5
	4	0.41	—	—	42.7	17.7
	5	1.00	27.5	27.5	40.7	40.7
	6	1.00	21.0	21.0	39.2	39.2
	7	2.41	24.0	57.9	41.3	99.8
	8	2.41	18.0	43.4	39.3	95.0
Rb-I (s=10)	1	0	16.5	—	29.9	—
	2	0	17.0	—	28.2	—
	3	0.41	15.0	6.2	20.6	8.52
	4	0.41	11.0	4.6	20.5	8.48
	5	1.00	13.3	13.3	21.2	21.2
	6	1.00	—	—	20.3	20.3
	7	2.41	18.5	44.6	27.5	66.5
	8	2.41	11.0	26.6	27.9	67.5
	9	∞	—	—	—	186
	10	∞	—	—	—	114
Rb-II (s=10)	1	0	21.0	—	35.8	—
	2	0	23.0	—	38.7	—
	3	0.41	20.5	85.	32.1	13.3
	4	0.41	19.0	7.9	31.6	13.1
	5	1.00	23.0	23.0	34.1	34.1
	6	1.00	28.0	28.0	32.9	32.9
	7	2.41	12.5	30.2	31.9	17.1
	8	2.41	—	—	32.6	78.8
	9	∞	—	—	—	171
	10	∞	—	—	—	141

Series No.	Beam No.	M/T	Visco-elastic limit (t·cm) T_B	Visco-elastic limit (t·cm) M_B	Ultimate strength (t·cm) T_u	Ultimate strength (t·cm) M_u
Rc-I (s=15)	1	0	25.0	—	31.6	—
	2	0	22.0	—	30.7	—
	3	0.41	20.3	8.4	27.8	11.5
	4	0.41	22.0	9.1	28.0	11.8
	5	1.00	18.5	18.5	27.1	27.1
	6	1.00	20.0	20.0	29.3	29.3
	7	2.41	22.3	53.8	27.3	66.0
	8	2.41	12.5	30.1	26.2	63.4
Rc-II (s=15)	1	0	24.8	—	32.9	—
	2	0	24.0	—	34.9	—
	3	0.41	21.5	8.9	31.8	13.1
	4	0.41	21.0	8.7	31.1	12.9
	5	1.00	22.0	22.0	32.6	32.6
	6	1.00	24.5	24.5	32.3	32.3
	7	2.41	—	—	32.5	78.6
	8	2.41	—	—	33.7	81.5
P_s (P.S.)	1	0	38.5	—	43.6	—
	2	0	34.3	—	45.3	—
	3	0.41	32.0	13.2	44.2	18.3
	4	0.41	32.0	13.2	41.6	17.2
	5	1.00	32.0	32.0	40.4	40.4
	6	1.00	31.7	31.7	39.0	39.0
	7	2.41	29.0	70.0	33.7	81.5
	8	2.41	24.0	57.9	34.0	82.7
	9	∞	—	—	—	163
	10	∞	—	—	—	151
Rb-III (P.S.) (s=10)	1	0	44.5	—	54.0	—
	2	0	45.3	—	54.7	—
	3	0.41	42.5	17.5	47.2	19.5
	4	0.41	44.5	18.4	49.7	20.5
	5	1.00	43.0	43.0	48.4	48.4
	6	1.00	37.5	39.0	46.1	46.1
	7	2.41	37.3	36.3	50.3	122
	8	2.41	29.0	34.5	44.6	108
	9	∞	—	—	—	247
	10	∞	—	—	—	257
Ra-I	9	6.00	7.0	42.0	20.7	124
Rb-I	11	6.00	7.5	45.0	21.8	131
Rc-I	9	6.00	9.3	55.8	21.4	128
Ra-II	9	6.00	—	—	32.1	193
Rb-II	11	6.00	10.0	60.0	32.5	195
Rc-II	9	6.00	15.0	90.0	31.8	191

note ; Visco-elastic limits were determined by reading measured by strain gage.

intersections of the failure surface with sides of the beam showed almost straight lines in pure torsion and in the combined loading of $M/T=0.41$, but the inclination to the beam axis was different between two sides, the one was almost 45 degree and the other was more inclined as illustrated in Photos 2. and 3. probably due to the prestressing effect. On some beams subject to the combined loadings of $M/T=1.0$ and 2.41, occurred a wedgeshaped spalling of concrete in compressive zone. (photo. 4)

(3) Rb-III Series (prestressed reinforced beams)

The initial crack could be seen with naked eye before the failure occurred. Except in the pure bending and $M/T=2.4$, the bearing capacity of the beams beyond the first cracking load up to the ultimate failure was fairly small, the first visible crack being found at 70 to 80% of the ultimate strength. Compressive zone was crushed in the beams at failure in pure bending and $M/T=2.4$.

(4) Ra, Rb, Rc, Series (reinforced concrete beams)

The frequency of cracks in each of the beams was influenced by the spacing of stirrups except $M/T=6.0$ and pure bending tests. The spacing of cracks was nearly in proportion to that of stirrups. Generally, cracks inclined at 45 degree to the beam axis were formed on the mid points of the sides of beam in pure torsion and under the combined loading of $M/T=2.4$, the initial cracks being found on both sides and bottom of beam. In $M/T=6.0$ nearly vertical cracks on the sides of beam occurred and finally the top surface was crushed.

4. ANALYTICAL STUDIES

Inclination of cracks

The inclination of tensile crack on each beam tested was varied with M/T ratio. Angle ψ of the crack with regard to the horizontal axis of beam is calculated from the fundamental elastic relationship;

$$\tan 2\psi = 2\,\tau/\sigma \qquad \cdots\cdots (1)$$

where τ is shearing stress and σ is normal stress in the cross section. If shearing stress is expressed (cf. Fig. 8) as follows;

$$\tau = KT, \quad K = \left\{\frac{1}{2}b_0^2\left(d_0 - \frac{1}{3}b_0\right)\right\}^{-1} \qquad \cdots (2)$$

then ψ on the sides near the top and the bottom faces of beam, is given by eq. (3);

$$\psi = \frac{1}{2}\tan^{-1}\left(\pm 2KW\frac{T}{M}\right) \qquad \cdots\cdots (3)$$

where W is the section modulus.

Comparisons between the theory and the measurements are drawn in Fig. 5 for the reinforced concrete beams series. It shows that the agreement is fairly good, although the theoretical values are calculated under rather poor assumption.

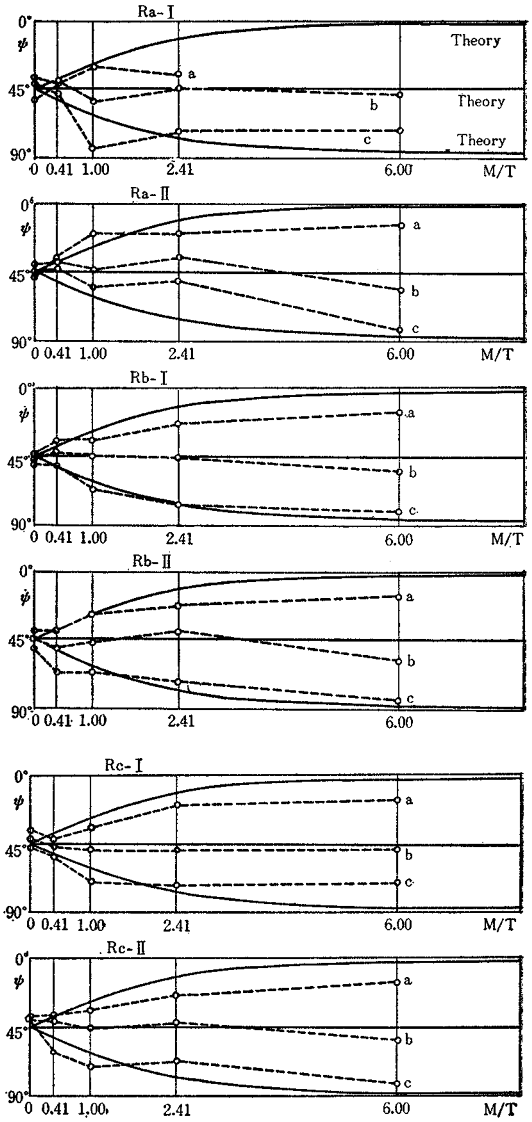
Modulus of Rigidity

In rectangular cross section having the ratio of $d_0/b_0=2$ the modulus of rigidity G is calculated from the elastic theory(1);

$$G = \frac{1}{4.58}\frac{T}{\theta} \qquad \cdots\cdots (4)$$

where θ is angle of rotation.

In this study the angle of rotation θ was measured by two methods, one by the dial gages attached as shown in Fig. 3 and the other the



note a : near the top surface of longer side
b : mid point of longer side
c : near the bottom surface of longer side

Fig. 5 Inclination of Cracks.

electric strain gages, having inclination of 45 degree to the beam axis also as shown in Fig. 3.

From dial gage reading (δ_i);

$$\theta = 1.25(\delta_1 + \delta_3 - \varepsilon_2 - \delta_4) \times 10^{-6} \ \text{(Rad/cm)} \qquad \cdots\cdots (5)$$

From the strain reading (ε_b) of electric strain gages;[1),3)]

$$\theta = 2\,\varepsilon_d / f(x, y) \ \text{(Rad/cm)} \qquad \cdots\cdots (6)$$

where

$$f(x,y) = \frac{-8\,b_0}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2}(-1)^{(n-1)/2} \times \left\{1 - \frac{\cosh(n\pi y/b_0)}{\cosh(n\pi d_0/2b_0)}\right\}\sinh\frac{n\pi x}{b_0}$$

and x-y coordinate is taken as shown in Fig. 8.

Practically, considering gage length (6.8 cm) of the electric strain gage, the value of $f(x, y)$ is taken as a mean of $f(5, 0)$, $f(5, 1.2)$ and $f(5, 2.4)$, and substituting it into eq. (6) gives;

$$\theta = 0.220\varepsilon_d \quad \text{for each beam} \cdots\cdots\cdots\cdots (7)$$

Thus, $T\sim\theta$ curve is drawn from the measurements, and Fig. 6 shows some examples of curves measured by dial gages. It can be seen that each curve describes large hysteresis loop. No reliable value of θ could not be obtained near the failure of beam under combined loading.

In general, $T\sim\theta$ curve has an almost straight part at begining followed by a curving part. On $T\sim\theta$ curve of Rc-II-3 beam in Fig. 6, point A shows the elastic limit, and point B, which is determined by extending line OA to the horizon having the ultimate torsion Tu and by intersecting the perpendicular with the original $T\sim\theta$ curve, is defined here as the visco-elastic limit, so called initial cracking torsional moment. The visco-elastic point B in each of the tested beams, measured from electiric strain gages are indicated in Fig. 4. Generally these points, A and B obtained by dial gage measurements are smaller than those by electric strain gages.

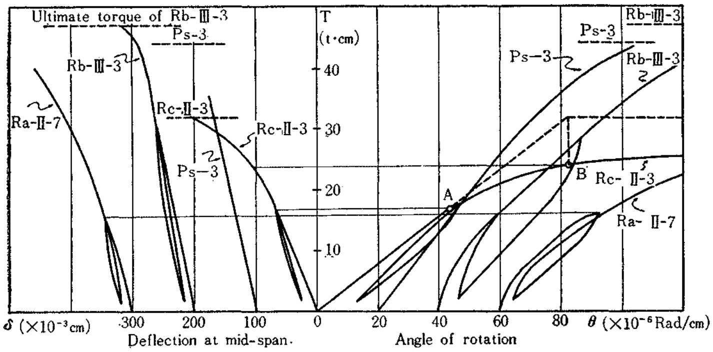


Fig. 6 $T\sim\theta$, $T\sim\delta$ Curves measured by dial gages ($M/T=0.41$)

Apparent modulus of rigidity (G_a or G_b to corresponding the elastic or visco-elastic limit) of beam under the combined loading is calculated using the slope of OA or OB in $T\sim\theta$ curve. In order to eliminate the effects of variation of concrete strength in different series of tests, the ratios of G_a or G_b to G_c, the calculated value from Ec of the standard cylinder specimen and the assumed poisson's ratio (1/6), are shown in Table 4. It may be concluded from this table that; (1) Probably

Table 4 Summary of Modulus of Rigidity

Series No.	M/T	Strain gage		Dial gage	
		G_a/G_c	G_b/G_c	G_a/G_c	G_b/G_c
P	0	0.89	0.75	—	—
	0.41	0.88	0.71	0.83	—
	1.00	0.80	0.70	0.77	—
	4.80	—	—	—	—
P_s (P.S.)	0	1.06	0.87	0.64	0.56
	0.41	1.07	0.81	0.88	0.80
	1.00	1.04	0.84	0.83	0.81
	2.41	1.05	0.82	0.78	0.57
Rb-III (P.S.) (s=10)	0	1.01	0.85	0.77	0.63
	0.41	1.03	0.93	0.75	0.69
	1.00	0.99	0.84	0.74	0.61
	2.41	1.02	0.71	0.90	0.72
Ra-I (s=5)	0	1.09	0.62	—	—
	0.41	1.06	0.58	0.55	0.38*
	1.00	1.13	0.59	0.88	0.47
	2.41	1.01	0.42	0.81	0.46
	6.00	0.55*	0.27*	0.75*	0.36*
Ra-II (s=5)	0	0.88	0.52	0.50	0.36
	0.41	0.93	0.50*	0.56	0.41
	1.00	0.92	0.55	0.55	0.37
	2.41	0.73	0.38	0.62	0.33
	6.00	0.65*	—	0.82*	0.27

Series No.	M/T	Strain gage		Dial gage	
		G_a/G_c	G_b/G_c	G_a/G_c	G_b/G_c
Rb-I (s=10)	0	0.85	0.50	0.54	0.41
	0.41	0.76	0.50	0.66	0.39
	1.00	0.85	0.49*	0.85	0.49
	2.41	0.87	0.45	0.61	0.36
	6.00	0.62*	0.24*	0.78*	0.28*
Rb-II (s=15)	0	1.04	0.64	0.56	0.37
	0.41	0.83	0.53	0.64	0.45
	1.00	0.97	0.74	0.80	0.52
	2.41	0.83	0.33*	0.42	0.28*
	6.00	0.76*	0.24*	0.85*	0.28*
Rc-I (s=15)	0	0.85	0.67	0.72	0.52
	0.41	0.88	0.68	0.94*	0.72*
	1.00	0.84	0.58	0.80	0.56
	2.41	0.81	0.50	0.69	0.41
	6.00	0.61*	0.30*	0.66*	0.28*
Rc-II (s=15)	0	0.68	0.49	0.61	0.41
	0.41	0.68	0.49	0.57	0.42
	1.00	0.72	0.55	0.60	0.51
	2.41	0.68	—	0.59	0.34
	6.00	0.72*	0.35*	—	—

Note : Values are averages of two specimens respectively except * of one specimen

Table. 5 Mean values of Gb/Ga for $M/T=0$, 0.41 and 1.0

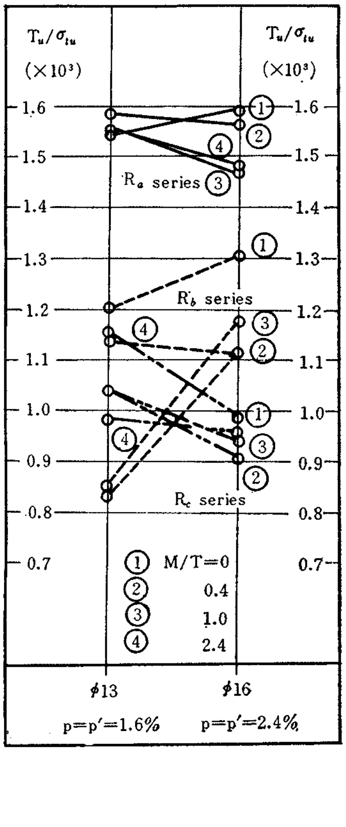
Spacing of stirrup	$s=5$ (cm)		$s=10$ (cm)		$s=15$ (cm)		plain	prestressed	
Series No.	Ra-I	Ra-II	Rb-I	Rb-II	Rc-I	Rc-II	P	P_s	Rb-III
Ga/Gb (%)	56	57	60	67	74	74	82	71	86

because of lower accuracy in measurements by dial gages, the values of G/G_c by dial gages seem to give smaller values than those by electric strain gages, though both being essectially same. (2) Transfer of prestress gives higher G_a/G_c and G_b/G_c. (3) In each series both G_a/G_c and G_b/G_c are nearly constant so far as $M/T=0.0\sim1.0$ and decrease as M/T increases further. (4) The effect of amount of horizontal-bar on G_b/G_c or G_a/G_c is not clearly shown, while the effect of stirrup:pitch is fairly clear. The mean values of G_b/G_a of each series for $M/T=0.0\sim1.0$ are shown in Table 5. Table 5 illustrates that the value of G_a/G_b increases as stirrup spacing becomes larger, that is, the mode of failure of beam becomes resembling that of plain or prestressed beam with increasing stirrup pitch.

Effects of reinforcement on ultimate strength ($M/T=0.0\sim2.4$)

It may be impossible in this study to examine

Fig. 7 Effects of Reinforcement.



accurately the effects of reinforcement on the ultimate strength of beam because of different strength of concrete used in each beam. So, in order to simplify the results of tests and to eliminate the variations in concrete strength, the ratio of T_u (the ultimate twisting moment) to σ_{tu} (the tensile strength of concrete by the split test in each series) are calculated and the results are drawn in Fig. 7.

(1) **Effect of horizontal-bar**

The values of T_u/σ_{tu} in Ra series (stirrup pitch of 5 cm) is almost same independently of the amount of reinforcement. In R_b series (stirrup pitch of 10 cm) the heavily reinforced beams ($p=p'=2.4\%$) show higher T_u/σ_{tu} values for $M/T=0.4$ and 1.0, while in Rc series (stirrup pitch of 15 cm) the beams with heavy reinforcement ($p=p'=2.4\%$) show lower values of T_u/σ_{tu} than those with $p=p'=1.6\%$ reinforcement. In general it can be said that the effect of horizontal-bar on T_u is rather small.

(2) **Effect of spacing of stirrup**

Beams with 5 cm spacing stirrup (Ra series) have greater resistance to torsion compared with beams having 10 or 15 cm spacing stirrups. However it may be said that stirrup-spacing of 15 cm is still effective as the torsional reinforcement. But it is desirable to use stirrups spaced not more than a half of the depth of the member section.

5. CALCULATION OF TORSIONAL STRENGTH OF BEAMS

For pure torsion

The following assumptions are made concerning the elastic limit, the visco-elastic limit and the failure of beam subject to pure torsion;

(1) The elastic limit (T_a) attains when the shearing stress at the surface of mid-depth of the section has reached σ_{tu}.

(2) The visco-elastic limit (T_b) attains where the shearing stress in concrete has been uniformly distributed on the section, being equal to σ_{tu}, but the longitudinal and the stirrup steel

are still in the elastic range.

(3) The failure of beam occurs when the shearing stress σ_{tu} has been uniformly distributed on the concrete and the stress in the stirrup has attained the yield point.

In general the resisting torsional moment of reinforced concrete beam under pure torsion is expressed in the next form;

$$T_o=[T_0]_p+\boxed{T_v}+\boxed{T_l} \quad \cdots\cdots\cdots\cdots\cdots(8)$$

where

$[T_0]_p$: resisting torsional momet of plain concrete section under pure torsion

$\boxed{T_v}$: resisting torsional moment of stirrup

$\boxed{T_l}$: resisting torsional moment of horizontal bar

$[T_0]_p$ is given by eq. (9).

$$[T_0]_p=\sigma_{tu}/K_i \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

where σ_{tu} : direct tensile strength of concrete

$$K_1 : \frac{1}{\alpha b_0{}^2 d_0} \text{ for elastic range } (i=1) \quad \cdots\cdots\cdots(10)$$

$$K_2 : \frac{1}{\beta b_0{}^2 d_0} \text{ for perfect plastic range } (i=2) \quad \cdots\cdots\cdots\cdots\cdots(11)$$

α, β are coefficients depending on the ratio b_0/d_0 (Fig. 8), and for the cross-sectional dimensions of test beams; $\alpha=0.246$ and $\beta=0.417$. When the web reinforcement is in the form of tie, Cowan[5] obtained the resisting moment of the tie from the strain energy method as follows;

$$\boxed{T_v}=\frac{1.6\,bda_{sv}\sigma_s}{s} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(12)$$

where a_{sv} : cross-sectional area of one stirrup

σ_s : stress in the stirrup

s : pitch of stirrups

Next, for the section reinforced as shown in Fig. 8, the shearing stresses on the centroid of each reinforcing bar are given by eq. (13)[6],

$$\left.\begin{aligned}\tau_{yz}&=G_s\theta\left[\frac{\partial F}{\partial y}-x\right]_{\substack{x=\frac{1}{2}b'\\ y=\frac{1}{2}d'}}\\ \tau_{xz}&=G_s\theta\left[\frac{\partial F}{\partial x}+y\right]_{\substack{x=-\frac{1}{2}b'\\ y=\frac{1}{2}d'}}\end{aligned}\right\} \quad \cdots\cdots\cdots(13)$$

where G_s : modulus of elasticity of the steel

F : a stress function

θ : angle of rotation due to the twisting moment

Therefore $\boxed{T_l}$ can be expressed approximately as follows;

$$\boxed{T_l}=2A_s\left(\tau_{yz}\frac{b'}{2}-\tau_{xz}\frac{d'}{2}\right) \quad \cdots\cdots\cdots(14)$$

where A_s is the cross-sectional area of two horizontal bars. For Fig. 8, $\boxed{T_l}$ is calculated;

$$\left.\begin{aligned}\boxed{T_l}&=78.2\times10^6(\text{kg cm}) \text{ for } 13\text{ mm bar } (p=p'=1.6\%)\\ &=118.0\times10^6(\text{kg cm}) \text{ for } 16\text{ mm bar } (p=p'=2.4\%)\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots(15)$$

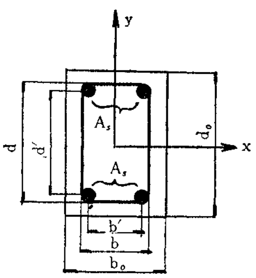
Thus, the elastic limit T_a can be computed by using eqs. (8), (10), (12), (15) and the visco-elastic limit T_b by eqs. (8), (11), (12) and (15). Figs. 9 and 10 show the computed values of T_a and T_b for the reinforced concrete beams tested, respectively. It is to be noted here that $\frac{E_s}{E_c}\sigma_{tu}$ is substituted for σ_s in eq. (12) and $\theta=20\times10^{-6}$ Rad./cm is used in eq. (15) which is determined by the $T\sim\theta$ relationship from the test results.

Similarly for the plain and prestressed beams T_a and T_b are also calculated by substituting $\sigma_{tu}{}^*$, obtained by Cowan and given by eqs. (16) and (17), for σ_{tu} in eq. (9).[3][7]

$$\sigma_{tu}{}^*=\sigma_{tu}\sqrt{1+\sigma/\sigma_{tu}} \quad \text{for } -\sigma_{tu}\geqq\sigma\geqq\sigma_{cu}-5.03\,\sigma_{tu} \cdots(16)$$

$$\sigma_{tu}{}^*=\sigma_{tu}\sqrt{0.040(\sigma_{cu}/\sigma_{tu})^2+0.120\sigma\sigma_{cu}/(\sigma_{tu})^2-0.159(\sigma/\sigma_{tu})^2} \quad \text{for } \sigma_{cu}-5.03\,\sigma_{tu}\geqq\sigma\geqq\sigma_{cu} \cdots\cdots(17)$$

Fig. 8 Cross section.



Cowan suggested in 1954 eqs. (16) and (17) as a dual criterion of failure for concrete, the failure envelop being shown in Fig. 11.

In Figs. 9 and 10 the measured values of T_a and T_b by both the electric strain gages and dial gages are illustrated for comparison. It is clearly shown that except for P or P_s series the theoretical values coincide well with the measurements.

As described previously the effect of longi-

Table. 6 Comparison of theoretical T_{ou} with test result

Series No.	Ra-I	Ra-II	Rb-I	Rb-II	Rc-I	Rc-II	P	P_s	Rb-III
Theoretical T_{ou} / Test result	1.16	0.970	1.02	0.889	0.922	1.03	1.01	1.08	1.02

Fig. 9 Elastic limit in pure torsion.

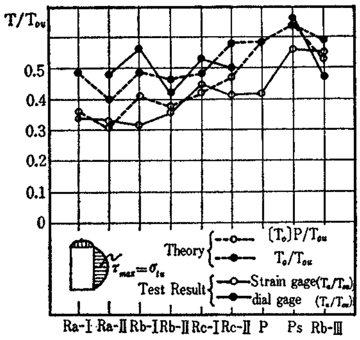


Fig. 10 Visco-elastic limit in pure torsion.

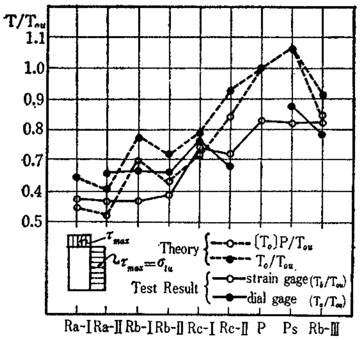
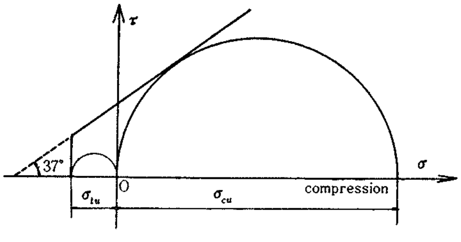


Fig. 11 Cowan's criterion.



tudinal reinforcement on the ultimate torsional strength T_u of beam is quite small (cf. Fig. 7), T_u of the reinforced concrete beam may be calculated as follows;

R.C. series (Ra, Rb and Rc) :

$$[T_{0u}]_{RC}=[T_{0u}]_P+\boxed{T_v}^{*} \quad \cdots\cdots\cdots\cdots\cdots(18)$$

Prestressed reinforced series (Rb-III) :

$$[T_{0u}]_{PRC}=[T_{0u}]_{PS}+\boxed{T_v}^{*} \quad \cdots\cdots\cdots\cdots(19)$$

$[T_{0u}]_P$ and $[T_{0u}]_{PS}$ are the visco-elastic limit T_b calculated before for the plain and the prestressed beams, respectively, and $\boxed{T_v}^{*}$ is given by eq. (12) except that the yielding point σ_{sy} of stirrup is substituted for σ_s as advocated by P.A. Zia.[3] The ratios of the theoretical values of ultimate torsion to the measurements for eath series are given in Table 6. It can be seen that the ultimate strength can be predicted by eqs. (18) and (19) with considerable accuracy and the effect of horizontal bar on $[T_{0u}]$ may be ignored for the reinforced beams.

For combined bending and torsion

(1) Plain and Prestressed Concrete Member

Part 1. Initial Cracking Moment; $[T_c]$

Cowan's eq.[7] for the initial cracking moment T_c for Ps series is,

$$[T_c]_{PS}=[T_{0u}]_{PS}\sqrt{1-\frac{K_b'[M_c]_{PS}}{K'[T_{0u}]_{PS}}} \quad \cdots\cdots\cdots\cdots\cdots(20)$$

where $[T_{0u}]_{PS}$: ultimate twisting strength of prestressed beam in pure torsion

$K_b'=\sigma_{tu}/W(\sigma_{Bu}+\sigma_{PS})$

$K'=K_2(1+\sigma_{PS}/\sigma_{tu})^{-1/2}$

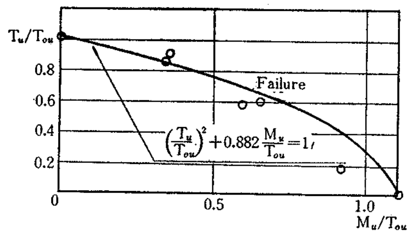
σ_{Bu} : flexural strength of concrete

σ_{PS} : compressive stress due to the effective prestressing force

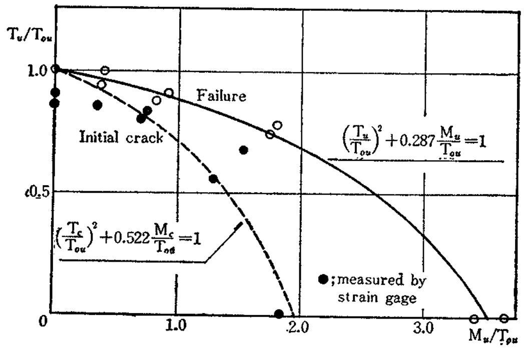
M_c : initial cracking bending moment

Fig. 12 Strength of beams for combined load

(a) P series



(b) P_s series



This relation may be used for Rb-Ⅲ series (prestressed reinforced beam) when $[T_{0u}]_{PRC}$ is substituted for $[T_{ou}]_{PS}$ in the above equation.

Agreement of test results with theoretical values is very good as shown in Figs. 12 (a) and 15 (b).

Part 2. Ultimate Strength; $[T_u]$

For plain concrete member Cowan derived eq. (27)[7] by assuming that failure of beam under combined bending and torsion $[M_u]$, $[T_u]$, is determined by the criterion of a constant maximum tensile stress;

$$[T_u]_P=[T_{0u}]_P\sqrt{1-\frac{K_b[M_u]_P}{K_2[T_{0u}]_P}} \quad \cdots(21)$$

where

K_b : a constant depending on b_0 and d_0, and for a rectangular section

$$K_b=\left(\frac{b_0 d_0{}^2}{4.23}\right)^{-1}$$

K_2 : this is given by eq. (11)

For uniformly prestressed member, the ultimate strength $[T_u]_{PS}$ may be determined by the following assumptions; (1) Failure is determined by the criterion given by eq. (16). (2) The critical point for failure under combined loading is at the middle of the longer side ($x=b/2$, $y=0$ in Fig. 8) of the section where the tensile stress σ due to bending moment $[M_u]_{PS}$ is given by eq. (22) (Fig. 13),

$$\sigma=\sigma_{tu}{}^*\frac{[M_u]_{PS}}{[M_{0u}]_{PS}} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(22)$$

where

$[M_{0u}]_{PS}$: ultimate flexural strength of prestressed concrete beam in pure bending

And shearing stress τ due to torsion $[T_u]_{PS}$ at the critical point is,

$$\tau=K_2[T_u]_{PS}\gtrless\sigma_{tu}{}^* \quad \cdots\cdots\cdots\cdots\cdots\cdots(23)$$

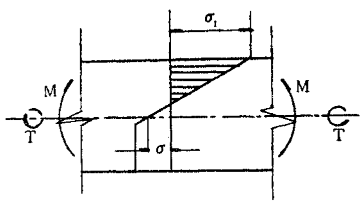
Rewriting of eq. (16) in the following form;

$$\tau=\sigma_{tu}{}^*\sqrt{1-\sigma/\sigma_{tu}{}^*}$$

and substituting eqs. (22) and (23) into the above gives the ultimate $[T_u]_{PS}$ as follows;

$$[T_u]_{PS}=[T_{0u}]_{PS}\sqrt{1-[M_u]_{PS}/[M_{0u}]_{PS}} \quad \cdots\cdots\cdots\cdots\cdots(24)$$

Fig. 13 Assumed stress distribution



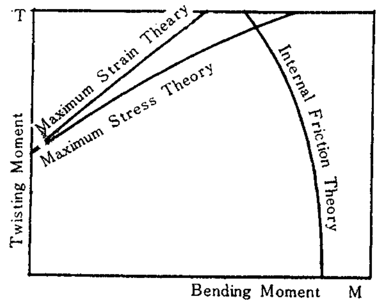
The computed ultimate strengths of the plain and prestressed concrete beams agree well with the results of tests as shown in Fig. 12.

(2) Reinforced Concrete Member

Part 1. Visco-elastic Limit; $[T_e]$

Cowan derived the equation of interaction curve for the visco-elastic limit of reinforced concrete beam under the combined bending and torsion as shown in Fig. 14. For instance for cace of small ratio of M/T, following equation is given from the maximum stress theory;[7]

Fig. 14 Visco-elastic limit of reinforced concrete beam.



$$[T_e]_{RC}=[T_{0e}]_{RC}\sqrt{1+\frac{K_B{}'}{K''}\frac{[M_e]_{RC}}{[T_{0e}]_{RC}}} \quad \cdots\cdots\cdots\cdots\cdots(25)$$

where

$[T_e]_{RC}$: twisting moment at the visco-elastic limit in combined loading

$[T_{0e}]_{RC}$: twisting moment at the visco-elastic limit in pure torsion

$$K''=\frac{\dfrac{r_2}{r_1}\dfrac{d}{b}}{\dfrac{1}{K_1}+\dfrac{1.6b'd'}{s}\dfrac{E_s}{E_c}a_{sv}}$$

$K_b{}'=1/W$ and r_1, r_2 are coefficients depending on the ratio of b_0/d_0.

For reference, the calculated values $[T_e]_{RC}$ of reinforced concrete members based on the above equation are plotted in Fig. 4 (a)～(g). The figures show that agreement of test results with the theoretical values is not good in this study.

Part-2. Ultimate Strength; T_u

The ultimate strength of reinforced concrete beam under combined bending and torsion can not be so easily determined as that of plain concrete beam, since the reinforced concrete member is still able to carry further load after a formation of bending or torsional cracks in the section. However under combined bending and torsion, two essentially different modes of failure are possible. That is, for very small ratios of M/T, failure is in general due to diagonal tension due to shear and sudden, while for large ratios of M/T, failure results by the crushing of concrete on the compressive zone with or without steel yielding. Many studies ever done show that in normal reinforced concrete beams bending does not so reduce and even to increases resistance to torsional deformation over a range of M/T ratios up-to 2.5 to 3 beyond which any increase in bending actually reduces the torsional strength.

Torsional failure

For small ratios of M/T, the effect of the compressive stress due to bending is to reduce the principal tension due to torsional shear and thereby the beam is generally able to sustain comparatively high torsional moment in spite of a decrease in the effective depth due to bending cracks in the section. Thus the resistant torsional moment $[T_u]_{RC}$ of reinforced concrete beam may be calculated by

$$[T_u]_{RC}=[T_u]_P^*+\boxed{T_v} \cdots\cdots\cdots\cdots\cdots\cdots (26)$$

where

$\boxed{T_v}$: apparent resisting torsional moment due to shear reinforcement

$[T_u]_P^*$: resisting torsional moment of the apparent uncrackd section where the bending cracks have not penerated into yet

It is assumed that the failure of concrete is determined by the criterion of a constant maximum tensile stress given by eq. (16).

The distribution of normal stress due to bending on the uncracked section varies from tensile stress σ_{tu} at the lower edge to compressive stress σ_c at the upper fibre. In this region, although the normal stress distribution is not uniform, the apparent tensile strength τ is, by referring to eq. (16), assumed to be

$$\tau=\sigma_{tu}\sqrt{1+\frac{1}{\sigma_{tu}}\left(\frac{\sigma_{cu}}{2}\frac{[M_u]_{RC}}{[M_{0u}]_{RC}}\right)} \cdots\cdots (27)$$

Further, let assume the depth (jd_0) of the uncracked section is equal to

$$j_{d_0}=\left(1-\frac{[M_u]_{RC}}{[M_{0u}]_{RC}}\right)d_0 \leq \frac{1}{2}d_0 \cdots\cdots\cdots (28)$$

and $[T_u]_P^*$ is given by eq. (29)

$$[T_u]_P^*=\tau/K_2' \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (29)$$

where

$$K_2'=\left\{\frac{1}{2}b_0^2\left(jd_0-\frac{1}{3}b_0\right)\right\}^{-1}$$

Next, $\boxed{T_v}$ may be calculated by eq. (30) assuming the stirrup is yielding at the time of beam failure.

$$\boxed{T_v}=\boxed{T_v}^*=\frac{1.6\,b'd'a_{sv}\sigma_{sy}}{s} \cdots\cdots\cdots (30)$$

Quite similarly for uniformly prestressed reinforced concrete beam, the ultimate strength $[T_u]_{PRC}$ can be calculated by eq. (31).

$$[T_u]_{PRC}=\frac{\sigma_{tu}}{K_2'}\sqrt{1+\frac{1}{\sigma_{tu}}\left(\sigma_{PS}+\frac{\sigma_{cu}}{2}\frac{[M_u]_{PRC}}{[M_{0u}]_{PRC}}\right)} +\boxed{T_v}^* \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (31)$$

Bending Failure

For large ratios of M/T, the diagonal compression due to torsional shear increases the direct compression due to bending so that the effect of the addition of torsion is to reduce the bending strength. And, as bending effects predominate, the ultimate strength of beam is not so remarkably affected by the shear reinforcement as seen in Fig. 4 (j), as far as the appropriate shear reinforcement is provided.

The effective cross section which may resist to torsional moment at the time of beam failure is naturally smaller than the original full section, but it can not be determined numerically now. Now let make following three assumptions;

(1) Failure occurs on the compression face, and is governed by the internal friction theory expressed by eq. (17).

(2) In calculating the shearing stress τ by eq. (17) the normal compressive stress σ_c due to bending $[M_u]_{RC}$ at the compressive fibre is given by

$$\sigma_c=\sigma_{cu}\frac{[M_u]_{RC}}{[M_{0u}]_{RC}}$$

where $[M_{0u}]_{RC}$: strength of beam in pure bending.

Thus, τ is expressed as follows

$$\tau=\sigma_{cu}\sqrt{0.04+0.12\frac{[M_u]_{RC}}{[M_{0u}]_{RC}}-0.159\left(\frac{[M_u]_{RC}}{[M_{0u}]_{RC}}\right)^2} \cdots\cdots\cdots\cdots\cdots (32)$$

(3) Considering the combined action of the uncracked concrete and the reinforcement, the eqs. (9) and (10) which were used to calculate the torsional strength in pure torsion by the elastic theory, can also be used to determine the shearing strength τ at the time of failure of beam subjected to predominant bending moment.

Then the resisting torsional moment $[T_u]_{RC}$ of beam is shown by

$$[T_u]_{RC}=\tau/K_1 \quad \cdots\cdots(33)$$

Eqs. (32) and (33) show the conditons of beam failure under large ratios of M/T.

Quite similarly for uniformly prestressed reinforced concrete beam, the ultimate strength $[T_u]_{PRC}$ is given by

$$[T_u]_{PRC}=\frac{\sigma_{cu}}{K_1}\sqrt{0.04+0.12\,k-0.159\,k^2} \quad \cdots\cdots(34)$$

where

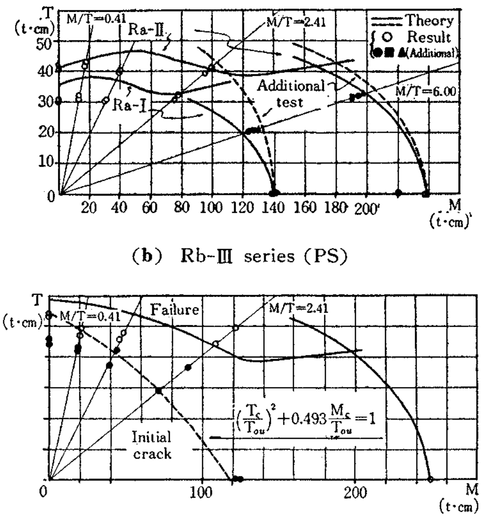
$$k=\frac{1}{\sigma_{cu}}\left\{\sigma_{ps}+(\sigma_{cu}-\sigma_{ps})\frac{[M_u]_{PRC}}{[M_{0u}]_{PRC}}\right\}$$

Calculated ultimate strengths of beams of Ra-I, Ra-II, Rd-III and the additional test series are compared with the measurements and illustrated in Figs. 15 (a) and (b). The figures show that the agreement between both values is fairly good although the theoretical ones are derived under poor assumptions.

Strictly speaking, between the torsional and bending failure considered above there is a transition stage. Failure in this intermediate stage is very complicated and is needed to be studied further.

Fig. 15 Strength of beams for combined load

(a) Reinforced concrete.



6. SUMMARY

Totally eighty rectangular plain, prestressed and reinforced concrete beams were tested in pure torsion, as well as in combined bending and torsional moment to determine the visco-elastic limit and the ultimate strength. Strain gages and dial gages were used to obtain the angle of rotation or the apparent modulus of rigidity of beam in torsion. The results of these tests may be summarized as follows;

(1) The measuring method of angle of rotation by electric resistant gages selected in this test is superior to that by dial gages so far as the member behaves still elastically.

(2) The maximum spacing of stirrup in the rectangular reinforced concrete member subjected to small ratio of M/T is to be limited to one half of the depth.

(3) The conventional apparent modulus of rigidity to be used for calculating angle of rotation of section is almost identical in both plain and reinforced concrete member, but is increased by introducing prestress. The reason for this phenomenon is to be cleared by further study.

(4) Cowan's equation of calculating torsional resistant moment due to shear reinforcement (eq. (12)) can be applied to failure of member, though this equation was derived from the elastic energy theory.

(5) The ultimate torsional strength of plain concrete member calculated by assuming concrete to be plastic agrees with the test result in pure torsion.

(6) As for the ultimate strength of plain and prestressed concrete beams, the interaction curve of torque and bending moment is clearly represented by a adequate parabolic relationship.

(7) The ultimate strength of reinforced concrete and prestressed reinforced concrete beams in combined bending and torsion can be estimated fairly well by eqs. (26) and (31) or eqs. (32), (33) and (34) suggested here. However more studies are needed, especially for the transition stage of failure.

Acknowledgement

This project was made possible by a grant given to one of the authors from the Ministry of Education, and the tests were carried in the Civil Engineering Laboratory of Kyoto University.

The authors wish to express their thanks to the stuffs and the research fellows in the laboratory, in particular to Mr. T. Yoshitake (B.S.), for their help in preparing and testing of the specimens.

Reference

1) Timoshenko: "Theory of elastity." McGRAW-HILL 1951, 2nd edit. pp. 275-278
2) Arockiasamy; "Torsional strength (elastic and plastic) of reinforced concrete members." Indian Concrete Journal, Nov. 1964, pp. 433-440
3) Paul Zia; "Torsional strength of prestressed concrete members." Dr. Dissertation, Univ. of Florida (1960)
4) Nadai; "Theory of flow and fracture of solids." Mc GRAW-HILL, 1950, pp. 490-500
5) H.J. Cowan; An elastic theory for the torsional strength of rectangular reinforced concrete beams." Magazine of Concrete Research, July 1950, pp. 3-8
6) H.J. Cowan; "The theory of torsion applied to reinforced concrete design." (Part 1) Magazine of Concrete Research, Sept. 1953, pp. 827-829 (continued)
7) H.J. Cowan; "The strength of plain, reinforced and prestressed concrete under the action of combined stresses, with particular reference to the combined bending and torsion of rectangular sections." Magazine of Concrete Research, Dec. 1953, pp. 75-86
8) L.E. Payne; "Torsion of composite sections. "Iowa State College, Journal of Science (1949) pp. 381-395
9) H. Gesund et al; "Ultimate strength in combined bending and torsion of concrete beams containing both longitudinal and transverse reinforcement", Journal of A. C. I. Dec. 1964, pp. 1950-1522

(*Received Dec. 2, 1965*)

昭和 41 年 7 月 15 日 印 刷
昭和 41 年 7 月 20 日 発 行

土木学会論文集 第 131 号 定価 200 円 (〒 20 円)

編集兼発行者 東京都新宿区四谷一丁目 社団法人 土 木 学 会 羽 田 巌
印 刷 者 東 京 都 港 区 赤 坂 1—3—6 株式会社 技 報 堂 大 沼 正 吉

発 行 所 社団法人 土 木 学 会 振替東京 16828 番
東京都新宿郵便局区内 新 宿 区 四 谷 一 丁 目 電話 (351) 代表 5138 番

昭和三十七年五月二十八日第三種郵便物認可
昭和四十一年七月十五日印刷
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