ERROR IN FINITE DIFFERENCE SOLUTIONS OF LOCAL BUCKLING STRENGTH*

*By Fumio Nishino***

ABSTRACT

The paper presents a summary of analyses of the error which is inherent in a solution of local buckling strength obtained by the finite difference method. Approximate amounts of error which appear in the final results of this buckling analysis are explored on each source of error. Basic difference error was illustrated analytically for a simply supported plate at the unloaded edges. Similarly, the error due to approximate representation of boundary conditions was analyzed on a plate fixed at the two unloaded edges. Error limits by concentrating the non-uniform stress distribution at a mesh point were explored. It was concluded that the error present in a solution of this buckling problem by the finite difference method can be estimated.

1. INTRODUCTION

1.1 Objective of the Paper

The strength of steel columns has been investigated in great deal and variety by many investigators, where the residual stresses were introduced as the main factor influencing the buckling strength of centrally loaded columns defined by the Euler load and the yield load. As a rule, most of the cross sections of steel columns consist of plate elements. It is possible, therefore, that even before instability of a colnmn takes place, the component plates may buckle locally.

The effect of residual stress on the elastic buckling of steel plates were studied and presented in Refs. 1 and 2, where the analyses were made on elastic plates with the perturbation method and with the aid of integral equations, respectively. The analysis was further developed into the inelastic range, where it was shown that integral equations could be used to solve the inelastic buckling problem[3]. Analytical solutions were obtained in Ref. 3 for simply supported, fixed and elastically restrained plates at the unloading edges, and simply supported at the loading edges, together with a numerical solution for simply supported plates with a particular distribution of residual stresses. The existence of residual stresses makes it difficult to obtain analytical solutions of local buckling from two reasons; non-uniform stress distribution and partial yielding which may take place at a certain loading due to the existence of compressive residual stress. The author obtained solutions for local buckling strength of steel colnmns with varieties of residual stress distributions by a finite difference approximation of a differential equation[4]. The bonndary conditions considered are simply supported at the two loading edges and at the other two edges which are free of loading, they are simply supported, fixed, free and elastically restrained for rotation.

The finite difference method has first been suggested by Richardson[5] for the determination of the eigenvalue, investigated independently by Collatz[6] and it was presented in a very broad and clear manner by Salvadori[7,8] with examples on the buckling of columns and plates and with brief discussions on the error involved. With the use of a large scale digital computer of greatly improved speed and computational efficiency, the finite difference method is one of the easiest approach to the problem governed by a differential equation if a rigorous solution

---

* The paper was read at the 23rd annual meeting of Japan National Congress for Applied Mechanics at Univ. of Tokyo on September 1965.

** Lecturer, Structural Testing Laboratory, Engineering Research Institute, Faculty of Engineering, University of Tokyo; Formerly, Research Instructor, Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pa., U.S.A.

is a quite difficult task. One of the defects is the error inherent in the method. While the energy method, for instance, results in very small error in safe side for a buckling problem, very little is known on the error in a finite difference solution.

The paper presents an analysis of the error existing in this particular plate buckling problem dealt by the author[4]. Nevertheless, a similar analysis could be applied on other problems. With the analysis, it was possible to guarantee the required accuracy in numerical solutions with the minimum computing time. This paper is based on a dissertation[9] to which reference may be made for detailed information.

1.2 Analysis of Plate Buckling by Finite Difference Method

The governing equation of plate buckling which is applicable both in the elastic and in the inelastic domains of a plate is a forth order partial differential equation with variable coefficients. Analysis of local buckling of a column can be made under the conditions that the thrust is applied at the two opposite edges of a plate element in the middle plane and that the residual stress is present only along the same direction as the thrust and its magnitude is constant in that direction[4]. Taking the coordinate system as shown in Fig. 1, the basic equation can be shown by Eq. 1[4)10)].

$$E\left[\frac{\partial^2}{\partial z^2}\left(Ik_1\frac{\partial^2 w}{\partial z^2}+Ik_2\frac{\partial^2 w}{\partial y^2}\right)+4\frac{\partial^2}{\partial z\partial y}\left(Ik_4\frac{\partial^2 w}{\partial z\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(Ik_2\frac{\partial^2 w}{\partial z^2}+Ik_3\frac{\partial^2 w}{\partial y^2}\right)\right]+t\sigma_z\frac{\partial^2 w}{\partial z^2}=0 \quad \cdots\cdots(1)$$

where k_1 to k_4 are defined as follows by Poisson's ration ν, tangent modulus E_t and by the secant modulus E_s.*

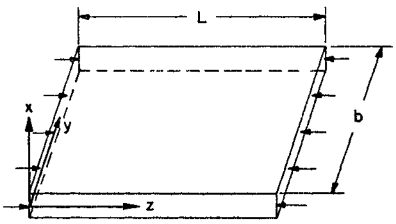

Fig. 1 Coordinate Axes for Analysis of Local Buckling.

$$\left.\begin{aligned}k_1&=\frac{1+3\left(\frac{E_t}{E_s}\right)}{(5-4\nu+3e)-(1-2\nu)^2\left(\frac{E_t}{E}\right)}\\ k_2&=\frac{2-2(1-2\nu)\left(\frac{E_t}{E}\right)}{(5-4\nu+3e)-(1-2\nu)^2\left(\frac{E_t}{E}\right)}\\ k_3&=\frac{4}{(5-4\nu+3e)-(1-2\nu)^2\left(\frac{E_t}{E}\right)}\\ k_4&=\frac{1}{2+2\nu+3e}\\ e&=\frac{E}{E_s}-1\end{aligned}\right\} \quad \cdots\cdots(2)$$

E, w, t are the modulus of elasticity, deflection of a plate and thickness, respectively. I is the moment of inertia of the plate defined by its thickness

$$I=\frac{t^3}{12} \quad \cdots\cdots(3)$$

In the analysis, the deflected shape of the plate is assumed by the following product function which satisfies the boundary conditions at the loading edges[4]

$$w=Y\sin\frac{\pi}{L}z \quad \cdots\cdots(4)$$

where Y is a function of the coordinate y alone and L is the buckling length of the plate as shown in Fig. 1. Substituting Eq. 4 into Eq. 1 and dividing by a constant I_0, the basic differential equation can be shown finally in the following form

$$\frac{d^2}{dy^2}\left(\frac{I}{I_0}k_3\frac{d^2Y}{dy^2}-\frac{\pi^2}{L^2}\frac{I}{I_0}k_2Y\right)-4\frac{\pi^2}{L^2}\frac{d}{dy}\left(\frac{I}{I_0}k_4\frac{dY}{dy}\right)-\frac{\pi^2}{L^2}\frac{I}{I_0}k_2\frac{d^2Y}{dy^2}+\frac{\pi^2}{L^2}\left(\frac{\pi^2}{L^2}\frac{I}{I_0}k_1-\frac{t\sigma_z}{EI_0}\right)Y=0 \quad \cdots\cdots(5)$$

A finite difference analogue of a differential equation can be obtained directly from the given differential equation by replacing the derivatives with difference quotients. When the coefficients of the differential equation are not constant,

* There are two main current trends in the development of inelastic buckling of plates, one based on Henkey's total strain theory and the other on Prandtl-Reuiss' incremental theory. Eq. 2 is the results by the total strain theory, however, substitution of e in Eq. 2 by 0 corresponds to the results by the incremental theory.

average values replace the variable coefficients. It is also necessary to express boundary conditions by finite differences in solving the difference equations.

It is customary and convenient to end a plate on a mesh point(the integer stations) or at the middle of mesh points (the half-integer stations). The boundary conditions for fixed, simply supported, free ends and so on are similarly obtained by replacing the expressions of boundary conditions by finite difference quotients and are summarized in Table 1 for simply supported and fixed ends. The finite difference expressions at the simply supported edge are obtained by the following relationship at the mesh point i

$$\left(\frac{d^2Y}{dy^2}\right)_i=\frac{1}{r^2}(Y_{i+1}-2Y_i+Y_{i-1})+E_i \qquad \cdots\cdots\cdots\cdots\cdots(5)$$

Table 1 Boundary Conditions for Fixed and Simply Supported Edges.

End Condition	For Differential Equation	For Difference Equation	
		End on Integer Station i	End on Half-Integer Station $i+\frac{1}{2}$
Fixed	$Y=0$	$Y_i=0$	$Y_i=0$
	$\frac{dY}{dy}=0$	$Y_{i+1}=Y_{i-1}$	$Y_{i+1}=0$
Simply Supported	$Y=0$	$Y_i=0$	$Y_{i+1}=-Y_i$
	$\frac{d^2Y}{dy^2}=0$	$Y_{i+1}=-Y_{i-1}$	$Y_{i+2}=-Y_{i-1}$

where r is the width of the mesh. E_i is the error involved and is of the following type for an evenly spaced mesh[(7)].

$$E_i=a_1r^2\frac{d^4Y}{dy^4}+a_2r^4\frac{d^6Y}{dy^6}+\cdots \qquad \cdots\cdots\cdots(6)$$

where a_1, a_2, and so on are constants. It is noted that the forth derivative of Y, the sixth' and so on are equal to zero so that no error is involved for the finite difference expressions of boundaries at the simply supported edges.

Finite difference equations considered at each mesh point form homogeneous simultaneous equations together with suitable boundary conditions. The approximate buckling strength is obtained as an eigenvalue of these equations, which can be determined in such a way that a non-trivial solution exists.

2. ERROR IN FINITE DIFFERENCE SOLUTIONS

It is quite difficult to analyze the error which is inherent in finite difference solutions, when the finite difference method is used as an approximate method for a problem governed by the differential equation; thus a general discussion is out of the scope of this paper. In this paper, however, sources of error are briefly reviewed and then approximate amounts of error which appear in the final results of this buckling problem are explored analytically as well as by comparing the results of the finite difference method with the exact solutions or with any other solutions which can be used for the comparison. The error, thus evaluated, would give some kind of index for the accuracy of the approximate solutions, for which no exact solution is available.

The main sources of errors are:

(1) Basic difference error which is due to the basic difference approximation to the derivatives in basic differential equations.

(2) Boundary condition error which is due to the representation of boundary conditions in the difference form.

(3) Averaging error of variable coefficients in the basic differential equation.

(4) Round off error.

2.1 Basic Difference Error

In general, the error terms in finite difference quotients of the i-th derivative is of the following type for an evenly spaced mesh[7)],

$$E_i=f_2(Y)r^2+f_4(Y)r^4+f_6(Y)r^6+\cdots \quad (7)$$

where the function $f_n(Y)$ is the product of a constant and the $(n+2)$'th derivative of the original function Y, and thus it is independent of r.

Equation 7 shows that when the Taylor series of $Y(y)$ is converging rapidly (small values of r and rapidly decreasing values of the successive derivatives of $Y(y)$), the series in the error expression will also converge rapidly.

The main interest of this paper is the error present in the eigenvalue itself, which is obtained by solving the finite difference equation. Thus the error of this problem is not identical with that shown in Eq. 7, although some kind of relationship may exist.

Salvadori has presented a theorem which is of importance in evaluating the error in the eigenvalue[7]. The theorem states as follows:

"The error in the eigenvalue of linear ordinary differential equations with constant coefficient evaluated by finite differences is of the r^2-type."

It should be noticed in the theorem that no error in the expressions of boundary conditions is considered. Nonetheless, his analysis proves that the basic difference error in the eigenvalue is of the same type as the error in the finite difference quotients and that the error diminishes with increasing number of cells.

It has been pointed out previously that there is no error in representing boundary conditions in difference form at simply supported edges. Thus, the entire error in a solution of a plate, which consists of these boundaries only, is due to the basic difference approximation, provided that the plate is free of residual stress and that the plate is compressed uniformly so as to be free of averaging error and thus to satisfy the condition of constant coefficients.

To illustrate the basic difference error in critical width-thickness ratio, consider a simple case; a plate simply supported at both the unloaded edges. The analytical expression of error term can be obtained for this case and it may suggest the characteristic of the error for the other cases.

The difference quotients which replaces the derivatives in the basic differential equation can be considered identical with the derivatives if the error terms are included. Under the assumption that the Taylor series of $Y(y)$ is converging rapidly, the first error term may represent sufficiently the error involved in the quotients. In this meaning, the relationship between derivatives and differential quotients with first error term included is not an approximation, but an identity.

The basic differential equation, Eq. 5 can be simplified as follows, when a plate free of residual stress is compressed uniformly

$$k_3\frac{d^4Y}{dy^4}-2\frac{\pi^2}{L^2}(k_2+2k_4)\frac{d^2Y}{dy^2}+\frac{\pi^2}{L^2}\left(\frac{\pi^2}{L^2}k_1-\frac{12\sigma_z}{Et^2}\right)Y=0 \quad\cdots\cdots(8)$$

of which the solution is given in Ref. 11 as

$$Y=C_4\sin\beta y \quad\cdots\cdots(9)$$

$$\beta=\frac{\pi}{b} \quad\cdots\cdots(10)$$

$$\lambda_e^2=\frac{\pi^2}{12}\left(\frac{\sigma_Y}{\sigma_z}\right)\left[\frac{L^2}{b^2}k_3+2(k_2+2k_4)+\frac{b^2}{L^2}k_1\right]=0 \quad\cdots\cdots(11)$$

where C_4 is a constant, b is the width of the plate and σ_Y is the static yield strength of the material. λ is the non-dimensionalized width-thickness ratio defined by

$$\lambda=\frac{b}{t}\sqrt{\frac{\sigma_Y}{E}} \quad\cdots\cdots(12)$$

The subscript e to λ shows that it is the exact solution. Replacement of the derivatives with difference quotients with error terms included does not change the equation and consequently the following equation must have the same solution as Eq. 8.

$$k_3\left[\frac{1}{r^4}(Y_{i+2}-4Y_{i+1}+6Y_i-4Y_{i-1}+Y_{i-2})-\frac{r^2}{6}\left(\frac{d^6Y}{dy^6}\right)_i\right]-2\frac{\pi^2}{L^2}(k_2+2k_4)\times\left[\frac{1}{r^2}(Y_{i+1}-2Y_i+Y_{i-1})-\frac{r^2}{12}\left(\frac{d^4Y}{dy^4}\right)_i\right]+\frac{\pi^2}{L^2}\left(\frac{\pi^2}{L^2}k_1-\frac{12\sigma_z}{Et^2}\right)Y_i=0 \quad\cdots\cdots(13)$$

Substituting the known solution of Eq. 8, that is Eqs. 9, 10 and 11, into Eq. 13, the following relation is obtained

$$\frac{k_3}{r^4}(Y_{i+2}-4Y_{i+1}+6Y_i-4Y_{i-1}+Y_{i-2})-2\frac{\pi^2}{L^2}\frac{(k_2+2k_4)}{r^2}(Y_{i+1}-2Y_i+Y_{i-1})+\frac{\pi^4}{L^4}k_1Y_i=\left\{12\cdot\frac{\pi^2}{L^2b^2}\cdot\frac{\sigma_z}{\sigma_Y}\lambda_e^2-\frac{b^2}{6n^2}\left[k_3\beta^6+\frac{\pi^2}{L^2}(k_2+2k_4)\beta^4\right]\right\}Y_i \quad\cdots\cdots(14)$$

where n is the number of mesh cells in the width of the plate.

The difference equation corresponding to Eq. 8 is obtained by replacing the derivatives of

Eq. 8 by the difference quotients excluding the error terms, thus

$$\frac{k_3}{r^4}(Y_{i+2}-4Y_{i+1}+6Y_i-4Y_{i-1}+Y_{i-2})$$
$$-2\frac{\pi^2}{L^2}\frac{(k_2+2k_4)}{r^2}(Y_{i+1}-2Y_i+Y_{i-1})$$
$$+\frac{\pi^4}{L^4}k_1Y_i=12\frac{\pi^2}{L^2b^2}\frac{\sigma_z}{\sigma_Y}\lambda^2 Y_i \quad\cdots\cdots\cdots(15)$$

Since it is known that the solution Y of an ordinary difference problem with constant coefficients is idential with solution Y of the corresponding differential problem[13], the eigenfunction Y_i in Eq. 14 is identical with that in Eq. 15. Therefore, the left hand sides of Eqs. 14 and 15 are identical and hence the right hand sides can be equated

$$12\frac{\pi^2}{L^2b^2}\frac{\sigma_z}{\sigma_Y}\lambda^2=12\frac{\pi^2}{L^2b^2}\frac{\sigma_z}{\sigma_Y}\lambda_e^2$$
$$-\frac{\pi^4}{6n^2b^2}\left[k_3\frac{\pi^2}{b^2}+\frac{\pi^2}{L^2}(k_2+2k_4)\right]$$
$$\cdots\cdots\cdots\cdots\cdots(16)$$

Finally Eq. 16 leads to the following form, so as to be able to evaluate the error term

$$\lambda=\lambda_e\left\{1-\frac{\pi^2}{12}\frac{\frac{L^2}{b^2}+\frac{k_2+2k_4}{k_3}}{\frac{L^2}{b^2}+2\frac{k_2+2k_4}{k_3}+\frac{\pi^2}{L^2}\cdot\frac{k_1}{k_3}}\left(\frac{1}{n^2}\right)\right\}$$
$$\cdots\cdots\cdots\cdots\cdots(17)$$

where λ is the width-thickness ratio by the finite difference method and λ_e is the exact value of the width-thickness ratio as shown in Eq. 11. From Eq. 17, the error term is shown to be

$$E_r=-\frac{\pi^2}{12}\left\{\frac{\frac{L^2}{b^2}+\frac{k_2+2k_4}{k_3}}{\frac{L^2}{b^2}+2\cdot\frac{k_2+2k_4}{k_3}+\frac{b^2k_1}{L^2k_3}}\right\}\left(\frac{1}{n^2}\right)$$
$$\cdots\cdots\cdots\cdots\cdots(18)$$

Nothing that r is proportional to the reciprocal of the number of mesh cells, n, it is seen that the error is of the r^2-type and tends toward zero for increasing number of mesh cells. Since the error term is negative, the finite difference solution gives smaller width-thickness ratios. The error is also a function of the aspect ratio and it increases when a longer plate is considered, and approaches the following value for the infinitely long plate.

$$E_r=-\frac{\pi^2}{12}\left(\frac{1}{n^2}\right)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(19)$$

The error is constant for the elastic buckling of a plate no matter what the thrust is. To the contrary in the inelastic range, the coefficients, k's, are the function of thrust according to the total strain theory of plasticity and consequently a function of loading. Nevertheless, the variations of numerical value of k's are not expected to be large as is understood by their definition, Eq. 2, and similarly for the variation of errors under different loading. Table 2 shows the variation of errors for different critical stresses computed for a mesh of 10 cells, which confirms that the difference in errors for different buckling stresses is negligibly small for practical purpose. Figure 2 shows the basic difference error and aspect ratio relationship for the elastic buckling of simply supported plates at the unloaded edges.

The width-thickness ratios are computed for the plate changing the width of mesh. As pointed out previously, two mesh systems are possible; one in which the ends of the plate occur at the integer stations and the other at the half-integer stations. Since both meshes introduce no error in representing the boundary conditions by finite difference equations, it is expected that both methods result in the same amount of error for the same width of mesh.

The results for the plate by two different mesh systems are tabulated in Table 3 and are plotted in Fig. 3. The table also includes the

Table 2 Variation of Basic Difference Errors for Different Critical Stresses and Aspect Ratios (Simply Supported Plates at Unloaded Edges)

Aspect Ratio	Basic Difference Error %			
	Elastic Buckling	Inelastic Buckling* (Total Strain Theory)		
		$\frac{\varepsilon_{cr}}{\varepsilon_Y}=1.0$	$\frac{\varepsilon_{cr}}{\varepsilon_Y}=2.0$	$\frac{\varepsilon_{cr}}{\varepsilon_Y}=4.0$
0.5	−0.16449	−0.32813	−0.32220	−0.31926
1.0	−0.41123	−0.49433	−0.50026	−0.50321
1.5	−0.56940	−0.59361	−0.60347	−0.60824
2.5	−0.70902	−0.70304	−0.71152	−0.71549
5.0	−0.79083	−0.78532	−0.78872	−0.79026

* The errors in solutions by the incremental theory of plasticity are functions of only aspect ratio and coincide with those for $\frac{\varepsilon_{cr}}{\varepsilon_Y}=1.0$ in the Table.

NOTE : The above table is computed for a mesh with 10 cells.

Table 3 Errors in Width-Thickness Ratio by Finite Difference Solusion (Simply Supported Plate at Unloaded Edges and Square Tube; Free of Residual Stress)

Number of Cells	Solutions by Finite Difference Method						Error by Eq. 18
	Ends on Ineger Stations		Ends on Half-Integer Stations		Square tube		
	λ	Error %	λ	Error %	λ	Error %	
8	2.671 7640	−0.639 32	2.671 7640	−0.63932	2.671 7644	−0.639 30	−0.642 55
9	2.675 3572	−0.505 69	— —	—	— —	—	−0.507 70
10	2.677 9306	−0.409 99	2.677 9306	−0.40999	2.677 9305	−0.409 99	−0.411 23
11	2.679 8370	−0.339 09	— —	—	— —	—	−0.339 86
12	2.681 2861	−0.285 19	2.681 2865	−0.28519	2.681 2868	−0.285 12	−0.285 58
13	2.682 4168	−0.243 15	— —	—	— —	—	−0.243 33
14	2.683 3109	−0.209 89	2.683 3109	−0.20989	2.683 3152	−0.209 74	−0.209 81
15	2.684 0285	−0.183 21	— —	—	— —	—	−0.182 77
16	2.684 6130	−0.161 47	2.684 6127	−0.16149	2.684 6139	−0.161 44	−0.160 64
17	2.685 1063	−0.143 13	— —	—	— —	—	−0.142 30
18	2.685 5099	−0.128 12	2.685 5090	−0.12815	2.685 5158	−0.127 80	−0.126 92
19	2.685 8522	−0.115 39	— —	—	— —	—	−0.113 91
20	2.686 1607	−0.103 92	2.686 1598	−0.10395	2.686 1647	−0.103 77	−0.102 81
30	2.687 5173	−0.053 47	2.607 5117	−0.05367	— —	—	−0.045 69
Exact	2.688 9550	±0.000 01	2.688 9550	±0.00001	2.688 9550	±0.000 01	

NOTE: The non-dimensionalized width-thickness ratios are for the case

$$\frac{\sigma_{cr}}{\sigma_Y}=0.5 \quad \text{and} \quad \frac{L}{b}=1$$

The errors are constant for elastic buckling of the plate with the same aspect ratio.

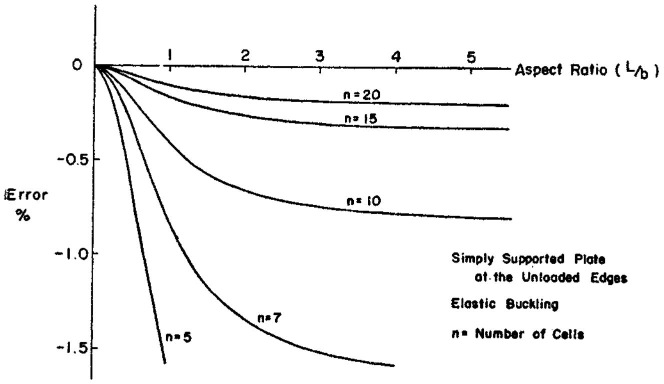

Fig. 2 Basic Difference Error and Aspect Ratio Relationship.

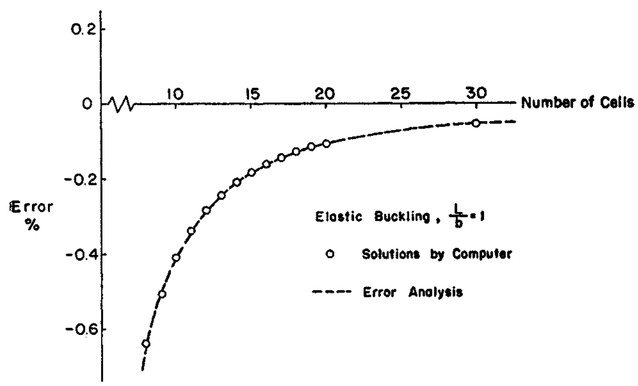

Fig. 3 Basic Difference Error S.S. Plates at Unloaded Edges.

expected errors computed by Eq. 18 and the numerical results for local buckling of a square box-column with four idential component plates, where, again, no boundary error is involved and thus the same results with simply supported plates are expected. All numerical results for three different cases are in good agreement with each other and with the expected errors by error analysis.

2.2 Boundary Condition Error

It has been seen that an analysis of simply supported plates at the unloaded edges with various numbers of cells directly gives the error due to the basic difference approximation. In general cases, the solutions by finite difference methods include error due to the approximate representation of boundary conditions in difference form in addition to the error due to basic difference approximation.

The error present in the solution of the finite difference equation due to approximate representation of boundary conditions must be the same as the error which is found in a solution of original differential equation with the same boundary conditions used to solve the finite difference equation. Therefore, if the original differential equation can be solved with the same boundary conditions used for the finite difference equation, the error may be evaluated comparing the result thus obtained with the exact solution.

Consider a plate fixed at the unloaded edges to see how the error behaves. The general solution of the basic differential equation, Eq. 8, is given by Eq. 20[11),14)].

$$Y = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 \cos \beta y + C_4 \sin \beta y \quad \cdots\cdots (20)$$

where C_1 through C_4 are constants and α and β are defined as

$$\left.\begin{aligned} \alpha &= \sqrt{\sqrt{\frac{(k_2+2k_4)^2-k_1k_3}{k_3^2}\cdot\frac{\pi^2}{L^2}+\left(\frac{12}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)\left(\frac{\sigma_z}{\sigma_Y}\right)\lambda^2}+\left(\frac{k_2+2k_4}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)} \\ \beta &= \sqrt{\sqrt{\frac{(k_2+2k_4)^2-k_1k_3}{k_3^2}\cdot\frac{\pi^2}{L^2}+\left(\frac{12}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)\left(\frac{\sigma_z}{\sigma_Y}\right)\lambda^2}-\left(\frac{k_2+2k_4}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)} \end{aligned}\right\} \quad \cdots\cdots (21)$$

By substituting the above solution into the boundary condition at the fixed edges, the condition that a non-trivial solution exists results in the following transcendental equation

$$\alpha_e \tanh\frac{\alpha_e b}{2} + \beta_e \tan\frac{\beta_e b}{2} = 0 \quad \cdots\cdots (22)$$

where α_e and β_e are defined as

$$\left.\begin{aligned} \alpha_e &= \sqrt{\sqrt{\frac{(k_2+2k_4)^2-k_1k_3}{k_3^2}\cdot\frac{\pi^2}{L^2}+\left(\frac{12}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)\left(\frac{\sigma_z}{\sigma_Y}\right)\lambda_e^2}+\left(\frac{k_2+2k_4}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)} \\ \beta_e &= \sqrt{\sqrt{\frac{(k_2+2k_4)^2-k_1k_3}{k_3^2}\cdot\frac{\pi^2}{L^2}+\left(\frac{12}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)\left(\frac{\sigma_z}{\sigma_Y}\right)\lambda_e^2}-\left(\frac{k_2+2k_4}{k_3}\right)\left(\frac{\pi^2}{L^2}\right)} \end{aligned}\right\} \quad \cdots\cdots (23)$$

The errors due to two different representations of approximate boundary conditions are solved; one corresponds to plate which ends on integer stations for the unloaded edges and the other for plate which ends on half-integer stations. Assume that the width between mesh points is sufficiently narrow, compared with the width of the plate and that the approximate solution of the width-thickness ratio is close to the exact value. By using this assumption, it may be sufficient for the error analysis to consider only the first error term in the series of the Taylor expansion.

The origin of the coordinate y is taken at one edge of the plate; then the boundary conditions for the differential equation are given at $y=0$ and $y=\frac{b}{2}$, where b is the width of the plate.

ENDS ON INTEGER STASIONS

The approximate boundary conditions are shown in Table 1, from which they are shown for the case

$$\left.\begin{aligned} Y_{y=0} &= 0 \\ Y_{y=r} &= Y_{y=-r} \end{aligned}\right\} \quad \cdots\cdots (24)$$

and the other conditions are symmetrical at $y=b/2$, where no error is involved. By substituting the general solution, Eq. 20, into the boundary conditions, Eq. 24, the condition that a non-trivial solution exists results in the following transcendental equation which corresponds to the exact solution, Eq. 22

$$\sinh \alpha r \tanh\frac{\alpha b}{2} + \sin \beta r \tan\frac{\beta b}{2} = 0 \quad \cdots (25)$$

Assume that both the exact solution λ_e from Eq. 22 and the approximate solution λ from Eq. 25 are known and the following relation exists

$$\lambda_e = \lambda(1+E_{rb}) \quad \cdots\cdots (26)$$

where E_{rb} is the error due to approximate boundary conditions. Noting that E_{rb} is small compared with unity, substitution of Eq. 26 and 23 into Eq. 21 results in

$$\left.\begin{aligned} \alpha &= \alpha_e\left\{1+\left(\frac{1}{\alpha_e^2}\right)\frac{\frac{12}{k_3}\cdot\frac{\pi^2}{L^2}\cdot\frac{\sigma_z}{\sigma_Y}\cdot\lambda_e^2}{\frac{\pi}{L}\sqrt{\frac{(k_2+2k_4)^2-k_1k_3}{k_3^2}+\frac{12}{k_3}\frac{\sigma_z}{\sigma_Y}\lambda_e^2}}\left(\frac{E_{rb}}{2}\right)\right\} \\ \beta &= \beta_e\left\{1+\left(\frac{1}{\beta_e^2}\right)\frac{\frac{12}{k_3}\cdot\frac{\pi^2}{L^2}\cdot\frac{\sigma_z}{\sigma_Y}\cdot\lambda_e^2}{\frac{\pi}{L}\sqrt{\frac{(k_2+2k_4)^2-k_1k_3}{k_3^2}+\frac{12}{k_3}\frac{\sigma_z}{\sigma_Y}\lambda_e^2}}\left(\frac{E_{rb}}{2}\right)\right\} \end{aligned}\right\} \quad \cdots\cdots (27)$$

The second terms in the above equations are the errors in α and β, which will be denoted by E_{rb}'/α_e^2 and E_{rb}'/β_e^2, respectively, where E_{rb}' is defined by

$$E_{rb}'=\frac{\frac{12}{k_3}\frac{\pi^2}{L^2}\frac{\sigma_z}{\sigma_Y}\lambda_e^2}{\frac{\pi}{L}\sqrt{\frac{(k_2+2k_4)-k_1k_2}{k_3^2}+\frac{12}{k_3}\frac{\sigma_z}{\sigma_Y}\lambda_e^2}}\times\left(\frac{E_{rb}}{2}\right) \quad \cdots\cdots(28)$$

and thus

$$\left.\begin{aligned}\alpha&=\alpha_e\left(1+\frac{E_{rb}'}{\alpha_e^2}\right)\\ \beta&=\beta_e\left(1+\frac{E_{rb}'}{\beta_e^2}\right)\end{aligned}\right\} \quad \cdots\cdots(29)$$

With Eq. 29, the terms in Eq. 25 can be expanded into a Taylor series. Taking the first error terms only, the following approximate relationships are obtained

$$\left.\begin{aligned}\tanh\frac{\alpha b}{2}&=\tanh\frac{\alpha_e b}{2}\left[1-\frac{\alpha_e b}{\sinh\alpha_e b}\left(\frac{E_{rb}'}{\alpha_e^2}\right)\right]\\ \tan\frac{\beta b}{2}&=\tan\frac{\beta_e b}{2}\left[1+\frac{\beta_e b}{\sin\beta_e b}\left(\frac{E_{rb}'}{\beta_e^2}\right)\right]\end{aligned}\right\} \quad \cdots\cdots(30)$$

$$\left.\begin{aligned}\sinh\alpha r&=\alpha_e r\left[1+\left(\frac{E_{rb}'}{\alpha_e^2}\right)+\frac{1}{6}(\alpha_e^2r^2)\right]\\ \sin\beta r&=\beta_e r\left[1+\left(\frac{E_{rb}'}{\beta_e^2}\right)-\frac{1}{6}(\beta_e^2r^2)\right]\end{aligned}\right\} \quad \cdots\cdots(31)$$

Substitution of Eqs. 30 and 31 together with Eq. 22 into Eq. 25 results in the following relationship between E_{rb}' and r, from which the boundary condition error is determined as a function of r

$$\frac{r^2}{6}(\alpha_e^2+\beta_e^2)=E_{rb}'\left[\frac{1}{\beta_e^2}\left(1+\frac{\beta_e b}{\sin\beta_e b}\right)-\frac{1}{\alpha_e^2}\left(1+\frac{\sinh\alpha_e b}{\alpha_e b}\right)\right] \quad \cdots\cdots(32)$$

With

$$\alpha_e^2+\beta_e^2=\frac{2\pi}{L}\sqrt{\frac{(k_2+2k_4)^2-k_1k_3}{k_3^2}+\frac{12}{k_3}\frac{\sigma_z}{\sigma_Y}\lambda_e^2}$$

and

$$r=\frac{b}{n}$$

The following expression for the error results

$$E_{rb}=-\frac{2}{3}\left\{\frac{1+\left[\frac{(k_2+2k_4)^2}{k_3}-k_1\right]\frac{1}{12\lambda_e^2}\left(\frac{\sigma_Y}{\sigma_z}\right)}{\frac{1}{\alpha_e^2b^2}+\frac{1}{\alpha_e b\sinh\alpha_e b}-\frac{1}{\beta_e^2b^2}-\frac{1}{\beta_e b\sin\beta_e b}}\right\}\left(\frac{1}{n^2}\right) \quad \cdots\cdots(33)$$

Thus, the error is proportional to the square of the width of the mesh cells. In the elastic range Eq. 33 reduces to

$$E_{rb}=-\frac{2}{3}\left\{\frac{1}{\frac{1}{\alpha_e^2b^2}+\frac{1}{\alpha_e b\sinh\alpha_e b}-\frac{1}{\beta_e^2b^2}-\frac{1}{\beta_e b\sin\beta_e b}}\right\}\left(\frac{1}{n^2}\right) \quad \cdots\cdots(34)$$

ENDS ON HALF-INTEGER STATIONS

The approximate boundary conditions for this case are given in Table 1, from which they are shown for the case

$$Y_{r/2}=Y_{-r/2}=0 \quad \cdots\cdots(35)$$

and the other conditions are symmetrical at $y=b/2$.

Similarly as in the previous case, substitution of the general solutions, Eq. 20, into the boundary conditions results in the following buckling equation

$$\tanh\frac{\alpha r}{2}\tanh\frac{\alpha b}{2}+\tan\frac{\beta r}{2}\tan\frac{\beta b}{2}=0 \cdots(36)$$

Expanding the two new terms in the above equation in the Taylor series

$$\left.\begin{aligned}\tanh\frac{\alpha r}{2}&=\frac{\alpha_e r}{2}\left[1+\left(\frac{E_{rb}'}{\alpha_e^2}\right)-\left(\frac{\alpha_e^2r^2}{12}\right)\right]\\ \tan\frac{\beta r}{2}&=\frac{\beta_e r}{2}\left[1+\left(\frac{E_{rb}'}{\beta_e^2}\right)+\left(\frac{\beta_e^2r^2}{12}\right)\right]\end{aligned}\right\} \quad \cdots\cdots(37)$$

Then, the same procedure as in the previous case results in the following error-number of cells relationship

$$E_{rb}=\frac{1}{3}\left\{\frac{1+\left[\frac{(k_2+2k_4)^2}{k_3}-k_1\right]\frac{1}{12\lambda_e^2}\left(\frac{\sigma_Y}{\sigma_z}\right)}{\frac{1}{\alpha_e^2b^2}+\frac{1}{\alpha_e b\sinh\alpha_e b}-\frac{1}{\beta_e^2b^2}-\frac{1}{\beta_e b\sin\beta_e b}}\right\}\left(\frac{1}{n^2}\right) \quad \cdots\cdots(38)$$

For elastic buckling, the equation reduces to

$$E_{rb}=\frac{1}{3}\left\{\frac{1}{\frac{1}{\alpha_e^2 b^2}+\frac{1}{\alpha_e b\sinh\alpha_e b}-\frac{1}{\beta_e^2 b^2}-\frac{1}{\beta_e b\sin\beta_e b}}\right\}\left(\frac{1}{n^2}\right)\cdots\cdots(39)$$

It is noted that the boundary condition error in this case is one-half the magnitude and in opposite direction, compared with the previous case.

As seen in Eqs. 33 and 38, the boundary condition errors are also r^2-type in both mesh systems in this particular case. It may be noticed that the error in the analysis of the plate, of which the unloaded edges are on half-integer stations, are one-half the magnitude of the error for the other case with an opposite sign. Figure 4 shows the variation of the error under different aspect ratio. The maximum error occurs for an aspect ratio of about 1.75 and decreases with increasing aspect ratio. The variation of the error for different critical stresses in the inelastic range is negligibly small as it is for the basic difference error.

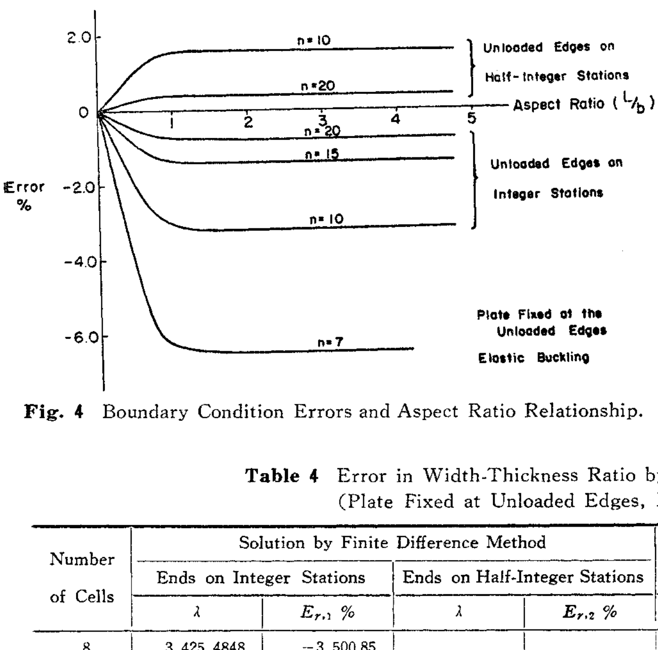

Fig. 4 Boundary Condition Errors and Aspect Ratio Relationship.

The width-thickness ratios were computed by the finite difference method on two different mesh systems and they are compared in Table 4. The errors in the finite difference solutions are due to the sum of errors by the basic difference approximation and by the approximation of boundary conditions. Thus, if the errors due to averaging and due to rounding-off can be neglected, the error involved in this case may be expressed as

$$E_r=E_{rt}+E_{rb}\cdots\cdots(40)$$

where E_{rt} and E_{rb} denote the errors due to the basic finite difference approximation and due to the approximation of boundary conditions, respectively. Since the error E_{rt} is a function only of the width of the mesh for a solution of the plate, the solutions obtained under two different approximations of boundary conditions may make it possible to check the analysis shown in Eqs. 33 and 38. Denoting the error obtained on a mesh in which plate ends on integer stations by subscript 1 and that by the other approximation by sub-

Table 4 Error in Width-Thickness Ratio by Finite Difference Solution (Plate Fixed at Unloaded Edges, Free of Residual Stress)

Number of Cells	Solution by Finite Difference Method				Estimation of Boundary Condition Error		Comparison with Analysis	
	Ends on Integer Stations		Ends on Half-Integer Stations					
	λ	$E_{r,1}$ %	λ	$E_{r,2}$ %	$E_{rb,1}$ %	$E_{rb,2}$ %	$E_{r,1}-E_{r,2}$	$E_{rb,1}-E_{rb,2}$
8	3.425 4848	−3.500 85						
10	3.465 5122	−2.373 24	3.592 9297	1.216 23	−2.621 40	1.310 70	−3.589 47	−3.932 10
12	3.489 2756	−1.703 80	3.580 1354	0.855 80	−1.820 42	0.910 21	−2.559 60	−2.730 63
14	3.504 3854	−1.278 15	3.572 2525	0.633 71	−1.337 45	0.668 72	−1.911 86	−2.006 17
16	3.514 5316	−0.992 32	3.567 0639	0.487 50	−1.023 99	0.511 99	−1.479 88	−1.535 98
18	3.521 6482	−0.791 83	3.563 4735	0.386 42	−0.809 07	0.404 53	−1.178 25	−1.213 61
20	3.526 8222	−0.646 08	3.560 8876	0.313 57	−0.655 35	0.327 68	−0.959 65	−0.983 03
30	3.539 3250	−0.293 87	3.554 6676	0.138 34	−0.145 63	0.145 63	−0.432 21	−0.436 90
Exact	3.549 7565	±0.000 01	3.549 7565	±0.000 01				

NOTE: The non-dimensionalized width-thickness ratios are for a case

$\frac{\sigma_{cr}}{\sigma_Y}=0.5$ and $\frac{L}{b}=0.66$

However, the errors are identical for any elastic buckling of a plate with the same aspect ratio.

script 2, then

$$\left.\begin{array}{l}E_{r,1}=E_{rt,1}+E_{rb,1}\\E_{r,2}=E_{rt,2}+E_{rb,2}\end{array}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(41)$$

If the solutions are obtained for the mesh of the same width, then,

$$E_{rt,1}=E_{rt,2}$$

and thus,

$$E_{r,1}-E_{r,2}=E_{rb,1}-E_{rb,2}\quad\cdots\cdots\cdots\cdots\cdots\cdots(42)$$

The left-hand term of the above equation is tabulated in Table 4 from the actual solutions of finite difference method and it is compared with the right hand term computed by Eqs. 34 and 39. The errors computed by the analysis are slightly larger in magnitude than those present in finite difference solutions as seen in the last two columns in Table 4. The difference is larger for solutions by smaller number of cells and it is decreasing with increasing number of cells. This tendency suggests that the second and the higher order error terms, which have been neglected in the analysis, contribute to the error, since the higher order terms play a bigger role for solutions by coarser mesh systems. Nonetheless, the error analysis gives agreements good enough to conclude that the method is valid for use in exploring the error due to the approximate representation of boundary conditions. Figure 5 shows the error-number of cells relationship for the plate.

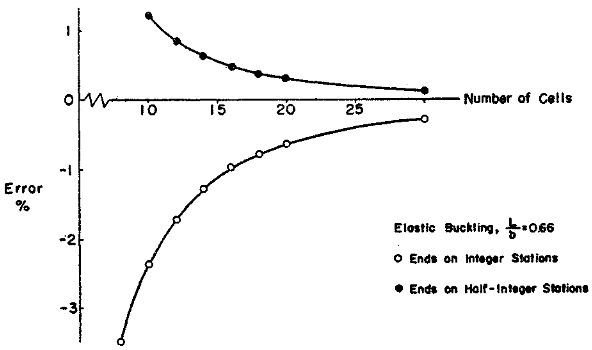

Fig. 5 Error and Number of Cells Relationship Fixd Plate at Unloaded Edges.

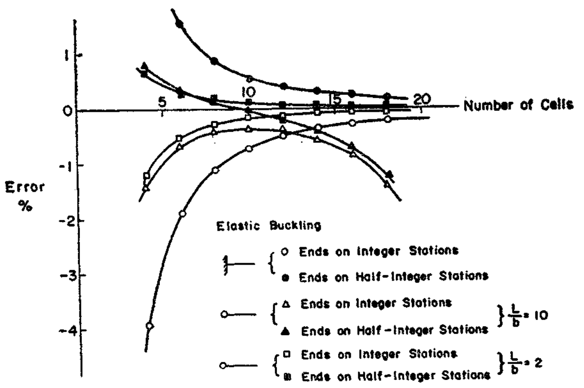

Fig. 6 Error and Number of Cells Relationship Plates Free at One Unloaded Edge and S.S. and Fixed at the Other Edge.

No analysis is made for the other boundary conditions. However, the characteristic of the error-number of cells relationship may be similar. Fig. 6 shows the relationship for the plates free at one of the unloaded edges, and fixed or simply supported at the other unloaded edges. It may be noticed in Fgs. 5 and 6 that the total errors due to basic difference approximation and due to approximate representation of boundary conditions are smaller for the mesh system in which the plates end on half-integer stations in all three cases.

2.3 Averaging Error

In the preceding discussions, only the equation with constant coefficients was considered. When a buckling problem of a plate including residual stress is considered, the coefficients of the basic differential equation are no longer constant. If it comes to the local buckling of columns, the thickness of each plate element may be different and this causes a sudden jump for one of the coefficients at the intersection of the plate elements. Thus, the basic differential equation has variable coefficients and proper assumptions have to be made concerning the averaging of the coefficients. However, the error analysis on this problem is out of the scope of this paper, instead some of the numerical results are presented so that the order of the error involved may be estimated.

Consider first the local buckling of square box sections which consist of two pairs of plates of different thickness. If a mesh system is selected such that the corner of the section is on an integer station, then the moment of inertia is different at both sides of the point and the averaging method comes into the picture. The numerical solutions are obtained for the case, free of residual

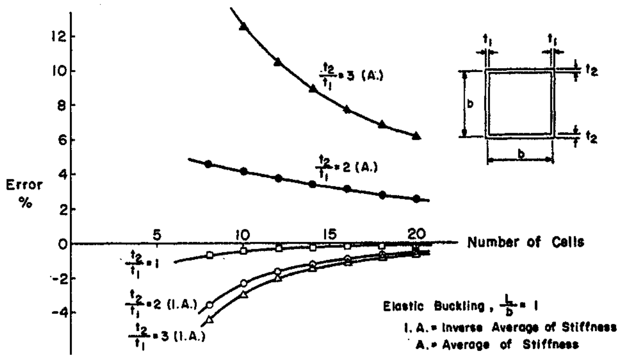

Fig. 7 Difference in Error Due to Different Averaging Methods of Moment of inertia of Plate.

stress and uniformly compressed so as to be able to see only the difference due to different assumptions on the averaging of stiffness. Two different methods are considered; one is to take the average of inverse stiffness defined by Eq. 43 for the average stiffness at a cell,

$$\frac{1}{I_i}=\frac{1}{r}\int_{i-r/2}^{i+r/2}\frac{dy}{I} \quad\cdots\cdots(43)$$

and the other is simply to take the average of stiffness defined by

$$I_i=\frac{1}{r}\int_{i-r/2}^{i+r/2}Idy \quad\cdots\cdots(44)$$

The actual solutions of a finite difference equation with these two different average values of stiffness are compared with exact solutions and the error involved is plotted in Fig. 7. The results are for elastic buckling of square tubes, of which buckling length is equal to the width of the component plates. The difference of errors in both averaging methods are not very significant for small numbers of cells. However, the difference is maked when a finer mesh is used; the inverse average method approaches rapidly to the exact solution, but the simple average method shows quite poor accuracy. Figure 7 also shows that the total error increases when the ratio of the plate thicknesses increases.

It may be understood readily from the last term in the basic differential equation, Eq. 5, that a variation of the term "$t\cdot\sigma_z$" may play an identical role as the reciprocal of variation of moment of inertia of a plate plays for the accuracy of the finite difference solution. Thus the assumption of the averaging of stress as shown in Eq. 44 with I replaced by $t\sigma_z$ is expected to result in good accuracy. Further, it is a natural assumption to concentrate to its center the thrust acting in a cell. However, no exact solution is at hand for the buckling of plates with residual stress, and consequently it is not possible to evaluate the error due to this assumption on the averaging of stress. The consideration of this error problem is limited only to present results for a particular case. The same method, however, can be used for other cases to check the accuracy. A simply supported plate with residual stress distribution as shown in Fig. 8 a is considered as an example.

Several different models are considered such that a bound of error due to the averaging is obtained. These models, the same plates with different residual stress distributions, are shown in Figs. 8 b through 8 f among which Fig. 8 d corresponds to the average model of Fig. 8 a. It is easily understood from the nature of the problem that, when a concentrated thrust is applied at a point as shown in Fig. 8 g, the closer the point locates to the center of the plate, the more severe the effect of the thrust

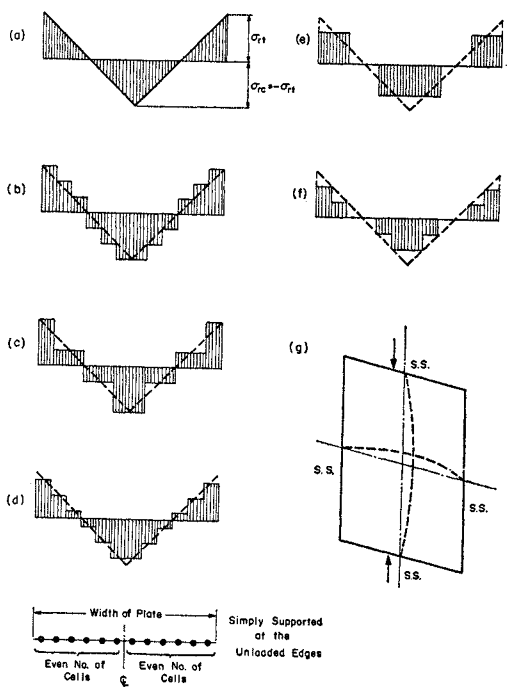

Fig. 8 Sketch of Residual Stress Distribution.

for buckling of the plate will be. With this effect in mind, it is obvious that the model "b" gives the most conservative result and next in line is the model "c". Similarly model "e" results in an unconservative solution and "f" gives the most unconservative result. Although the amount of thrust acting on a mesh point for model "d" is the same as with the plate shown in Fig. 8 a, it may be noticed that the center of thrust of model "d" is exactly on a mesh point in each cell; however, if the original distribution "a" is divided into the same mesh system, the center of thrust is shifted slightly to an edge of the plate from the center of each cell. This difference is the cause of the error. A similar consideration as given previously leads to a conclusion that the model "d" gives a conservative solution for the residual stress distribution of "a". Thus there is the following relation among the solutions of the models and the exact solution of Fig. 8 a.

$$\lambda^b < \lambda^c < \lambda^d < \lambda_e < \lambda^e < \lambda^f \quad \cdots\cdots (45)$$

where superscripts show the width-thickness ratio obtained by the corresponding models in Fig. 8. If the width-thickness ratios for models "d" and "e" are computed, the exact solution is between these two values and thus these two values give the bounds of error due to the averaging of the non-uniform stress distribution. The error limit due to the averaging of the stress, E_{rl}, is given for this particular example as follows,

$$E_{rl} = \frac{\lambda^d}{\lambda^e} - 1 \quad \cdots\cdots (46)$$

Figure 9 shows the width-thickness ratios obtained on models "d" and "e" for different

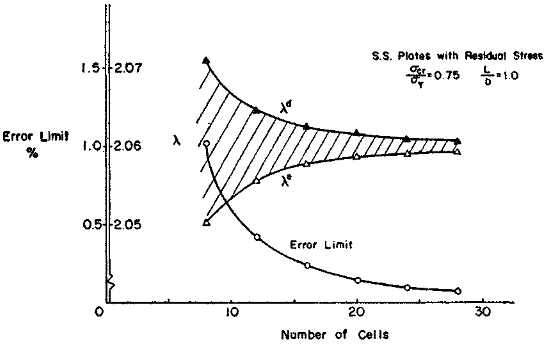

Fig. 9 Error Limit and Bounds of Width-Thickness Ratio.

number of cells. The values are obtained for a plate which buckles under a thrust of three quarters of the yield load. The solutions by the exact averaging method, if they exist, are in the hatched area, in which it may be expected that they lie close to the line for model "d" from the physical interpretation of the problem. The figure also shows the error bounds in percent, which are, as seen, fairly small.

2.4 Round-Off Error

It is said[15] that no really satisfactory theory is available at present on this round-off problem, instead actual practice has produced several methods of employing the computer itself to give some indication of the error. One of the favorite methods is the so-called "double precision" method, which is to run the problem in both single- and double-precision arithmetic and to draw the conclusion that the number of places which agree in the single- and double-precision answers is the number which are right[15]. However, no attempt was made to check the round-off error in this study, partly because of the difficulty in coding the problems and partly because the results obtained in most cases were good.

All three independent results in Table 3 agree quite well with each other and their errors coincide with the analytical estimation of errors. These facts, in turn, may imply that the round-off errors are insignificant compared with the error due to basic finite difference errors. It may be noticed here that the size of the matrices involved is the same as the number of cells for the computation of tubes and is onehalf of the number of cells for the other two cases of plate elements. Similar conclusions can be drawn from Table 4 where the behavior of boundary condition error is in good agreement with the analytical prediction. The above mentioned cases are for the plate elements where the aspect ratio is close to unity, namely from 0.6 to 1. A question arises as to what will be the case with longer plates since the coefficients in the basic differential equation for the terms to the shear rigidity and to the bending rigidity along the z-axis are functions of the aspect ratio. For a plate with

Table 5 Comparison of Error for Different Aspect Rations.

$\frac{L}{b}$	λ_e	λ	Error	Error by Eq. 18
0.5	3.361 1938	3.355 6322	−0.163 98	−0.164 49
1	2.688 9550	2.677 9306	−0.409 99	−0.411 23
2	3.361 1938	3.339 1440	−0.656 01	−0.657 97
4	5.714 0294	5.669 9245	−0.771 87	−0.774 09
6	8.260 9446	8.224 7801	−0.798 03	−0.800 24
8	10.923 880	10.835 665	−0.807 54	−0.809 81
10	13.579 223	13.468 935	−0.812 18	−0.814 32
20	26.956 774	26.736 240	−0.817 99	−0.820 42
50	67.250 765	66.699 388	−0.819 88	−0.822 14
100	134.461 20	133.358 37	−0.820 18	−0.822 38
200	268.902 22	266.696 76	−0.820 37	−0.822 45

NOTE : The non-dimensionalized width-thickness ratios are obtained for simply supported plates ;
$\frac{\sigma_{cr}}{\sigma_Y}=0.5, \quad n=10$

a large aspect ratio, it is possible that these terms become comparatively small and they will fade away into the round-off error. This was checked on a plate simply supported at both the unloaded edges and is summarized in Table 5, of which the computation has been made for a matrix of small size, namely 5. Comparing the actual errors in computer solutions with the error computed analytically, it can be concluded that no significant round-off error is involved even for a simply supported plate with large aspect ratio.

The only case, in which round-off error seems to play a role, is a plate simply supported and free at the unloaded edges, respectively. The critical width-thickness ratios have been computed for the aspect ratios of 2 and 10 with the number of cells from 4 to 20. The sizes of the matrix are the same as the number of cells. The results show no significant round-off error for a plate with aspect ratio of 2, while the longer plate shows a peculiar behavior for a number of cells larger than 10, which cannot be expected from the preceeding analysis of errors to be due to the basic difference approximation ; and thus it may be the round-off error playing a role.

3. SUMMARY AND CONCLUSION

The paper presents the results of an exploration of error which is inherent in a solution of local buckling strength of centrally loaded columns obtained by the finite difference method. Basic difference error was analyzed and illustrated for a simply supported plate at the two unloaded edges. The results of error analysis agreed well with the errors present in numerical solutions by the finite difference method. Similarly errors due to an approximate representation of a boundary condition in a difference form was analyzed for a fixed plate at the two unloaded edges and compared with the errors in numerical solutions. In both cases, it was confirmed that the errors are of r^2–type. Further, the errors present in numerical solutions for plate with other boundary conditions considered, decreased in a similar manner with increasing number of mesh cells. With use of 8 digit numbers in computation, the round off error seemed to be of minor importance for the size of matrices of less than 20 by 20, but a few special cases.

When plate thickness changes, an average rigidity replaces, at each mesh point, the variable rigidity in the original differential equation. It was found that an greatly improved accuracy was obtained for local buckling of a square box column with the inverse average rigidity of stiffiness.

The non-uniform distribution of stress due to the existence of residual stress for this particular problem is the essential reason to employ an approximate method for the analysis of local buckling. No attempt was made to analyze the error which arises by replacing the variable stress intensity by a concentrated load at each mesh point. Instead several models with various distribution of residual stresses were considered such that the finite difference solutions on the models gave a bound of error. The example dealt in the paper showed that the limits of error were resonably narrow that the method could be used to estimate the accuracy of the numerical results.

The most important conclusion is that the error inherent in the finite difference solutions of local buckling of a column with residual stress can be estimated or at least can be bounded so that the finite difference method gives solutions with known and required accuracy for engineering purpose.

4. ACKNOWLEDGEMENT

The work described in this paper was conducted as a part of a dissertation presented to the Graduate Faculty of Lehigh University, Bethlehem, Pennsylvania, U.S.A. The author wishes to acknowledge particularly the advice and assistance, that he received during the progress of the study from Professor Lambert Tall, the supervisor of the dissertation. The author expresses his thanks to the Institute of Research and to the Computer Laboratory, Lehigh University, for granting time for use of the GE 225 computer and for their help in programming.

Sincere appreciation is expressed to all the author's associates at Fritz Engineering Laboratory, in particular to Mr. Tsuneo Tsuiji who assisted running the programs.

The author is also appreciative of the encouragement by Professor Toshie Okumura of University of Tokyo in the preparation of this paper.

5. NOMENCLATURE

a_1, a_2 : constants
b : width of a plate
c_1 to c_4 : constants
E : modulus of elasticity
E_i : error in finite difference approximation at mesh i
E_r : error
E_{rb} : error due to approximate representation of boundary conditions
E_{rt} : error due to basic difference approximation
E_{rl} : error limit due to averaging of the variable coefficients of basic differential equation
E_s : secant modulus of elasticity
E_t : tangent modulus of elasticity
e : $\frac{E}{E_s}-1$, a subscript to show exact solution
$f_n(Y)$: a function of y
I : moment of inertia
I_0 : a constant
i : a subscript to show a mesh point
k_1 to k_4 : coefficients relating to stress-strain relationship
L : buckling length of a plate
n : number of mesh cells
r : width of mesh cells
t : thickness of plate
w : deflection of plate
xyz : cartesian coordinates
Y : a function of y
α, β : a notation in the solution of differential equation governing plate buckling
ϵ_{cr} : strain at buckling load
λ : non dimensionalized critical width-thickness ratio
ν : Poisson's ratio
σ_z : normal stress component parallel to z-axis

REFERENCES

1) Okumura, T. : Buckling of Rectangular Plates with Residual Stress, Jour. Ooyorikigaku Konwa-kai, Univ. of Tokyo, Vol. 2, No. 5, 1947. (Japanese)
2) Yoshiki, M., Fujita, Y. and Kawai, T. : Influence of Residual Stresses on the Buckling of Plates, Jour. Zosen Kyokai, Japan, Vol. 107, 1960 (Japanese with English Abstract).
3) Ueda, Y. : Elastic, Elastic-plastic and Plastic Buckling of Plates with Residual Stresses, Ph. D. Dissertation, Lehigh University, 1962.
4) Nishino, F. and Tall, L. : Residual Stress and Local Buckling Strength, Fritz Laboratory Report 290.8, Lehigh University, In Preparation.
5) Richardson, L.F. : The Approximate Arithmetical Solution by Finite Differences of Physical Problems Involving Differential Equations with an Application to the stresses in a Masonry Dam, Phil. Trans. Royal Soc., London, Vol. 210, 1911.
6) Collatz, L. : Eigenwert Probleme und Ihre numerishe Behandlung, Chelsea Publishing Co., New York, 1948.
7) Salvadori, M.G. : Numerical Computations of Buckling Loads by Finite Differences, Trans. ASCE, Vol. 116, 1951.
8) Salvadori, M.G. and Baron, M.L. : Numerical Method in Engineering, Prentice-Hall, Englewood Cliffs, N. J., 1952.
9) Nishino, F. : Buckling Strength of Columns and Their Component Plates, Ph. D. Dissertation, Lehigh University, 1964.
10) Bijlaard, P.P. : Theory and Tests on the Plastic Stability of Plates and Shells, Jour. Aero. Sci., Vol. 16, No. 9, September 1949.
11) Bijlaard, P.P. and Fisher, G.P. : Interaction of Columns and Local Buckling in Compression Members, NACA TN 2640, 1952.
12) Handelman, G.H. and Prager, W. : Plastic Buckling of a Rectangular Plate Under Edge Thrusts, NACA Rep. 946, 1949.
13) Bleich, F. and Melan, E. : Die Gewohnlichen und Partiellen Differenzengleichungen der Baustatic, Springer, Berlin, 1927.
14) Timoshenko, S. and Gere, J.M. : Theory of Elastic Stability, 2nd Ed., McGraw-Hill, New York, 1961.
15) Hamming, R.W. : Numerical Methods for Scientists and Engineers, McGraw-Hill, New York, 1962.

(*Received Oct.* 5, 1965)

論文集への討議について

論文集編集委員会では，論文集に掲載した全論文に対しての討議を受付けておりますので，討議をされる方は下記の要項をご参照のうえ論文集編集委員会あてご提出下さい。

記

1. 討議は論文集掲載全論文を対象とします。
2. 討議の受付は論文集掲載後6ヵ月以内とします。
3. 討議原稿を提出するときは学会原稿用紙に必要事項を記入のうえ論文集編集委員会あてご提出下さい。
4. 討議原稿の取扱いは論文集編集委員会にご一任下さい。
5. 討議に関する問合せは論文集編集委員会へご連絡下さい。

昭和41年3月15日印刷
昭和41年3月20日発行　　土木学会論文集 第127号　　定価 150円（〒20円）

編集兼発行者　東京都新宿区四谷一丁目　社団法人 土木学会　羽田　巌
印刷者　東京都港区赤坂溜池5　株式会社 技報堂　大沼正吉

発行所　社団法人 土木学会　振替東京 16828番
東京都新宿郵便局区内　新宿区四谷一丁目　電話（351）代表 5138番

土木学会出版物案内

<申込み次第図書目録進呈>

編著者名	図書名	判型	ページ数	会員特価	定価	送料	備考
土木学会編	日本の土木技術 —100年の発展のあゆみ—	A 5	488		**1 200**	150	箱入上製
同	土木学会誌・論文集総索引	B 5	252		**800**	100	写真植字オフセット
同	建設／創造／技術(写真集)	A 4	280		**3 800**	200	彰国社刊
同	土木技術者の活躍と大学土木教育	A 5	140	**250**	300	50	
同	コンクリート標準示方書 土木学会規準	B 6	234		**200**	50	
同	プレストレストコンクリート設計施工指針	B 6	124	**250**	350	50	
土木学会編	工事報告・川俣アーチダム	B 5	346	**1 600**	2 000	150	口絵6ページ
九州電力KK編	工事報告・一ツ瀬 杉安 アーチダム	B 5	550	**2 300**	2 900	150	口絵2ページ
電源開発KK編	工事報告・大鳥セミアーチダム	B 5	266	**1 200**	1 400	150	付図11枚
日本道路公団編	若戸橋調査報告・工事報告書 <昭和39年度土木賞受賞>	B 5	二冊合計 2 350		**30 000**	学会負担	タイプオフ限定出版
日本道路公団編	関門トンネル工事誌	B 5	692	**1 500**	2 000	200	
土木学会編	トンネル標準示方書	A 5	34		**100**	30	示方書条文
同	トンネル標準示方書解説	A 5	138	**500**	600	50	条文を詳細に解説
同	トンネル工学シリーズ 1 第1回トンネル工学シンポジウム	B 5	106	**300**	400	50	6編を収録
同	トンネル工学シリーズ 2 最近のトンネル工学 —工事の実例と話題—	B 5	136	**400**	500	50	6編を収録
同	昭和40年度夏期講習会資料 構造工学における最近の諸問題	B 5	320	**1 200**	1 500	150	12編を収録
同	水工学シリーズ<全9巻> 1964 A. ダム・河川コース B. 海岸・港湾コース	B 5	計 340		**A. 1 100** **B. 1 300**	100 100	
同	水工学シリーズ<全2巻> 1965 A. ダム・河川コース B. 海岸・港湾コース	B 5	計 300		**A. 2 000** **B. 1 500**	100 100	11編を収録
同	第12回海岸工学講演会講演集(1965)	B 5	258		**1 500**	100	39編を収録

◀振替東京 16828 番▶

編著者名	図書名	判型	ページ数	会員特価	定価	送料	備考
吉田徳次郎	コンクリート・ライブラリー ■第1号 コンクリートの話 ―吉田徳次郎先生御遺稿より―	B 5	48	**150**	200	20	
土木学会編	コンクリート・ライブラリー ■第2号 第1回異形鉄筋シンポジウム	B 5	98	**350**	450	20	10編を収録
同	コンクリート・ライブラリー ■第3号 異形鉄筋を用いた鉄筋コンクリート構造物の設計例	B 5	42	**300**	380	40	付図5枚付
国分・三村 上野・細谷	コンクリート・ライブラリー ■第4号 ペーストによるフライアッシュの使用に関する研究	B 5	22	**100**	120	20	吉田賞受賞
和仁・川口・菅原・野口・羽田野	コンクリート・ライブラリー ■第5号 小丸川 PC 鉄道橋の架替え工事ならびに，これに関連して行なった実験研究の報告	B 5	38	**150**	200	30	吉田賞受賞
川口輝夫	コンクリート・ライブラリー ■第6号 鉄道橋としてのプレストレストコンクリート桁の設計方法に関する研究	B 5	62	**220**	250	40	
村田二郎	コンクリート・ライブラリー ■第7号 コンクリートの水密性の研究	B 5	36	**100**	120	30	吉田賞受賞
山崎寛司	コンクリート・ライブラリー ■第8号 鉱物質微粉末がコンクリートのウォカビリチーおよび強度におよぼす効果に関する基礎研究	B 5	56	**160**	200	40	吉田賞受賞
石田一郎	コンクリート・ライブラリー ■第9号 添えばりを用いるアンダーピンニング工法の研究	B 5	18	**100**	120	20	吉田賞受賞
土木学会編	コンクリート・ライブラリー ■第10号 構造用軽量骨材シンポジウム	B 5	96	**400**	500	50	13編を収録
樋口芳朗	コンクリート・ライブラリー ■第11号 微細な空げきてん充のためのセメント注入における混和材料に関する研究	B 5	28	**100**	120	30	吉田賞受賞
岩間滋	コンクリート・ライブラリー ■第12号 コンクリート舗装の構造設計に関する実験的研究	B 5	32	**100**	120	30	吉田賞受賞
運輸省港研編	コンクリート・ライブラリー ■第13号 プレパックドコンクリート施工例集	B 5	330	**1 600**	2 000	150	
土木学会編	コンクリート・ライブラリー ■第14号 第2回異形鉄筋シンポジウム	B 5	240	**900**	1 100	100	19編を収録
同	土質実験指導書	B 5	65		**320**	60	データシート26枚付
同	土木材料実験指導書	B 5	90		**380**	60	データシート30枚付
同	水理公式集	A 5	603	**1 100**	1 400	150	
同	Coastal Engineering in Japan, 1965	B 5	151		**1 200**	100	11編を収録
同	Civil Engineering in Japan, 1965	A 4	138		**1 200**	100	口絵24ページ

◀申込先・土木学会・東京 351-5130▶

土木学会論文集在庫一覧

申込先：土木学会へ

編著者名	論文名	判型	ページ数	定価	送料
小西一郎	論文集9号 一般剛節構造物の解法およびその極限状態付近における性状について	B 5	10	**20**	10
猪股俊司	論文集17号 プレストレストコンクリート桁に関する研究	B 5	90	**250**	30
高野俊介	論文集26号 打込み温度がマッスコンクリートの強度におよぼす影響の研究	B 5	56	**180**	30
仁杉巌	論文集27号 支間 30 m のプレストレストコンクリート鉄道橋（信楽線第一大戸川橋梁）の設計，施工およびこれに関連して行った実験研究の報告	B 5	56	**160**	20
伊丹康夫	論文集37号 ブルドーザによる土工の設計に関する研究	B 5	50	**120**	30
猪股俊司	論文集48号 プレストレスト コンクリート スラブ式 2 ヒンジ ラーメン橋の設計法に関する研究	B 5	68	**200**	30
岩佐義朗	論文集59号（別冊 3-1） 巾の漸変する水路における水流の遷移現象と境界特性との関連に関する理論的研究	B 5	32	**150**	20
奥田秋夫	論文集59号（別冊 3-2） コンクリート舗装の施工合理化に関する研究	B 5	54	**250**	30
大久保・西原 相馬	論文集61号（別冊 3-1） 発電用河川流量の研究	B 5	32	**150**	20
太田誠一郎	論文集61号（別冊 3-3） 骨材の表面積と新面積法による構築混合物の検討とその応用に関する研究	B 5	48	**220**	30
三村・鈴木 上野・細谷	論文集65号（別冊 3-2） ロッドミルによる製砂方法に関する研究	B 5	15	**80**	20
永井荘七郎	論文集65号（別冊 3-3） 防波堤に働く砕波の圧力に関する研究	B 5	38	**160**	20
国分・河原・太斎	論文集68号（別冊 3-2） 各種フライアッシュの共通試験報告	B 5	32	**100**	20
井田至春	論文集69号（別冊 3-2） 広巾員開水路の定常流 —断面形の影響について—	B 5	18	**100**	20
室田明	論文集70号（別冊 1-1） 開水路分水工の研究	B 5	34	**200**	20
嶋・荻原	論文集71号（別冊 4-1） **On Water-Hammer Pressure due to Periodic Opening and Closure of Valve**（英文）	B 5	12	**70**	20
国分ほか 12 名	論文集71号（別冊 4-3） フライアッシュ・シンポジウム提出論文集録	B 5	66	**230**	35
森麟	論文集71号（別冊 4-4） 舗装後の路床状態の変化についての研究とそれにもとづく CBR 試験法の改善に対する一提案	B 5	16	**100**	20
島田静雄	論文集72号（別冊 3-1） 弾性針金の変形と応力	B 5	16	**100**	20
山本稔	論文集72号（別冊 3-3） 不完全合成 T 型桁橋の曲げ理論とその応用	B 5	24	**150**	20

昭和三十七年五月二十八日 第三種郵便物認可
昭和四十一年三月十五日印刷（毎月一回
昭和四十一年三月二十日発行 二十日発行）

土木学会論文集 第一二七号

定価百五十円