ON THE RESISTANCE COEFFICIENT AT THE INTERFACE BETWEEN SALT AND FRESH WATER

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1. INTRODUCTION

The evaluation of the resistance coefficient f_i at the interface in two-layered system is not only an important problem in the engineering field, but also an interesting one from the scientificial point of view, because many of the characteristics of the internal waves in the stratified flow are still unknown.

The resistance coefficient f_i, defined by $f_i = \frac{\tau_i}{1/2\,\rho\,U^2}$, where τ_i is the shearing stress at the interface, ρ is the density of the fluid and U is the scale of velocity, may be calculated theoretically, if the fluid motion is laminar in this system[1].

The effect of the turbulence, or turbulent mixing associated with the mean motion, to the stability of the interface has to be small compared with that of the stability of the homogeneous flow. Where the strong vertical density gradients exist, the tubulence should be suppressed or damped out by the gravity effect caused by density gradients. It is well-known that the criterion from laminar to turbulent motion in heterogeneous flow is given not by Reynolds number, but by Richardson number defined by $R_i = \frac{-g/\rho \cdot \partial\rho/\partial y}{(\partial U/\partial y)^2}$. After Schlichting, every moiton is stable if $R_i > 0.0417$[6]. Therefore, it may be possible that the motion with very high Reynolds number can be laminar, or at least a regular motion in the neighbourhood of the interface. The discussions which will be developed in this paper should be valid in this region, and thus, the arguments are almost general in the sence that the turbulent motion is diminished near the interface, or in the region of large density gradients.

In case that the turbulent dissipation is strong enough to break up the interface against gravity force, the two-layered model should not be adequate and the conception of interfacial shear does not exist any more. In this case the theory of turbulence in homogeneous fluid may be useful, because the motion is not influenced by the density variation.

It should be emphasized again that the theory developed in this paper can be applicable even if Reynolds number is very large, as far as there exists the region of large density gradients, or of high Richardson number.

2. THE OUTLINE OF THE THEORY

The resistance coefficient at the interface may be calculated if we assume suitable velocity distribution in the two-layered system, where the interface is considered as a flat plane. However, such a situation does not seem to occur actually, because the existence of the internal waves is neglected. The writer considers that these internal waves may play a significant role in the resistance coefficient. The outline of calculation and assumptions are following.

First, assume the potential function ϕ of the internal waves.** As far as the equation of continuty, i.e. div $=0$, is satisfied***, there exists the stream function, which yields the existence of a complex potential. Consider the equation of continuity for density ρ and for concentration c.

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** The use of the stream function may be easier to understand than the use of the potential function. As far as the wave motion is concerned, the assumption of the existence of the velocity potential is feasible, if we assume the shape of the waves is not influenced by the viscosity very much. While this assumption is considered suitable, the writer should add some other explanations to avoid the misunderstanding that such a motion were the usual shear flow. What is concerned here is the energy dissipation caused by internal waves. Thus, it is clear that this motion is not an usual shear flow, but a virtual shear flow.

*** Even if the density gradient exists, we may obtain this equation.

$$\nabla\cdot(\rho V)=0,\quad \nabla\cdot(cV)=0$$

If there is any relationship between ρ and c, the following calculation is feasible in engineering sence.

$$\nabla\cdot(\rho V)=\rho\nabla\cdot V+V\cdot\nabla\rho=\rho\nabla\cdot V+\frac{d\rho}{dc}V\cdot\nabla c$$
$$=\rho\nabla\cdot V+\frac{d\rho}{dc}[\nabla\cdot(cV)-c\nabla\cdot V]$$
$$=\rho\nabla\cdot V-\frac{d\rho}{dc}c\nabla\cdot V$$
$$=\left[\rho-\frac{d\rho}{dc}c\right]\nabla\cdot V=0$$

Therefore, $\nabla\cdot V=0$, unless $\rho\cdot d\rho/dc$ or $\rho=$const. $\times c$., which is impossible usually, because if $c\neq 0$, then $\rho\neq 0$.

The dissipation function may be calculated from the potential function. Writing the energy dissipation per unit mass of fluid caused by the internal waves as ϵ, we are able to define the interfacial shear as $\tau_i=\epsilon/U$, which is virtual. Thus we can obtain f_i also.

3. THEORETICAL CALCULATION

Let us consider the two-layered system in which the upper layer moves while the lower is stationary.**** Co-ordinate system and symbols are expressed in Fig. 1. The meaning of symbols are listed on the nomenclature table. If the displacements at the free surface are negligible compared with the interfacial disturbances, we obtain the following set of functions assuming the sinus internal waves with respect to the upper layer.

$$\eta=a\sin mx \quad\cdots\cdots(1)$$
$$\phi=-(U_1-c)x+A\cosh m(y-h_1)\cos mx \quad\cdots\cdots(2)$$

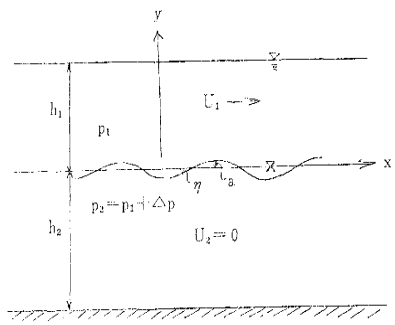

Fig. 1

$$A=\frac{(U_1-c)a}{\sinh mh_1} \quad\cdots\cdots(3)$$

Total energy dissipation ε across the interface can be expressed as follows with respect to the volume of the frame of reference.

$$\varepsilon=\mu\int_V\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right]dxdydz \quad\cdots\cdots(4)$$

As we are concerning about the interfacial energy dissipation caused by the wave motion, we write eq. (4) in the form of surface integral, assuming rot-$v=0$.*****

$$\varepsilon=-\mu\int_s\frac{\partial q^2}{\partial n}dxdz \quad\cdots\cdots(5)$$

where $q^2=u^2+v^2$ and n denotes the inward normal. As we are concerning about the energy dissipation at the interface, we may put $-n=y$ approximately. Therefore we can proceed calculations as follows.

$$\left.\frac{\partial q^2}{\partial y}\right|_{y=0}=2m^2(U_1-c)A\sinh(-mh_1)\sin mx+2m^3A^2\cosh(-mh_1)\sinh(-mh_1)\sin^2 mx+2m^3A^2\sinh(-mh_1)\cosh(-mh_1)\cos^2 mx \quad\cdots\cdots(6)$$
$$=-2m^2(U_1-c)A\sinh mh_1\sin mx-2m^3A^2\sinh mh_1\cosh mh_1 \quad\cdots\cdots(7)$$
$$\varepsilon=B\lambda\epsilon=-\mu\int_{-B/2}^{B/2}\int_0^{\lambda}\frac{\partial q^2}{\partial n}dxdz \quad\cdots(8)$$

where B is the width of the rectangular channel, or may be the equivalent width of any shape of channels. The effects of the walls may be neglected here.

The first term in eq. (7) vanishes if we integrate it from 0 to y with respect to x, and we obtain ϵ as the final form.

$$\epsilon=2\mu m^3A^2\sinh mh_1\cosh mh_1 \quad\cdots\cdots(9)$$

After substituting eq. (3) into eq. (5), we obtain

$$\epsilon=2\mu m^3(U_1-c)^2a^2\coth mh_1$$
$$=16\pi^3\mu\frac{U_1^2}{\lambda^3}(1-s)^2a^2\coth\frac{2\pi h_1}{\lambda} \quad\cdots(10)$$

where $\lambda=2\pi/m$ and $s=c/U_1$ $\cdots\cdots(11)$

Eq. (10) gives us the energy dissipation of the internal waves of which the first approximation is given by the potential flow.

If s is equal to unity in eq.-(10), $\epsilon=0$, which means there is no energy dissipation caused by

**** If we consider this motion as a relative motion, it will be general.

***** This assumption leads us to the existence of the velocity potential.

the internal waves in this case. From eq. (11) we see that s is a kind of Froude number.

Now that ϵ is obtained, we can define the resistance coefficient f_i as the usual form.

$$f_i = \frac{\tau_i}{1/2\,\rho\,U_1^2} = \frac{\epsilon}{1/2\,\rho\,U_1^3} \quad \cdots\cdots(12)$$

Substituting eq.-(10) into eq.-(12), we obtain

$$f_i = 32\,\pi^3 \frac{1}{R_e}\left(\frac{h_1}{\lambda}\right)\left(\frac{a}{\lambda}\right)^2 (1-s)^2 \coth 2\pi\frac{h_1}{\lambda}$$

where $R_e = U_1 h_1/\nu$.

λ/h_1 diminishing, $2\pi\cdot h_1/\lambda$ becomes larger and larger, and finally approaches unity rather rapidly. As a matter of fact if $h_1/\lambda = 0.5$, $\coth 2\pi\cdot h_1/\lambda$ is nearly equal to unity. Thus we may put $\coth 2\pi\cdot h_1/\lambda = 1$ as far as the concerned internal waves are of small amplitudes.

To determine s, we may only solve the following equation derived from the potential flow theory.*****.

$$\frac{g'}{mU_1} = (1-s)^2 \coth mh_1 + s^2 \coth mh_2 \quad \cdots\cdots(14)$$

From the above equation we obtain s as follows.

$$s = \frac{\coth mh_1 \pm \sqrt{\coth^2 mh_1 - (\coth mh_1 + \coth mh_2)\left(-g'/mU_1^2\right)}}{\coth mh_1 + \coth mh_2} \quad \cdots\cdots(15)$$

If m is sufficiently large, s will be read as follows.

$$s = \frac{1}{2}\left(1 \pm \sqrt{\frac{2g'}{mU_1^2} - 1}\right) \quad \cdots\cdots(16)$$

Therefore, the internal waves are stable under the condition expressed by eq. (17).

$$\frac{1}{\pi}\left(\frac{\lambda}{h_1}\right) \geq F_i^2 \quad \cdots\cdots(17)$$

In other words, the possible minimum λ, say, $\lambda_{\min}$, is to be determined if both F_i and h_1 are given.

$$\lambda = \lambda_{\min} = \pi\, h_1 F_i^2 \quad \cdots\cdots(18)$$

Assuming the existence of the possible maximum value of a/λ which may be written as $(a/\lambda)_{\max}$, we can write

$$\left(\frac{a^*}{\lambda_{\min}}\right) = \frac{a^*}{\pi\, h_1 F_i^2} \leq \left(\frac{a}{\lambda}\right)_{\max} \quad \cdots\cdots(19)$$

***** See, for instance, Milne-Thomson : Theoretical Hydrodynamics, p. 405.

$$a^* \leq \pi\, h_1 F_i^2 \left(\frac{a}{\lambda}\right)_{\max} = \lambda_{\min}\left(\frac{a}{\lambda}\right)_{\max} \quad \cdots\cdots(20)$$

where a^* means so-called minimax wave amplitude. Because from eq. (20) it is clear that for given $(a/\lambda)_{\max}$, $\lambda(a/\lambda)_{\max}$ gives us the possible maximum a, and $\lambda_{\min}(a/\lambda)_{\max}$ gives us the minimum of maximum a. Of course there can be a less than a^*, but this is to be so for all sets of a/λ which are less than $(a/\lambda)_{\max}$, because any λ cannot take less values than $\lambda_{\min}$ and therefore a/λ should become less. This fact should be emphasized, for $\lambda_{\min}$ gives us the maximum energy dissipation and the so-called roughness should mainly be determined by minimux a, *i. e.* a^*.

The extra importance is that the long internal waves are independent of the roughness at the interface; although such waves may have considerably large amplitude. Such long internal waves are not to be considered at the roughness as the interface. Though they contain large amount of energy, they have not intense energy dissipation rate. This fact is very much analogous to the contribution in energy of turbulent vortices to the power spectrum to a certain extent. Namely, long waves contain most of the total energy and dissipate only little energy, and the shorter waves are the energy dissipating waves. Thus, the waves which are able to exist stable must satisfy eq. (18).

$$\frac{\lambda}{h_1} = \pi\, F_i^2 \quad \cdots\cdots(18)$$

And s satisfies eq. (21) also.

$$s = \frac{1}{2} \quad \cdots\cdots(21)$$

The waves defined by eq. (21) give minimum possible wave length. They are the waves which must contribute to the energy dissipation at the interface.

Therefore, we obtain f_i as follows.

$$f_i = \frac{8\,\pi^2}{R_e F_i^2}\left(\frac{a}{\lambda}\right)^2 \quad \cdots\cdots(22)$$

If we use Keulegan number θ in eq. (22), it may be read

$$f_i = 8\,\pi^2\left(\frac{a}{\lambda}\right)^2 \theta \quad \cdots\cdots(23)$$

where θ is defined by $\theta = 1/R_e F_i^2$.

Eq. (23) means that if the motion is not

complete laminar, in other words even if it may be in a transition zone, the friction coefficient caused by the internal waves is proportional to Keulegan number θ. Locally, say, within the range of Reynolds number from 10 to 40000, f_i may be determined by R_e only. However, from its generality, f_i expressed by eq. (23) may cover considerably wide range, as far as there exists the possible maximum value of a/λ. The measured maximum value of a/λ is 0.323 for large wave length[5]. From this fact a/λ of the internal waves may be much larger than that of surface waves.

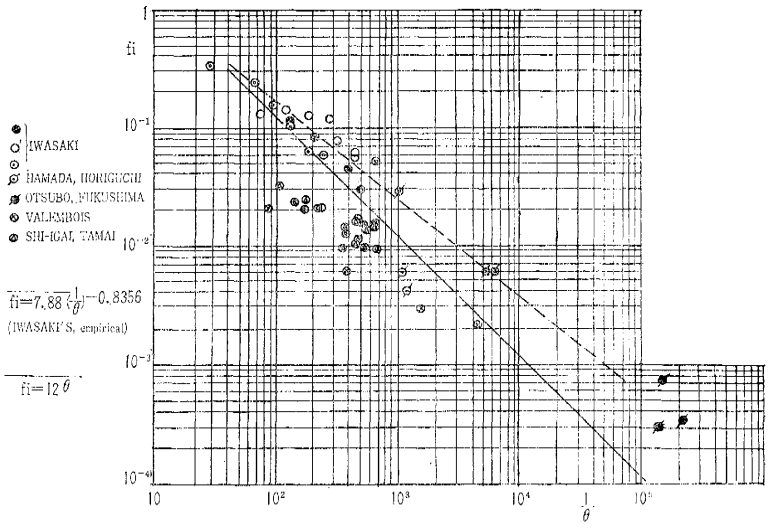

Fig. 2

On Fig. 2 experimental and field data and the theoretical curves are shown, where the parameter $8\pi^2(a/\lambda)^2$ is deducted as 12, or $a/\lambda=0.389$. The broken line on Fig. 2 shows the empirical formula obtained by Dr. Iwasaki in Tohoku University[3]. The tendency of scattering of measured points agrees with the theoretical curve, $f_i=12\theta$, fairly well. Accounting the facts that the laminar resistance coefficient at the interface is neglected in this theory, and that the experimental and field data plotted here were obtained at several laboratories by slightly different methods, namely, Dr. Iwasaki and Dr. Hamada calculated it from the mean slope of the interface, while the writer's data are the mean values of the local resistance coefficient, this theory gives us fairly satisfactory result.

The meaning of the numerical constant 12 should be considered not as a fixed value, but as a flexible one. The value 12 was determined by the data obtained by now. This is to be rechecked in future. Of course it may be possible to choose several possible values with respect to each experiment. However, one can easily find that it is meaningless to do so, for it only derives the increase of the number of constants which should be rechecked.

4. CONCLUSION

The resistance coefficient at the interface, f_i, defined by eq. (12), is governed by Keulegan number θ, if the interface becomes rough by the internal waves. This is derived theoretically. This theoretical derivation covers the tendency of field and experimental data fairly well, if the value of a/λ is well defined. The further collection of field and experimental data is expected to check the value of numerical constant which was determined as 12 in this paper.

NOMENCLATURE

The values with suffix 1 and 2 are referred to the upper and the lower layer, respectively.

V : velocity vector
U : mean velocity
h : depth of the layer
a : amplitude of internal waves
c : celerity of internal waves
λ : wave length
m : $2\pi/\lambda$
ϕ : velocity potential
ε : total energy dissipation
ϵ : energy dissipation
μ : dynamic viscosity
f_i : resistance coefficient at the interface
R_e : Reynolds number
F_i : densimetric Froude number $\left(=\frac{U}{\sqrt{g'h}}\right)$
ρ : density
$\Delta\rho$: density difference
g' : densimetric gravity acceleration
τ_i : shearing stress at the interface
θ : Keulegan number $(=1/R_e F_i^2)$

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(Received Sept. 25,1964)