A STUDY ON ELECTRONIC COMPUTATION OF CLOTHOIDAL ALIGNMENT

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INTRODUCTION

It is only recently that electronic digital computors have been used for computing planimetric alignments for new highway projects, and, therefore, the tendency has been only to apply the conventional manual methods and for the calculations, and merely to use the computor as a speedier method of carrying them out. However, this is not to say that manual methods are always the best methods to use in conjunction with a computor : the capability of manual methods is so limited that, if always referred to, the development of new methods for use in connection with electronic computation is restricted.

In order to employ electronic computors to their full advantage, it is necessary to develop new mathematical approaches to the problem and ways of handling the calculations that are especially suited to the equipment being used. This paper describes such a new method of handling the calculations for planimetric alignments of new highways by electronic means.

Usually planimetric alignments have been computed as follows : firstly, using a clothoid table and a set of clothoid and circular arc rulers, the alignments were selected graphically on the map to satisfy several control points. Then, by electronic computor, each clothoid and circular arc was connected to satisfy the boundary conditions of each alignment, and the correct co-ordinates of each station were determined. To carry out this computation for connecting the alignments smoothly, three methods have been used as follows :

1) Fixing the tangents and varying the circular arc.

In this method the clothoid parameters and the radius of the circular arc which were graphically selected are not changed during the calculation.

Figs. 1 shows an example of this type.

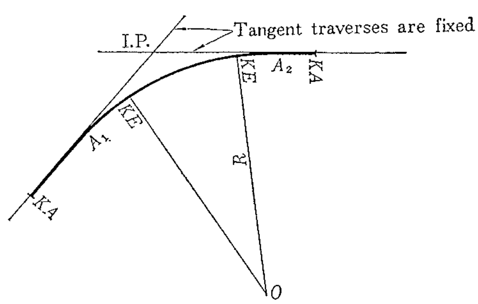

Fig. 1

2) Using three circular arcs, fixing the centres of the end circles and varying that of the intermediate circle.

Also in this method the clothoid parameters

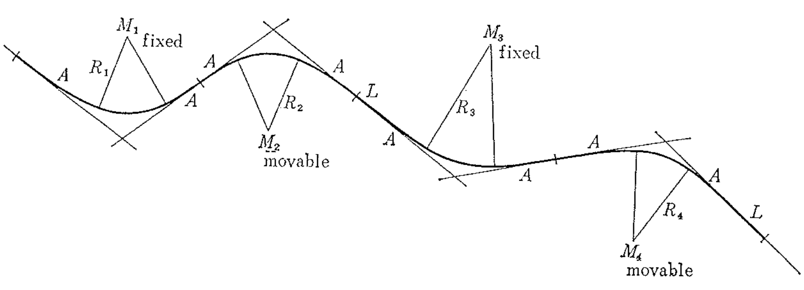

Fig. 2

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and the radii of the circles are not changed throughout the calculation.

Fig. 2 illustrates an example of this type.

3) Adjusting the clothoid parameter or radius of the circle instead of varying the co-ordinates of the centres of the circles, as in cases 1 and 2.

In this method the boundary conditions of alignments are given exactly. Fig. 3 shows an example of this type.

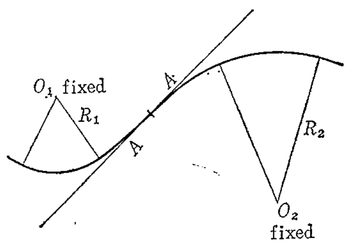

Fig. 3

In the methods 1) and 2), if the alignments thus calculated cannot satisfy the given control points, a new selection of alignment parameters must be made and the computation must be done all over again. In addition to this repetition, when the location engineer prefers certain clothoid parameters which are neither listed on the table nor are given by one of the rulers in the set, he must repeat the troublesome computation to select a suitable parameter. Such difficulties usually occur in the calculation of alignments, especially at interchanges. In order to resolve such difficulties, the basic approach adopted in the new method which is described in this paper is based on that of method 3 and the alignment parameters are not selected graphically, but are calculated by the computor to satisfy the given boundary conditions exactly. These will be in the form of a solution to a set of simultaneous equations, and to solve them new mathematical approaches suitable for electronic computation will be introduced.

1. DEFINITION

To solve the above mentioned problem, it is inconvenient to introduce the clothoids in their usual form into the simultaneous equations.

Therefore, the clothoid function is transformed into what is referred to in this paper as "clothoidal sine" (sincl) and "clothoidal cosine" (coscl) which are defined as follows :

$$\text{sincl}\,\tau=\tau-\frac{3}{5}\frac{\tau^3}{3!}+\frac{5}{9}\frac{\tau^5}{5!}-\frac{7}{13}\frac{\tau^7}{7!}+ \quad\cdots\cdots\cdots(1\text{-}1)$$

$$\text{coscl}\,\tau=\frac{2}{3}\frac{\tau^2}{2!}-\frac{4}{7}\frac{\tau^4}{4!}+\frac{6}{11}\frac{\tau^6}{6!}-\frac{8}{15}\frac{\tau^8}{8!}+ \quad\cdots\cdots\cdots(1\text{-}2)$$

According to these equations, the co-ordinates (x, y) of a point on the clothoid curve is given by the radius R and tangential angle τ of the point,

$$x=2R\,\text{sincl}\,\tau \cdots\cdots\cdots(1\text{-}3)$$

$$y=2R\,\text{coscl}\,\tau \cdots\cdots\cdots(1\text{-}4)$$

Here it must be noticed that the expansions of sincl τ and coscl τ have similar forms to the expansions of $\sin\theta$ and $\cos\theta$, and comparing the above equations, (1–3) and (1–4), to the general equations of a circle, (1–5) and (1–6), one can see that they can be considered to correspond to the equations of a circle with a radius $2R$.

$$x=r\sin\theta \cdots\cdots\cdots(1\text{-}5)$$

$$y=r\cos\theta \cdots\cdots\cdots(1\text{-}6)$$

From eqns. (1–5) and (1–6), the length of arc is given by $l=r\theta$ and similarly, the length of a clothoid from its origin is given by $L=2R\tau$.

When calculating the clothoid function by computer, it is necessary to determine automatically the quadrant in which the point lies. This problem is settled by giving a sign to R and A^2.

In the case when the curve is clockwise (see Fig. 4)

$R>0$

In the anti-clockwise case

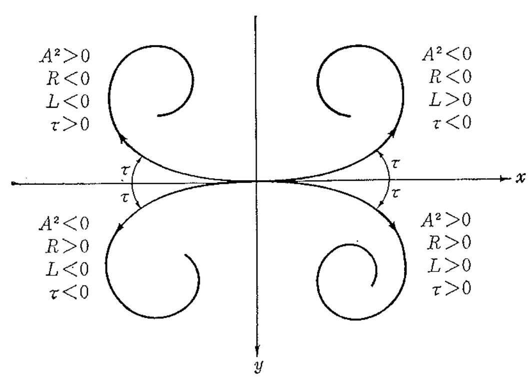

Fig. 4

$R < 0$

and, if the curvature increases along the curve, then :

$A^2 > 0$

and if the curvature decreases, then :

$A^2 < 0$

Denoting the curvatures of two points (P_1, P_2) on the clothoid by $1/R_1$ and $1/R_2$:

and if $1/R_2 - 1/R_1 > 0$,

then the curvature always increases and the sign of A^2 is positive, to the above definitions. This is called a real clothoid.

if $1/R_2 - 1/R_1 < 0$

the curvature decreases and a sign of A^2 must be negative. This is called an imaginary clothoid.

Following these definitions, the quadrant of a point $P(x,y)$ on the clothoid can be determined and these relations can be represented in a table as follows :

quadrant	A^2	R	τ	sincl τ	coscl τ	$x=2R$ sincl τ	$y=2R$ coscl τ
1	$+v_e$	$+v_e$	$+v_e$	$+v_e$	$+v_e$	$+v_e$	$+v_e$
2	$-v_e$	$+v_e$	$-v_e$	$-v_e$	$+v_e$	$-v_e$	$+v_e$
3	$+v_e$	$-v_e$	$+v_e$	$+v_e$	$+v_e$	$-v_e$	$-v_e$
4	$-v_e$	$-v_e$	$-v_e$	$-v_e$	$+v_e$	$+v_e$	$-v_e$

Next, the question of the centre of curvature will be considered. The locus of the centre is given by the formulae;

$$x_m = 2R \text{ sincl } \tau - R \sin \tau \quad \cdots\cdots (1\text{-}7)$$

$$y_m = 2R \text{ coscl } \tau + R \cos \tau \quad \cdots\cdots (1\text{-}8)$$

Here, two more functions are introduced : "hybrid sine" (sinhb), and "hybrid cosine" (coshb) defined as;

$$\text{sinhb } \tau = 2 \text{ sincl } \tau - \sin \tau \quad \cdots\cdots (1\text{-}9)$$

and

$$\text{coshb } \tau = 2 \text{ coscl } \tau + \cos \tau \quad \cdots\cdots (1\text{-}10)$$

Then the locus of the centre is given by the hybrid sine and hybrid cosine :

$$x_m = R \text{ sinhb } \tau \quad \cdots\cdots (1\text{-}11)$$

$$y_m = R \text{ coshb } \tau \quad \cdots\cdots (1\text{-}12)$$

In addition two more functions will frequently appear, referred to as the clothoidal chord (chordcl) and hybrid chord (chordhb) defined as :

$$\text{chordcl } \tau = \sqrt{\text{sincl}^2 \tau + \text{coscl}^2 \tau} \quad \cdots\cdots (1\text{-}13)$$

$$\text{chordhb}^2 \tau = \sqrt{\text{sinhb}^2 \tau + \text{coshb}^2 \tau} \quad \cdots\cdots (1\text{-}14)$$

Here it must be noted that the co-ordinate systems which are used in this paper are all clockwise systems which are the same as the national geodetic co-ordinate system.

2. THE SOLUTION OF SOME FUNDAMENTAL EXAMPLES.

In this section the solution of several kinds of fundamental problem will be explained according to the definitions of preceding section.

On Fig. 5 the notation which will be used in following sections is shown.

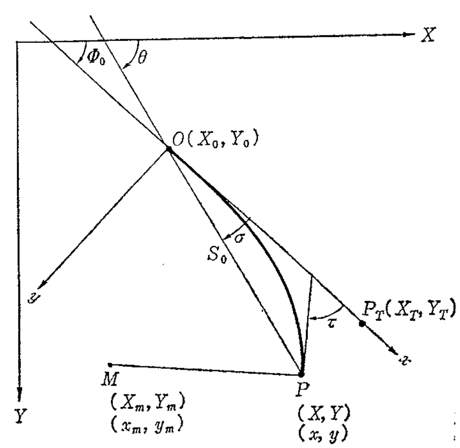

Fig. 5

2-1. A clothoid defined by the origin, any one point on the clothoid and the azimuth of the back tangent Φ_0.

Designating the National plane co-ordinate of the clothoid origin by (X_0, Y_0), the azimuth of back tangent by Φ_0 and the National co-ordinates of any point on the clothoid by (X, Y). Then among these points, there exists the following relationships :

$$(X - X_0)\cos \Phi_0 + (Y - Y_0)\sin \Phi_0 = x = 2R \text{ sincl } \tau \quad \cdots\cdots (2\text{-}1)$$

$$-(X - X_0)\sin \Phi_0 + (Y - Y_0)\cos \Phi_0 = y = 2R \text{ coscl } \tau \quad \cdots\cdots (2\text{-}2)$$

In these equations if (X, Y) and (X_0, Y_0) are known, substituting

$$(X - X_0) = S_0 \cdot \cos \theta \quad \cdots\cdots (2\text{-}3)$$

$$(Y - Y_0) = S_0 \cdot \sin \theta \quad \cdots\cdots (2\text{-}4)$$

into (2-1) and (2-2), one finds that

$$S_0 \cdot \cos(\theta - \Phi_0) = 2R \text{ sincl } \tau \quad \cdots\cdots (2\text{-}5)$$

Table 1

deg	sincl	coscl	tancl	chordcl	rad
40° 51′ 30″	0.6776 9239	0.1634 5134	0.2411 8809	0.6971 2503	0.7131 1244
31	0.6776 9653	0.1634 5348	0.2411 8978	0.6971 2956	0.7131 1729
32	0.6777 0067	0.1634 5562	0.2411 9147	0.6971 3408	0.7131 2214
33	0.6777 0480	0.1634 5776	0.2411 9316	0.6971 3861	0.7131 2699
34	0.6777 0894	0.1634 5990	0.2411 9484	0.6971 4313	0.7137 3184
35	0.6777 1308	0.1634 6204	0.2411 9653	0.6971 4765	0.7131 3668
36	0.6777 1721	0.1634 6418	0.2411 9822	0.6971 5218	0.7131 4153
37	0.6777 2135	0.1634 6633	0.2411 9990	0.6971 5670	0.7131 4638
38	0.6777 2549	0.1634 6847	0.2412 0159	0.6971 6123	0.7131 5123
39	0.6777 2962	0.1634 7061	0.2412 0328	0.6971 6575	0.7131 5608
40 51 40	0.6777 3376	0.1634 7275	0.2412 0497	0.6971 7027	0.7131 6093
41	0.6777 3790	0.1634 7489	0.2412 0665	0.6971 7480	0.7131 6577
42	0.6777 4204	0.1634 7703	0.2412 0834	0.6971 7932	0.7131 7062
43	0.6777 4617	0.1634 7918	0.2412 1003	0.6971 8384	0.7131 7547
44	0.6777 5031	0.1634 8132	0.2412 1172	0.6971 8837	0.7131 8032
45	0.6777 5445	0.1634 8346	0.2412 1341	0.6971 9289	0.7131 8517
46	0.6777 5858	0.1634 8560	0.2412 1509	0.6971 9742	0.7131 9001
47	0.6777 6272	0.1634 8774	0.2412 1678	0.6972 0194	0.7131 9486
48	0.6777 6686	0.1634 8988	0.2412 1847	0.6972 0646	0.7131 9971
49	0.6777 7099	0.1634 9203	0.2412 2016	0.6972 1099	0.7132 0456
40 51 50	0.6777 7513	0.1634 9417	0.2412 2184	0.6972 1551	0.7132 0941
51	0.6777 7927	0.1634 9631	0.2412 2353	0.6972 2003	0.7132 1425
52	0.6777 8340	0.1634 9845	0.2412 2522	0.6972 2456	0.7132 1910
53	0.6777 8754	0.1635 0059	0.2412 2691	0.6972 2908	0.7132 2395
54	0.6777 9168	0.1635 0273	0.2412 2859	0.6972 3361	0.7132 2880
55	0.6777 9581	0.1635 0487	0.2412 2859	0.6972 3813	0.7132 3365
56	0.6777 9995	0.1635 0702	0.2412 3197	0.6972 4265	0.7132 3850
57	0.6778 0409	0.1635 0916	0.2412 3366	0.6972 4718	0.7132 4334
58	0.6778 0822	0.1635 1130	0.2412 3535	0.6972 5170	0.7132 4819
59	0.6778 1236	0.1635 1344	0.2412 3703	0.6972 5622	0.7132 5304
40 52 00	0.6778 1650	0.1635 1558	0.2412 3872	0.6972 6075	0.7132 5789

Table 2

deg	sincl	coscl	tancl	chordcl	rad
29° 21′ 00″	0.4989 7467	0.0858 4230	0.1720 3738	0.5063 0487	0.5122 5414
01	0.4989 7915	0.0858 4389	0.1720 3903	0.5063 0955	0.5122 5898
02	0.4989 8362	0.0858 4548	0.1720 4069	0.5063 1423	0.5122 6383
03	0.4989 8810	0.0858 4708	0.1720 4234	0.5063 1891	0.5122 6868
04	0.4989 9257	0.0858 4867	0.1720 4399	0.5063 2359	0.5122 7353
05	0.4989 9704	0.0858 5027	0.1720 4564	0.5063 2827	0.5122 7838
06	0.4990 0152	0.0858 5186	0.1720 4730	0.5063 3295	0.5122 8322
07	0.4990 0599	0.0858 5346	0.1720 4895	0.5063 3763	0.5122 8807
08	0.4990 1047	0.0858 5505	0.1720 5060	0.5063 4231	0.5122 9292
09	0.4990 1494	0.0858 5665	0.1720 5225	0.5063 4699	0.5122 9777
29 21 10	0.4990 1942	0.0858 5824	0.1720 5391	0.5063 5167	0.5123 0262
11	0.4990 2389	0.0858 5983	0.1720 5556	0.5063 5635	0.5123 0747
12	0.4990 2836	0.0858 6143	0.1720 5721	0.5063 6103	0.5123 1231
13	0.4990 3248	0.0858 6302	0.1720 5887	0.5063 6571	0.5123 1716
14	0.4990 3731	0.0858 6462	0.1720 6052	0.5063 7039	0.5123 2201
15	0.4990 4179	0.0858 6621	0.1720 6217	0.5063 7507	0.5123 2686
16	0.4990 4626	0.0858 6781	0.1720 6382	0.5063 7975	0.5123 3171
17	0.4990 5073	0.0858 6940	0.1720 6548	0.5063 8443	0.5123 3655
18	0.4990 5521	0.0958 7110	0.1720 6713	0.5063 8911	0.5123 4140
19	0.4990 5968	0.0858 7259	0.1720 6878	0.5063 9379	0.5123 4625
29 21 20	0.4990 6415	0.0858 7419	0.1720 7043	0.5063 9846	0.5123 5110
21	0.4990 6863	0.0858 7578	0.1720 7209	0.5064 0315	0.5123 5595
22	0.4990 7310	0.0858 7738	0.1720 7374	0.5064 0782	0.5123 6079
23	0.4990 7758	0.0858 7897	0.1720 7539	0.5064 1250	0.5123 6564
24	0.4990 8205	0.0858 8056	0.1720 7705	0.5064 1718	0.5123 7049
25	0·4990 8653	0.0858 8216	0.1720 7870	0.5064 2186	0.5123 7534
26	0.4990 9100	0.0858 8375	0.1720 8035	0.5064 2654	0.5123 8019
27	0.4990 9547	0.0858 8535	0.1720 8200	0.5064 3122	0.5123 8504
28	0.4990 9995	0.0858 8694	0.1720 8366	0.5064 3590	0.5123 8988
29	0.4991 0442	0.0858 8854	0.1720 8531	0.5064 4058	0.5123 9473
29 21 30	0.4991 0890	0.0858 9013	0.1720 8696	0.5064 4526	0.5123 9958

$$S_0 \cdot \sin(\theta - \Phi_0) = 2R \operatorname{coscl} \tau \cdots\cdots\cdots (2\text{-}6)$$

Dividing (2-6) by (2-5)

$$\tan(\theta - \Phi_0) = \frac{\operatorname{coscl} \tau}{\operatorname{sincl} \tau} \cdots\cdots\cdots (2\text{-}7)$$

In this equation $(\theta - \Phi_0)$ is clearly a bearing angle of a point P between the abscissa and a segment from origin to the point P on the clothoid, and it is usually denoted by σ.

Therefore, if one defines

$$\frac{\operatorname{coscl} \tau}{\operatorname{sincl} \tau} = \operatorname{tancl} \tau \cdots\cdots\cdots (2\text{-}8)$$

then

$$\tan \sigma = \operatorname{tancl} \tau \cdots\cdots\cdots (2\text{-}9)$$

Table 1 and 2 is a new clothoid table made by the authors. An example of a problem using this table is given below :

Example 1.

Data :

	X	Y
P_0	65 381.256 m	38 109.125 m
P_1	62 996.825	38 581.362

	R	Φ
P_0	∞	182°21′35.″6
P_1	to be found	to be found

then :

$$\tan \theta = \frac{Y_1 - Y_0}{X_1 - X_0} = \frac{472\,237}{2\,384\,431}$$
$$= -0.198050185$$

and

$$\theta = 168°47'51''1$$
$$\sigma = \theta - \Phi_0 = -13°33'44''5$$
$$\operatorname{tancl} \tau = \tan \sigma = -0.24123020$$

then from the clothoid table 1 above;

$$\tau = -0.71323342 \quad (-40°51'55''0)$$
$$\Phi_1 = \Phi_0 + \tau = 223°13'30''6$$
$$\operatorname{sincl} \tau = -0.67779561$$
$$R = \frac{(X_1 - X_0)\cos \Phi_0 + (Y_1 - Y_0)\sin \Phi_0}{2 \operatorname{sincl} \tau}$$
$$= -1\,743.1243 \text{ m}$$
$$A^2 = 2\tau R^2 = -4\,334\,294.087$$

(imaginary clothoid)

and

$$A = 2\,081.8963$$

In order to do this calculation by electronic computor, it is necessary to use an inverse function of the clothoidal tangent. The inverse function of tancl is expanded by the Tchebychev interpolation as follows :

$$\begin{aligned}\tau = {} & 2.99999\ 99616 \tan\sigma \\ & -.77142\ 21291 \tan^3\sigma \\ & +.47323\ 21515 \tan^5\sigma \\ & -.32278\ 31681 \tan^7\sigma \\ & +.19801\ 37799 \tan^9\sigma \cdots\cdots\cdots (2\text{-}10)\end{aligned}$$

Provided that $|\tan \sigma| \leq 0.340182285$ and $|\tau| \leq 1.000000000$

2-2. The case when the curvature at any one point is given.

This is a case where Φ_0 and τ are unknown in the formulae (2-1) and (2-2). In order to find a solution to this case, equations (2-1) and (2-2) are squared and added :

$$S^2 = (X_1 - X_0)^2 + (Y_1 - Y_0)^2 = 4R^2 \times (\operatorname{sincl}^2 \tau + \operatorname{coscl}^2 \tau) \cdots\cdots\cdots (2\text{-}11)$$

according to the definition (1-13) :

$$\operatorname{chordcl} \tau = \frac{\sqrt{(X_1 - X_0)^2 + (Y_1 - Y_0)^2}}{2R} \cdots\cdots\cdots (2\text{-}12)$$

Then, τ can be found from (2-12) and σ can be found from (2-9).

And, as

$$\tan \theta = \frac{Y_1 - Y_0}{X_1 - X_0}$$

then

$$\Phi_0 = \theta - \sigma$$

Therefore, one can see, if the curvature of a point is given, the clothoid passing through this point and the origin can be determined uniquely together with azimuth of the back tangent Φ_0.

A numerical example of the this case is as follows :

Example 2.

Data :

	X	Y
P_0	65 381.256 m	38 109.125 m
P_1	62 996.825	38 581.362

	R	Φ
P_0	∞	To be found
P_1	2 400 m	To be found

from eqn. (2-12)

$$\frac{\sqrt{(X_1 - X_0)^2 + (Y_1 - Y_0)^2}}{2R}$$
$$= \frac{\sqrt{(2\,384.431)^2 + (472\,237)^2}}{2 \times 2\,400}$$
$$= 0.5064051105$$

looking up chordcl τ in table 2

$$\tau = 0.51235798 \quad (29°21'21''4)$$

$$\tan\theta = \frac{472.237}{-2\,384.431} = -0.19805019$$
$$\theta = 168°47'51''1$$
$$\tan\sigma = \text{tancl}\,\tau = 0.17207278$$
$$\sigma = 9°45'48''4$$
$$\Phi_0 = \theta - \sigma = 159°2'2''7$$
$$\Phi_1 = \Phi_0 + \tau = 188°23'24''1$$
$$A^2 = 2\tau R^2 = 5\,902\,363.93$$
$$A = 2{,}429.4781$$

2-3. The case when A is given in the problems 2-1 and 2-2.

In this problem, in addition to (2-1) and (2-2) following condition exists

$$2\tau R^2 = A^2 \quad \cdots\cdots(2\text{-}13)$$

namely, it is necessary to solve a set of simultaneous equations with three unknowns.

For example, the case where the clothoid origin is fixed at a certain point (X_0, Y_0) and the clothoid with a given parameter must pass through a point (X_1, Y_1) is of this type.

To solve the simultaneous equation, substituting (2-13) into (2-11)

$$S^2 = (X_1 - X_0)^2 + (Y_1 - Y_0)^2 = \frac{2A^2}{\tau}(\text{sincl}^2\tau + \text{coscl}^2\tau) \quad \cdots\cdots(2\text{-}14)$$

In order to solve this problem using table 1, first $S^2/2A^2$ must be computed, then one is able to find τ by looking in the column $\frac{\text{chordcl}^2\tau}{\tau}$.

To solve this problem using an automatic computor it is necessary to make an expansion series of the inverse function of $\frac{1}{\tau}$ chordcl$^2\tau$.

The same must be mentioned about the inverse function of chordcl τ for the case described in the last section.

They are expanded by the Tchebychev method as same as the inverse function of tancl τ, as follows :

putting

$$y = \text{chordcl}\,\tau$$
$$\tau = 1.0000000021\,y + 0.0444443728\,y^3 + 0.0052212011\,y + 0.0008016630\,y^7 + 0.0001499600\,y^9 + 0.0000160114\,y^{11} + 0.0000119564\,y^{13} \quad \cdots\cdots(2\text{-}15)$$

Provided

$$|\tau| \leq 1$$

and putting

$$x = \left(\frac{\text{chordcl}^2}{\sqrt{\tau}}\right)^{2/3}$$
$$\tau = +2.08008382638\,x + 0.42857078435\,x^3 + 0.228010647\,x^5 + 0.158127474\,x^4 + 0.134206515\,x^9 + 0.061859355\,x^{11} + 0.2216573x^{13} \quad \cdots\cdots(2\text{-}16)$$

Provided

$$|\tau| < 1$$

These approximation formulae are listed in the appendix with other various functions.

2-4. The case when *R* or *τ* is given with the clothoid parameter A.

This problem is the same as the usual method where the elements of the alignment were graphically determined.

Since the co-ordinates (x_1, y_1) of a point on a clothoid are given by

$$x_1 = 2R_1 \text{sincl}\,\tau_1$$
$$y_1 = 2R_1 \text{coscl}\,\tau_1$$

The problem is reduced to the simultaneous equations :

$$(X_1 - X_0)\cos\Phi_0 + (Y_1 - Y_0)\sin\Phi_0 = x_1 \quad \cdots\cdots(2\text{-}17)$$
$$-(X_1 - X_0)\sin\Phi_0 + (Y_1 - Y_0)\cos\Phi_0 = y_1 \quad \cdots\cdots(2\text{-}18)$$

This problem can be classified into two cases;

(a) How to determine X_1 and Y_1 when the azimuth of the back tangent Φ_0 is given and the origin of the clothoid is fixed, and

(b) How to determine X_0 and Y_0 when the location of the point (X_1, Y_1) on the clothoid and the azimuth of the back tangent Φ_0 are given

These two cases can be both solved using the equations;

$$X_1 - X_0 = x_1\cos\Phi_0 - y_1\sin\Phi_0 \quad \cdots\cdots(2\text{-}19)$$
$$Y_1 - Y_0 = x_1\sin\Phi_0 + y_1\cos\Phi_0 \quad \cdots\cdots(2\text{-}20)$$

2-5. The case where the back tangent is fixed.

In this case, the back tangent is fixed by giving Φ_0 and a point P_T on the tangent, and the clothoid origin $P_0(X_0, Y_0)$ is able to move along this line. To solve this problem, it is only necessary to introduce one more equation;

$$Y_0 - Y_T = (X_0 - X_T)\tan\Phi_0 \quad \cdots\cdots(2\text{-}21)$$

Then substituting (2-21) into (2-1) and (2-2), one finds

$$(X_1-X_T)\cos\Phi_0+(Y_1-Y_T)\sin\Phi_0$$
$$-(X_0-X_T)\sec\Phi_0=2R\,\mathrm{sincl}\,\tau\cdots(2\text{-}22)$$

and

$$-(X_1-X_T)\sin\Phi_0+(Y_1-Y_T)\cos\Phi_0$$
$$=2R\,\mathrm{coscl}\,\tau\cdots\cdots(2\text{-}23)$$

If R is given in these equations, τ can be found from equation (2-23) using the inverse function of coscl τ. Consequently, the co-ordinates of the clothoid origin (X_0, Y_0) can be determined by solving equations (2-22) and (2-21).

The solutions of these cases are summed up in table 3.

Table 3

	given conditions	solutions
2-1	X, Y, X_0, Y_0, Φ_0	τ, R
2-2	X, Y, X_0, Y_0, R	τ, Φ_0
2-3	$X, Y, X_0, Y_0\ A$	τ, R, Φ_0
2-4	$A, R(\tau)\Phi, X, Y, (X_0, Y_0)$	$X_1, Y_1(X_0, Y_0)$
2-5	X, Y, X_τ, Y_τ	τ, X_0, Y_0

3. THE CASE WHERE A CENTRE OF THE CONTACT CIRCLE IS GIVEN.

Let the National plane co-ordinates of the clothoid origin and the centre of the contact circle be denoted by (X_0, Y_0) and (X_m, Y_m) respectively. In this clothoid there exists following relations.

$$(X_m-X_0)\cos\Phi_0+(Y_m-Y_0)\sin\Phi_0$$
$$=x_m=R\,\mathrm{sinhb}\,\tau\ \cdots\cdots(3\text{-}1)$$
$$-(X_m-X_0)\sin\Phi_0+(Y_m-Y_0)\cos\Phi_0$$
$$=y_m=R\,\mathrm{coshb}\,\tau\ \cdots\cdots(3\text{-}2)$$

These equations have the same form as the equations mentioned in the proceding section, except that sincl τ and coscl τ are replaced by sinhb τ and coshb τ.

Therefore, one is able to use the same method as given in paragraphs (2-1) to (2-5). The solutions of this case are listed in table 4.

The case 3-5 can be used when the parameter of a clothoid which is approximately parallel to a given clothoid is required. Fig. 6 illustrates an example of this type.

4. A CLOTHOID PASSING THROUGH ANY TWO POINTS.

Letting the National co-ordinates of the clothoid origin be X_0, Y_0, the azimuth of the back tangent be Φ_0, and let the co-ordinates of two points P_1 and P_2 on the clothoid be (X_1, Y_1) and (X_2, Y_2), and the azimuths of the tangent, tangent angles and radii of curvature at these points be Φ_1, Φ_2; τ_1, τ_2 and R_1, R_2 respectively. (Fig. 7)

Then :

Table 4

	Given conditions	Unknowns	Inverse function to be used
3-1	$(X_0, Y_0), (X_m, Y_m), \Phi_0$	R, τ	cothb τ
3-2	$(X_0, Y_0), (X_m, Y_m), R$	τ, Φ_0	chordhb τ
3-3	$(X_0, Y_0), (X_m, Y_m), A$	R, τ, Φ_0	$\frac{\mathrm{chordhb}^2\tau}{\tau}$
3-4	(X_m, Y_m), R or τ, A	X_0, Y_0	$\tan(\theta-\Phi_0)$
4-5	$(X_T, Y_T), (X_m, Y_m), \Phi_0$, R or τ	X_0, Y_0, τ or R	coshb τ

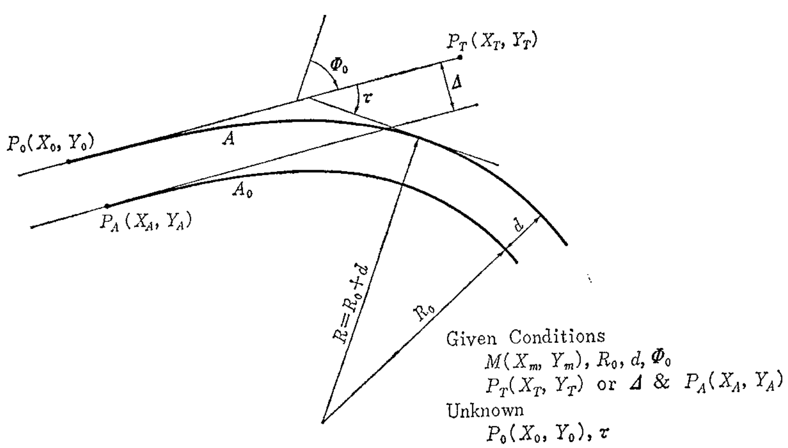

Fig. 6

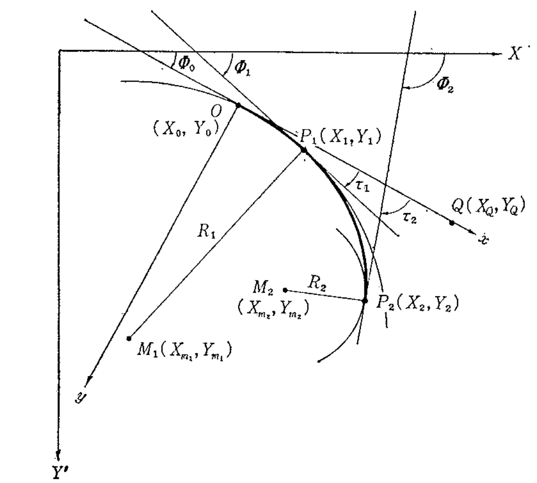

Fig. 7

$$(X_1-X_0)\cos\Phi_0+(Y_1-Y_0)\sin\Phi_0 =2R_1\text{sincl}\,\tau_1 \quad\cdots\cdots(4\text{-}1)$$

$$-(X_1-X_0)\sin\Phi_0+(Y_1-Y_0)\cos\Phi_0 =2R_1\text{coscl}\,\tau_1 \quad\cdots\cdots(4\text{-}2)$$

$$(X_2-X_0)\sin\Phi_0+(Y_2-Y_0)\sin\Phi_0 =2R_2\text{sincl}\,\tau_2 \quad\cdots\cdots(4\text{-}3)$$

$$-(X_2-X_0)\sin\Phi_0+(Y_2-Y_0)\cos\Phi_0 =2R_2\text{coscl}\,\tau_2 \quad\cdots\cdots(4\text{-}4)$$

$$\tau_1R_1^2=\tau_2R_2^2 \quad\cdots\cdots(4\text{-}5)$$

$$\Phi_1=\Phi_0+\tau_1 \quad\cdots\cdots(4\text{-}6)$$

and

$$\Phi_2=\Phi_0+\tau_2 \quad\cdots\cdots(4\text{-}7)$$

There are seven equations in 13 variables, so if the values of any six variables are known, the equations may be solved. This problem is separated into several kinds depending upon which seven are variables and which six are known.

Some typical problems in the case will be explained below and are listed in Table 5.

Table 5

	given conditions	solutions
4-1	X_0, Y_0, X_1, X_2, Y_1 Y_2	Φ_0, Φ_1, Φ_2, τ_1, τ_2, R_2, R_2
4-2	X_1, Y_1, R_1, X_2, Y_2, R_2	X_0, Y_0, Φ_0, Φ_1, τ_1, τ_2
4-3	X_1, Y_1, X_2, Y_2, Φ_0, A	X_0, Y_0, Φ_1, Φ_2, τ_1, τ_2, R_1, R_2
4-4	X_1, Y_1, X_2, Y_2, Φ_0, X_Q, Y_{Q0}	X_0, Y_0, Φ_1, Φ_2, τ_1, τ_2, R_1, R_2

4-1. Points (X_0, Y_0), (X_1, Y_1) and (X_2, Y_2) are given.

In this case it is enough to solve the set of five simultaneous equations and find τ_1, R_1, τ_2, R_2 and Φ_0, but it is impossible to obtain the rigorous solution of these equations directly and here one must try to solve them by successive approximation using the inverse functions which were mentioned previously. In this case one can usually find an approximate value of Φ_0, or in some cases of R_1 and R_2, and in a few cases, of A. Several cases with certain approximate values of some variables will be explained :

4-1-1. The case when the approximate value of Φ_0 can be found :

Putting

$$S_1\cos\theta_1=X_1-X_0 \qquad S_1\sin\theta_1=Y_1-Y_0$$

$$S_2\cos\theta_2=X_2-X_0 \quad\text{and}\quad S_2\cos\theta_2=Y_2-Y_0$$

then

$$\tan(\theta_1-\Phi_0)=\text{tancl}\,\tau_1 \quad\cdots\cdots(4\text{-}8)$$

$$\tan(\theta_2-\Phi_0)=\text{tancl}\,\tau_2 \quad\cdots\cdots(4\text{-}9)$$

As an approximate value of Φ_0 is given, approximate values of τ_1 and τ_2 can be found from the above equations.

Then, approximate values of R_1 and R_2 can also be computed from (4-1) and (4-2).

Denoting the corrections of these approximate values by $\Delta\Phi_0$, $\Delta\tau_1$, ΔR_1, $\Delta\tau_2$ and ΔR_2, respectively, and differentiating the equations (4-1), (4-2), (4-3), (4-4) and (4-5), one gets :

$$y_1\Delta\Phi_0=\Delta\tau_1 2R_1\frac{\partial}{\partial\tau_1}\text{sincl}\,\tau_1+2\Delta R_1\text{sincl}\,\tau_1 \quad\cdots\cdots(4\text{-}10)$$

$$-x_1\Delta\Phi_0=\Delta\tau_1 2R_1\frac{\partial}{\partial\tau_1}\text{coscl}\,\tau_1+2\Delta R_1\text{coscl}\,\tau_1 \quad\cdots\cdots(4\text{-}11)$$

$$y_2\Delta\Phi_0=\Delta\tau_2 2R_1\frac{\partial}{\partial\tau_2}\text{sincl}\tau_2+2\Delta R_2\text{sincl}\,\tau_2 \quad\cdots\cdots(4\text{-}12)$$

$$-x_2\Delta\Phi_0=\Delta\tau_2 2R_2\frac{\partial}{\partial\tau_2}\text{coscl}\,\tau_2+2\Delta R_2\text{coscl}\,\tau_2 \quad\cdots\cdots(4\text{-}13)$$

$$2\tau_1\Delta R_1-2\tau_2R_2\Delta R_2-R_1^2\Delta\tau_2-R_2^2\Delta\tau_2 =\tau_2R_2^2-\tau_1R_1^2 \quad\cdots\cdots(4\text{-}14)$$

and provided that

$$x_1=2R_1\text{sincl}\,\tau_1$$

$$y_1=2R_1\text{coscl}\,\tau_1$$

$$y_1=2R_1\text{sincl}\,\tau_2$$

$$y_2=2R_2\text{coscl}\,\tau_2$$

and

$$\frac{d}{d\tau}\text{sincl}\,\tau=1-\frac{3}{5}\frac{\tau^2}{2!}+\frac{5}{9}\frac{\tau^4}{4!}-\cdots\cdots$$

$$\frac{d}{d\tau}\text{coscl}\,\tau=\frac{2}{3}\tau-\frac{4}{7}\frac{\tau^3}{3!}+\frac{6}{11}\frac{\tau^5}{5!}-\cdots\cdots$$

One can obtain the corrections by solving the simultaneous equations (4-10) to (4-14). Each correction is derived as follows :

$$\Delta\Phi_0 = -\frac{\tau_2 R_2^2 - \tau_1 R_1^2}{R_2^2 U_2 - R_1^2 U_1 - 2\tau_2 R_2 V_2 - 2\tau_1 V_1 R_1} \quad \cdots\cdots (4\text{-}15)$$

$$\Delta\tau_1 = U_1 \Delta\Phi_0 \quad \cdots\cdots (4\text{-}16)$$

$$\Delta\tau_2 = U_2 \Delta\Phi_0 \quad \cdots\cdots (4\text{-}17)$$

$$\Delta R_1 = V_1 \Delta\Phi_0 \quad \cdots\cdots (4\text{-}18)$$

$$\Delta R_2 = V_2 \Delta\Phi_0 \quad \cdots\cdots (4\text{-}19)$$

where

$$U_i = \frac{x_i \operatorname{sincl} \tau_i + y_i \operatorname{coscl} \tau_i}{2R_i\left(\operatorname{coscl} \tau_i \frac{d}{d\tau_i} \operatorname{sincl} \tau_i - \operatorname{sincl} \tau_i \frac{d}{d\tau_i} \operatorname{coscl} \tau_i\right)} \quad \cdots\cdots (4\text{-}20)$$

$$V_i = \frac{y_i \frac{d}{d\tau_i} \operatorname{coscl} \tau_i + x_i \frac{d}{d\tau_i} \operatorname{sincl} \tau_i}{2\left(\operatorname{sincl} \tau_i \frac{d}{d\tau_i} \operatorname{coscl} \tau_i - \operatorname{coscl} \tau_i \frac{d}{d\tau_i} \operatorname{sincl} \tau_i\right)} \quad \cdots\cdots (4\text{-}21)$$

Repeating these calculations until $\tau_2 R_2^2 - \tau_1 R_1^2$ vanishes, one can obtain the solutions with a sufficient accuracy.

4-1-2. The case when approximate values of R_1 and R_2 are given :

The following two equations can be derived as in paragraph 2-2,

$$(X_1 - X_0)^2 + (Y_1 - Y_0)^2 = 4R_1^2 \operatorname{chordcl}^2 \tau_1 \quad \cdots\cdots (4\text{-}22)$$

$$(X_2 - X_0)^2 + (Y_2 - Y_0)^2 = 4R_2^2 \operatorname{chordcl}^2 \tau_2 \quad \cdots\cdots (4\text{-}23)$$

so that solving the inverse function of chordcl τ, one obtains the approximate values of τ_1 and τ_2, and these approximate values of τ_1 and τ_2 lead to two different values of Φ_0 from equations (4-8) and (4-9).

Taking their mean as the approximate value of Φ_0 and denoting the corrections for the approximate values of the unknowns as ΔR_1, ΔR_2, $\Delta\tau_1$, $\Delta\tau_2$ and $\Delta\Phi_0$ respectively, one can apply, the same method as in the last paragraph to solve this problem.

4-1-3. The case when the approximate value of A is given.

As in paragraph 2-3,

$$(X_1 - X_0)^2 + (Y_1 - Y_0)^2 = 2A^2 \frac{\operatorname{chordcl}^2 \tau_1}{\tau_1} \quad \cdots\cdots (4\text{-}24)$$

$$(X_1 - X_0)^2 + (Y_1 - Y_0)^2 = 2A^2 \frac{\operatorname{chordcl}^2 \tau_2}{\tau_2} \quad \cdots\cdots (4\text{-}25)$$

Consequently, from the inverse function of $\frac{\operatorname{chordcl}^2 \tau}{\tau}$, one can obtain the approximate values of τ_1 and τ_2. Putting these in the equations (4-1) and (4-3), the approximate values of R_1 and R_2 are obtained.

To this end, the method of solution for this case becomes entirely the same as that of the last paragraph.

4-2. The case when P_1 : (X_1, Y_1, R_1) and P_2 : (X_2, Y_2, R_2) are given.

If R_1 and R_2 are known and denoting the curve length between P_1 and P_2 as L_{12}, the relation

$$L_{12} = A^2\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \quad \cdots\cdots (4\text{-}26)$$

can be used effectively. The curve length L_{12} can be substituted approximately by the chord length S_{12} between P_1 and P_2, that is

$$L_{12} \fallingdotseq \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} = S_{12} \quad \cdots\cdots (4\text{-}27)$$

Then from equation (4-26) one gets the approximate values of A^2, and τ_1 and τ_2, and the clothoid co-ordinates (x_1, y_1), (x_2, y_2) can be obtained successively.

The equations (4-1) to (4-4) can be converted as

$$\tan(\Phi_{12} - \Phi_0) = \frac{y_2 - y_1}{x_2 - x_1} \quad \cdots\cdots (4\text{-}28)$$

where

$$\tan \Phi_{12} = \frac{Y_2 - Y_1}{X_2 - X_1} \quad \cdots\cdots (4\text{-}29)$$

and getting approximate values of Φ_0 from the equations (4-28) and (4-29), the approximate values of X_0, Y_0 can be obtained by

$$\tan(\Phi_1 - \Phi_0) = \operatorname{tancl} \tau_1 \quad \cdots\cdots (4\text{-}30)$$

where

$$s_1^2 = x_1^2 + y_1^2 \quad \cdots\cdots (4\text{-}31)$$

$$X_1 - X_0 = s_1 \cos \Phi_1 \quad \cdots\cdots (4\text{-}32)$$

$$Y_1 - Y_0 = s_1 \sin \Phi_1 \quad \cdots\cdots (4\text{-}33)$$

Next, denoting the corrections of each approximate value of the unknowns as ΔX_0, ΔY_0, $\Delta\tau_1$, $\Delta\tau_2$ and $\Delta\Phi_0$, the following equations can be derived from equations (4-1) to (4-5),

$$-\Delta X_0\cos\Phi_0-\Delta Y_0\sin\Phi_0+\Delta\Phi_0\cdot y_1-2R_1\frac{d}{d\tau_1}\text{sincl}\,\tau_1\Delta\tau_1=0 \quad\cdots\cdots(4\text{-}34)$$

$$+\Delta X_0\sin\Phi_0-\Delta Y_0\cos\Phi_0-\Delta\Phi_0 x_1-2R_1\frac{d}{d\tau_1}\text{coscl}\,\tau_1\cdot\Delta\tau_1=0 \quad\cdots\cdots(4\text{-}35)$$

$$-\Delta X_0\cos\Phi_0-\Delta Y_0\sin\Phi_0+\Delta\Phi_0\cdot y_2-2R_2\frac{d}{d\tau_2}\cdot\text{sincl}\,\tau_2\Delta\tau_2=0 \quad\cdots\cdots(4\text{-}36)$$

$$+\Delta X_0\sin\Phi_0-\Delta Y_0\cos\Phi_0-\Delta\Phi_0 x_2-2R_2\frac{d}{d\tau_2}\text{coscl}\,\tau_2\Delta\tau_2=0 \quad\cdots\cdots(4\text{-}37)$$

$$R_2^2\Delta\tau_2-R_1^2\Delta\tau_1=\tau_1R_1^2-\tau_2R_2^2 \quad\cdots\cdots(4\text{-}48)$$

These simultaneous equations can be solved by interation, until $\tau_2R_2^2-\tau_1R_1^2$ vanishes with sufficient accuracy, and one obtains the final results.

The determinant of above simultaneous equations is

$$D=\begin{vmatrix} -\cos\Phi_0 & -\sin\Phi_0 & y_1 & -2R_1\frac{d}{d\tau_1}\text{sincl}\,\tau_1 & 0 \\ \sin\Phi_0 & -\cos\Phi_0 & -x_1 & -2R_1\frac{d}{d\tau_1}\text{coscl}\,\tau_1 & 0 \\ -\cos\Phi_0 & -\sin\Phi_0 & y_2 & 0 & -2R_2\frac{d}{d\tau_2}\text{sincl}\,\tau_2 \\ \sin\Phi_2 & -\cos\Phi_0 & -x_2 & 0 & -2R_2\frac{d}{d\tau_2}\text{coscl}\,\tau_2 \\ 0 & 0 & 0 & -R_1^2 & R_2^2 \end{vmatrix} \quad\cdots\cdots(4\text{-}39)$$

and as D does not vanish, the above equation has a solution.

Solving this equation one gets :

$$\Delta\tau_1=-\frac{\tau_2R_2^2-\tau_1R_1^2}{R_1}\,\frac{U_2}{U_2R_1-R_1U_2} \quad\cdots\cdots(4\text{-}40)$$

$$\Delta\tau_2=\frac{\tau_2R_2^2-\tau_1R_1^2}{R_2}\,\frac{U_1}{R_2U_1-R_1U_2} \quad\cdots\cdots(4\text{-}41)$$

where

$$U_i=(x_2-x_1)\frac{d}{d\tau_i}\text{sincl}\,\tau_i+(y_2-y_1)\frac{d}{d\tau_1}\text{coscl}\,\tau_1 \quad\cdots\cdots(4\text{-}42)$$

then, by equations (4-28) and succeeding equations one can get Φ_0, X_0 and Y_0, successively.

4-3. The case where (X_1, Y_1) and (X_2, Y_2), Φ_0 and A are given.

This problem seems to have some difficulties, in spite of its simple appearance. By substituting $\tau=A^2/2R^2$ in the equations (4-1) to (4-4),

$$\left.\begin{aligned}&(X_2-X_1)\cos\Phi_0+(Y_2-Y_1)\sin\Phi_0\\&\quad=2R_2\text{sincl}\frac{A^2}{2R_2^2}-2R_1\text{sincl}\frac{A^2}{2R_1^2}\\&-(X_2-X_1)\sin\Phi_0+(Y_2-Y_1)\cos\Phi_0\\&\quad=2R_2\text{coscl}\frac{A^2}{2R_2^2}-2R_1\text{coscl}\frac{A^2}{2R_1^2}\end{aligned}\right\} \quad\cdots\cdots(4\text{-}43)$$

Although these simultaneous equations could be solved theoretically for the two unknowns R_1 and R_2, practically it is not so easy, for it is difficult to find the first approximations of the two unknowns R_1 and R_2.

This point is considered below. There are two ways to find the approximate values which show themselves in the two quantities, i.e. the chord length between two known points on the curve, and the direction angle of this chord in the local clothoid co-ordinate system, that is

$$S^2=(X_2-X_1)^2+(Y_2-Y_1)^2 \quad\cdots\cdots(4\text{-}44)$$

$$\sigma_{12}=\arctan\frac{Y_2-Y_1}{X_2-X_1}-\Phi_0 \quad\cdots\cdots(4\text{-}45)$$

Neverthless, the clothoid curve has a distinct characteristic which reveals itself in the following relation

$$\frac{1}{R_2}-\frac{1}{R_1}=\frac{1}{R_{12}} \quad\cdots\cdots(4\text{-}46)$$

where $R_{12}=\dfrac{A^2}{L_{12}}$ is a characteristic quantity attached to any curve length between the two given points on the clothoid at which the curve has the radius R_1 and R_2, respectively. And this relation is valid everywhere on the curve, and the corresponding direction angle σ_{12} may have any value for any portion on the curve. This implies that the way to discern even approximately the places where two points concerned are lying, is in general, not accessible.

But from the view point of practical use, the portion of the clothoid which usually appears is confined to a small range, and we may estimate the first approximation with sufficient accuracy considering the above.

For example, we may assume that τ_1 or τ_2 may be approximately equal to σ_{12}, when it is clear that the chord length is quite short. And when chord length is correspondingly large, a good approximation is the assumption that the one of the two which lines near to the origin is equal to half of σ_{12}.

If one of the two τ's is obtained approximately, for example, τ_1, one is able to get at once the approximate value of R_1. Then R_2 can be obtained from the following equations as in the last paragraph,

$$S_{12}=\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2} \cdots\cdots (4\text{-}47)$$

$$\frac{1}{R_2}-\frac{1}{R_1}=\frac{A^2}{S_{12}} \cdots\cdots\cdots\cdots (4\text{-}48)$$

and the corresponding τ_2 can be derived. This means that the approximate values of the co-ordinates of the two points concerned in the local clothoid co-ordinates system have been found. Let these be denoted by (x_1,y_1) and (x_2,y_2), then the origin of clothoid co-ordinates system is found in two different ways as follows;

$$X_{01}=x_1\cos\Phi_0-y_1\sin\Phi_0+X_1 \cdots\cdots (4\text{-}49)$$

$$Y_{01}=x_1\sin\Phi_0+y_1\cos\Phi_0+Y_1 \cdots\cdots (4\text{-}50)$$

$$X_{02}=x_2\cos\Phi_0-y_2\sin\Phi_0+X_2 \cdots\cdots (4\text{-}51)$$

$$Y_{02}=x_2\sin\Phi_0+y_2\cos\Phi_0+Y_2 \cdots\cdots (4\text{-}52)$$

From the two values obtained above for the co-ordinates of the origin, the means are taken as :

$$X_0=\frac{1}{2}(X_{01}+X_{02}) \cdots\cdots\cdots\cdots (4\text{-}53)$$

$$Y_0=\frac{1}{2}(Y_{01}+Y_{02}) \cdots\cdots\cdots\cdots (4\text{-}54)$$

which are assumed to be nearer the true position of the co-ordinate origin. Once the tentative values of X_0 and Y_0 are determined, one can follow the procedures described in paragraph 2-1.

$$\tau_1=\text{arctancl}(\tan\sigma_1) \cdots\cdots\cdots\cdots (4\text{-}55)$$

$$\tau_2=\text{arctancl}(\tan\tau_2) \cdots\cdots\cdots\cdots (4\text{-}56)$$

and

$$R_1=\pm\sqrt{\frac{A^2}{2\tau_1}},\ R_2=\pm\sqrt{\frac{A^2}{2\tau_1}} \cdots (4\text{-}57)$$

where the signes of the square root term are to be as same as that of the original. The obtained new values (τ_1,R_1) and (τ_2,R_2) will be used for the iteration mentioned above, which is carried out until the differences between the two approximate values of the co-ordinate origins become sufficiently small, as they converge to the true position.

4-4. The case in which ground co-ordinates, P_1 : (X_1, Y_1), P_2 : (X_2, Y_2) of any two points on the clothoid, and the ground co-ordinates Q : (X_Q, Y_Q) of a point on its back tangent, the direction angle Φ_0 of which is known, are given.

Let the co-ordinates of the origin of the clothoid co-ordinate system be denoted by (X_0, Y_0), then the condition that the point Q lies on the back tangent whose direction angle is Φ_0 provides the equation

$$Y_0-Y_Q=(X_0-X_Q)\tan\Phi_0 \cdots\cdots\cdots (4\text{-}58)$$

which is another condition in addition to equations (4-1) to (4-5).

Of this case, one example where one point on the clothoid is given and another example where the centre of the contact circle is given were mentioned in the previous sections 2 and 3. In those examples there were only 3 condition equations, and one had to confine the number of unknowns to three, but, on the contrary, in the case when two points on a clothoid are given, since the number of equations for the 6 unknowns of τ_1, R_1, τ_2, R_2, X_0 and Y_0, are six as (4-1) to (4-5) and (4-58), the problem is solvable for 6 unknowns.

In the former case, substituting equation (4-58) in equations (4-1) and (4-2) one obtained the following equations :

$$(X_1-X_Q)\cos\Phi_0+(Y_1-Y_Q)\sin\Phi_0 + (X_Q-X_0)\sec\Phi_0=2R_1\text{sincl}\,\tau_1 \cdots\cdots (4\text{-}59)$$

$$-(X_1-X_Q)\sin\Phi_0+(Y_1-Y_Q)\cos\Phi_0 =2R_1\text{coscl}\,\tau_1 \cdots\cdots\cdots (4\text{-}60)$$

the latter of which enabled one to find τ_1 or R_1 corresponding to a given R_1 or τ_1, and putting them in equation (4-59) one got X_0. Then Y_0 was easily obtained from equation

(4-58).

Next the case in which two points on a clothoid are given is dealt with. The following two equations can be constructed for the second point, in the same way as for one point

$$(X_1-X_Q)\cos\Phi_0+(Y_1-Y_Q)\sin\Phi_0 + (X_Q-X_0)\sec\Phi_0=2R_2\mathrm{sincl}\,\tau_2 \quad\cdots\cdots(4\text{-}61)$$

$$-(X_2-X_Q)\sin\Phi_0+(Y_2-Y_Q)\cos\Phi_0 =2R_2\mathrm{coscl}\,\tau_2 \quad\cdots\cdots(4\text{-}62)$$

and, introducing one more condition equation

$$\tau_1R_1^2=\tau_2R_2^2 \quad\cdots\cdots(4\text{-}63)$$

one is able to find the 5 unknowns from 5 simultaneous equations.

To solve these equations, one has to start the computation with a certain approximate value which will be selected for one of the unknowns A, R_i, τ_i or X_0.

They will be dealt with in sequence below.

1) When A is given approximately, one finds τ_1, R_1, τ_2 and R_2, by using the inverse function of $\frac{1}{\sqrt{\tau}}$ coscl τ to obtain two different values for X_0. From the means of these one gets the approximate values of τ_1, R_1 and τ_2, R_2. These values will necessarily lead to two different values of A, the mean value of which enables one to make a calculation by interation approaching to the final value.

2) When R_1 is given approximately, one gets τ_1 from equation (4-60) and then X_0 from (4-59). On the other hand, the approximate values of τ_1 and R_1 will lead to an approximate value of A and from this, the approximate values of R_2, τ_2 and X_0 can be found from equations (4-61) and (4-62). The two different values of X_0 will make the problem as same as case (1). When an approximate value of τ_1 is given, the method is the same.

3) When an approximate value of X_0 is given, each pair of (R_1, τ_1) and (R_2, τ_2) will lead to two different values of A, the mean of which will enable one to make successive approximations.

5. THE CASE IN WHICH THE CENTRES OF TWO TANGENTIAL CIRCLES ARE GIVEN.

This problem is an expansion of the case described in paragraph 3, and there exists also the analogy of paragraph 4. The boundary condition equations are in general

$$(X_{m1}-X_0)\cos\Phi_0+(Y_{m1}-Y_0)\sin\Phi_0 =x_{m1}=R_1\mathrm{sinhb}\,\tau_1 \quad\cdots\cdots(5\text{-}1)$$

$$-(X_{m1}-X_0)\sin\Phi_0+(Y_{m1}-Y_0)\cos\Phi_0 =y_{m1}=R_1\mathrm{coshb}\,\tau_1 \quad\cdots\cdots(5\text{-}2)$$

$$(X_{m2}-X_0)\cos\Phi_0+(Y_{m2}-Y_0)\sin\Phi_0 =x_{m2}=R_2\mathrm{sinhb}\,\tau_2 \quad\cdots\cdots(5\text{-}3)$$

$$-(X_{m2}-X_0)\sin\Phi_0+(Y_{m2}-Y_0)\cos\Phi_0 =y_{m2}=R_2\mathrm{coshb}\,\tau_2 \quad\cdots\cdots(5\text{-}4)$$

$$\tau_1R_1^2=\tau_2R_2^2 \quad\cdots\cdots(5\text{-}5)$$

The method for solving the above equations is given in the Table 6.

6. CONCLUSION.

In this paper a new approach to the computation for a clothoid curve based on a new definition of clothoid functions is described. These new functions are summarized in Table 4.

Also in this paper, the inverse functions of each clothoid function are given in the form of a Tchebycheff's expansion which can be effectively used in computations by electronic computor.

In the usual computation of highway alignments using clothoid curves, each curve is con-

Table 6

	given conditions	solutions	approximate values	inverse functions
5-1	$(X_0, Y_0)(X_{m1}, Y_{m1})(X_{m2}, Y_{m2})$	$(\Phi_0, R_1, \tau_1, R_2, \tau_2)$	1) Φ_0 2) R_1, R_2 3) A	arc cothb τ chordhb τ $\frac{1}{\tau}$ chordhb^2 τ
5-2	$(R_1, X_{m1}, Y_{m1})(R_2, X_{m2}, Y_{m2})$	$(X_0, Y_0, \Phi_0, \tau_1, \tau_2)$	curve length	
5-3	$(X_1, Y_1)(X_2, Y_2)\Phi_0, A$	$(X_0, Y_0)(R_1, \tau_1)(R_2, \tau_2)$	τ	
5-4	$(X_1, Y_1)(X_2, Y_2)\Phi_0(X_Q, Y_R)$	$(X_0, Y_0)(R_1, \tau_1)(R_2, \tau_2)$	1) A 2) R 3) X_0	

nected by moving the circular arcs or the tangents to be satisfied by the given boundary conditions without changing the parameters which are graphically selected on the drawings. This method is sufficient when the location of the highway alignments may be selected rather freely.

However, there exist many cases where space is restricted and there is little or no room to adjust the location of the highway segments. Many sites for interchanges and urban highways are confronted by such lack of space where the location of arcs or tangents may not be adjusted. In such cases it is necessary to find the most suitable parameters which satisfy the given boundary conditions. The method which is discussed in this paper is surely an effective one for determining the most suitable parameters quickly and accurately by electronic computation.

The derivation of the coefficients of the Tchebycheff's expansions was undertaken by the first mentioned author.

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APPENDIX

APPROXIMATION FORMULAE OF VARIOUS FUNCTIONS

1. Tangentclothoid (tancl)

   Definition : tancl τ = coscl $\tau \div$ sincl τ

   Formula : x = tancl τ

   $$\tau = 2.99999\,99616\,x - 0.77142\,21291\,x^3 + 0.47323\,21515\,x^5 - 0.32278\,317\,x^7 + 0.19801\,38\,x^9$$

   Range of validity : $|x| \leq 0.34301\,82285$, $|\tau| \leq 1.00000\,00000$

   Maximum error : 1.3×10^{-9}

2. Chordclothid (chordcl)

   Definition : chordcl $\tau = (\text{sincl}\,\tau + \text{coscl}\,\tau)^{1/2}$

   Formula : x = chordcl τ

   $$\tau = 1.00000\,00021\,x + 0.04444\,43728\,x^3 + 0.00522\,12011\,x^5 + 0.00080\,16630\,x^7 + 0.00014\,99600\,x^9 + 0.00001\,60114\,x^{11} + 0.00001\,19564\,x^{13}$$

   Range of validity : $|x| \leq 0.95626$, $|\tau| \leq 1.00000$

   Maximum error : 4.2×10^{-9}

3. $x = \dfrac{\text{Chordclothoid}^2\tau}{\tau}$

   Formula :

   $$\tau = 1.00000\,00023\,x + 0.08888\,87214\,x^3 + 0.02032\,09304\,x^5 + 0.00605\,53079\,x^7 + 0.00223\,97671\,x^9 + 0.00030\,47543\,x^{11} + 0.00107\,39617\,x^{13} - 0.00060\,87792\,x^{15} + 0.00036\,87403\,x^{17}$$

   Range of validity : $|x| \leq 0.91443\,05160$, $|\tau| \geq 1.00000\,00000$

   Maximum error : 4×10^{-10}

4. Cosinclothoid

   Formula : $x = \text{coscl}^{1/2}\tau$

   $$\tau = 1.73205\,08085\,5\,x + 0.18557\,67459\,2\,x^3 + 0.05188\,25242\,x^5 + 0.01890\,7921\,x^7 + 0.00864\,2101\,x^9 + 0.00051\,7880\,x^{11} + 0.00807\,780\,x^{13} - 0.00425\,059\,x^{15}$$

   Range of validity : $|x| \leq 0.31026\,83017$, $|\tau| \leq 1.00000\,00000$

   Maximum error : $\sim 10^{-10}$

5. $\dfrac{\text{Cosineclothoid}\ \tau}{\tau^{1/2}}$

   Formula :

   $$x = \left(\frac{\text{coscl}}{\tau^{1/2}}\right)^{2/3}$$

   $$\tau = 2.08008\,38263\,8\,x + 0.42857\,07843\,5\,x^3 + 0.22801\,0647\,x^5 + 0.15812\,7474\,x^7 + 0.13420\,6515\,x^9 + 0.06185\,9355\,x^{11} + 0.22165\,73\,x^{13}$$

   Range of validity : $|x| \leq 0.31026\,83017$, $|\tau| \leq 1.00000\,00000$

   Maximum error : 5×10^{-10}

6. Cotangenthybrid (cothb)

   Definition : cothb τ = sinhb $\tau \div$ coshb τ

   Formula : x = cothb τ

   $$\tau = \frac{1.00000\,00052\,x - 1.13285\,19856\,x^3 + 0.32838\,15154\,x^5 - 0.01524\,68524\,x^7}{1.00000\,00000\,x^2 - 1.33285\,17446\,x^4 + 0.51516\,07337\,x^6 - 0.05161\,06533\,x^8}$$

   Range of validity : $|x| \leq 0.83351\,60378$, $|\tau| \leq 1.00000\,00000$

   Maximum error : 7×10^{-9}

7. Chordhybrid (chordhb)

   Definition : chordhb $\tau = \pm(\text{sinhb}^2\tau + \text{coshb}^2\,\tau)^{1/2}$

   Formula : $x = \pm(\text{chdhb}^2\tau - 1)^{1/2}$

   $$\tau = 0.86602\,54035\,x + 0.01237\,17922\,x^3 + 0.00039\,36517\,x^5 + 0.00001\,53088\,x^7 + 0.00000\,06450\,x^9 + 0.00000\,00319\,x^{11}$$

   Range of validity : $|x| \leq 1.13302\,80027$

$|\tau| \leqq 1.00000\,00000$

Maximum error : 2×10^{-10}

8. $\dfrac{\text{Chordhybrid}^2\,\tau}{\tau}$

Formula : $x=\dfrac{\tau}{\text{chdhb}^2\,\tau}$

$y=2.99990\,67417\,31(0.43961\,56871\,594-x)^{1/2}$
$-0.99452\,18953\,68$

$\tau=0.39942\,28136-0.40999\,14851\,y$
$+0.04894\,92429\,y^2-0.04041\,24947\,y^3$
$+0.00585\,30330\,1\,y^4-0.00577\,38835\,y^5$
$+0.00064\,31927\,y^6-0.00093\,82707\,y^7$
$+0.00005\,23247\,y^8-0.00018\,81463\,y^9$
$-0.00000\,69905\,y^{10}-0.00001\,35168\,y^{11}$
$+0.00000\,19115\,y^{12}-0.00001\,84593\,y^{13}$
$-0.00000\,41506\,y^{14}$

Range of validity : $|x| \leqq 0.43961\,56871\,594$
$y \leqq 0.99452\,18953\,683$
$|\tau| \leqq 0.90845\,35997\,959$

Maximum error : 5×10^{-10}

9. Cosinehybrid

Formula : $x=(\text{coshb}^2\tau-1)^{1/2}$

$\tau=2.44948\,97399\,x+0.26244\,56401\,x^3$
$+0.06500\,52333\,x^5+0.02025\,21109\,x^7$
$+0.00613\,39985\,x^9+0.00437\,95100\,x^{11}$

Range of validity : $x \leqq 0.40104\,72657$
$\tau \leqq 1.00000\,00000$
$\text{coshb}\,\tau \leqq 1.16083\,89093$

Maximum error : 1×10^{-10}

10. $\dfrac{\text{Hybridcosine}^2}{\tau}$

Formula : $x=\dfrac{\tau}{\text{coshb}^2\,\tau}$, $\tau=A/B$

$A=1.00000\,00092\,20\,x-3.22550\,59287\,12\,x^3$
$+3.41709\,06714\,73\,x^5-1.26894\,911777\,62\,x^7$
$+0.09318\,52305\,75\,x^9$

$B=1.00000\,00000\,00-3.55883\,86708\,68\,x^2$
$+4.36526\,35935\,22\,x^4-2.09179\,72875\,80\,x^6$
$+0.29786\,69266\,00\,x^8$

Range of validity : $|x| \leqq 0.86144\,59698$
$|\tau| \leqq 1.00000\,00000$

Maximum error : 1.3×10^{-9}

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