BENDING OF A SIMPLY SUPPORTED CIRCULAR PLATE UNDER AN ECCENTRIC LOAD

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Bending of a circular plate simple supported along its edge and loaded at an arbitral point by a force or a couple is treated in this paper by employing bipolar coordinates. The problem of a circular plate subjected to an eccentric load has been discussed in various manner, since Clebsch obtained a series solution in polar coordinates of this problem. But the case of simply supported edge is more complicated than that of clamped edge and also these discussions were restricted to the case of a concentrated force. In this paper, a series solution for the case of a concentrated force in bipolar coordinates is first derived, and then it is seen that solutions for both cases of a bending and a twisting couple can be expressed in modified forms of the first solution.

In the case of a couple, if considered as concentrated, the slope of the deflection surface under the load point becomes infinitely large, and to eliminate this difficulty a small circular portion to which the couple may be regarded as applied must be considered as absolutely rigid, and it is noted that such consideration is only practicable for the use of bipolar coordinates as worked in this paper. Values of deflections, bending moments and reactive forces for some typical load positions are calculated and illustrated in the case of a concentrated force, and relations between the eccentricity of the load and the deflection or the slope of the rigid portion are observed in the case of couples.

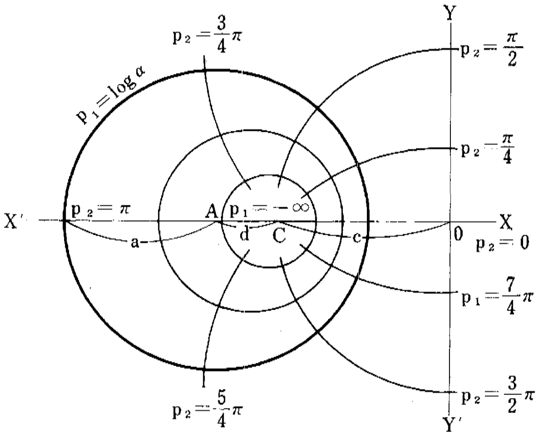

Fig. 1.

Let us employ bipolar coordinates given by

$$p_1=\tanh^{-1}\frac{2cx}{x^2+y^2+c^2}\ , \qquad p_2=\tan^{-1}\frac{-2cy}{x^2+y^2-c^2} \qquad \dots\dots\dots\dots\dots\dots(1)$$

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where $$c=-a\sinh p_{10}\quad,\qquad p_{10}=\log\alpha\quad,$$ $$\alpha=\frac{d}{a}\qquad\text{(modulus of eccentricity)},\qquad\ldots\ldots(2)$$

then it is seen that one of poles of these coordinates will coincide with the point where the load is concentrated. (as shown in Fig. 1) The general solution of $\Delta^2 w=0$, the fundamental differential equation concerning deflection of thin plates, with respect to bipolar coordinates can be written in the series form

$$w=\frac{1}{h}\sum_{n=0} f_n(p_1)\begin{matrix}\cos\\ \sin\end{matrix}\, np_2\quad,\qquad h=\frac{1}{c}(\cosh p_1+\cos p_2)\cdots\cdots\cdots\cdots\cdots(3)$$

As the deflection w is to vanish at the boundary $p_1=p_{10}$, $f_n(p_1)$ may be written in the form

$$\left.\begin{aligned}
&f_0(p_1)=A_0e^{p_1}(p_1-p_{10})+B_0e^{-p_1}(p_1-p_{10})\\
&\qquad +A_0{}^*e^{-p_1}(e^{2p_1}-e^{2p_{10}})\\
&f_1(p_1)=A_1(e^{2p_1}-e^{2p_{10}})\\
&\qquad +B_1(e^{-2p_1}-e^{-2p_{10}})+C_1(p_1-p_{10})\\
&n\geq 2\\
&f_n(p_1)=e^{(n-1)p_1}[A_n(e^{2p_1}-e^{2p_{10}})\\
&\qquad +B_n(e^{-2np_1}-e^{-2np_{10}})+C_n\{e^{-2(n-1)p_1}\\
&\qquad -e^{-2(n-1)p_{10}}\}]
\end{aligned}\right\}\qquad\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

In the case of a concentrated load the deflection under the load must be finite, and from this it follows that

$$B_0=B_1=B_n=C_n=0\ (n\geq 2)\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

The remaining constants A_0, C_1 will be determined from the conditions of equilibrium of reactive force along the edge, and $A_0{}^*, A_1, \ldots\ A_n, \ldots$ from the conditions of simply support. In this paper, three cases of i) a concentrated force, ii) a bending couple (see Fig. 2 (a) and iii) a twisting couple (see Fig. 2 (b)) are treated and from symmetry of the deflection surfaces it may be understood that in eq. (3) we take cos in cases i) and ii), and sin in case iii). The magnitude per unit length of the bending moments acting on a side parallel to direction p_1 = const is denoted by $M_{(11)}$, which may be expressed in the form

$$M_{(11)}=-\frac{D}{c}\sum_{n}m_n(p_1)\begin{matrix}\cos\\ \sin\end{matrix}\, np_2\qquad\cdots\cdots\cdots(6)$$

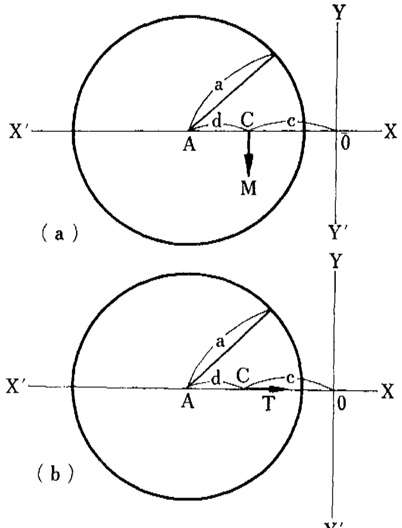


Fig. 2.

where D is the flexural rigidity of the plate and $\underset{1}{m}_n(p_1)$ are functions to be expressed in terms of $f_n(p_1)$. For other moments and forces, we can also give analogous expressions.

Now we consider the problem dividing into three cases:

i) Case of a concentrated force

If we denote the reactive force acting at the edge of the plate by $V_{(1)}$, the conditions of equilibrium of the reactive force are $P=\oint V_{(1)}ds$, $Pc=\oint V_{(1)}x\,ds$ in this case. From these conditions we find

$$A_0=-\sinh p_{10}\,K \quad , \quad C_1=0 \qquad \cdots\cdots(7)$$

From the conditions of simply supported edge $\underset{1}{m}_n(p_1)=0$, we have

$$A_0{}^*=\frac{1}{(1+\nu)\sinh p_{10}}\left(\frac{\beta}{2}A_0+\alpha A_1\right) \qquad \cdots\cdots(8)$$

$$\left.\begin{aligned}&\beta A_1+2\alpha A_2=H\\ &\alpha A_1+(\beta+2\cosh p_{10})A_2+3\alpha A_3=0\\ &(n-1)\alpha A_{n-1}+\{\beta+2(n-1)\cosh p_{10}\}A_n\\ &\qquad+(n+1)\alpha A_{n+1}=0\end{aligned}\right\} \qquad \cdots\cdots(9)$$

in which

$$\beta=2\cosh p_{10}-(1+\nu)\sinh p_{10} \qquad \cdots\cdots(10)$$

$$H=\alpha^{-1}\sinh p_{10}\,K \qquad \cdots\cdots(11)$$

Though analytical solutions of eqs. (9) are not obtainable, we can derive some approximate expression of A_n by treating the first two or three equations in (9), or letting

$$A_n=(-1)^n a_n K \qquad \cdots\cdots(12)$$

we can calculate values of a_n taking advantage of the following asymptotic formulas:

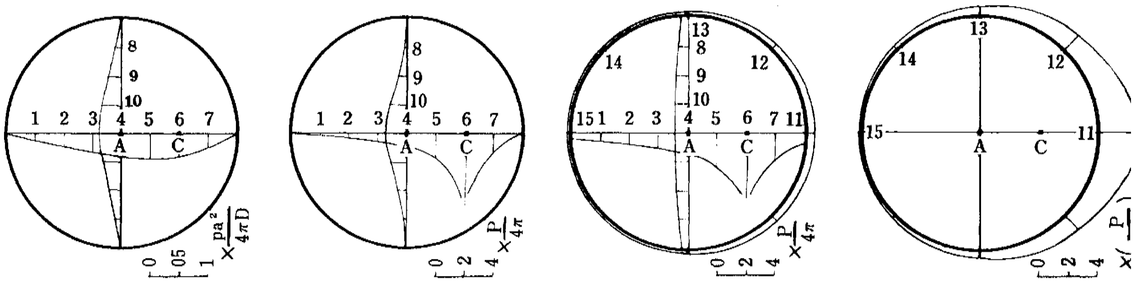
$$a_n=\lim_{N\to\infty}\underset{N}{a_n} \qquad \cdots\cdots(13)$$

$$\underset{N}{a_n}=\frac{-\sinh p_{10}}{n}\,\frac{\beta_{N,N-n}}{\beta_{N,N}} \qquad \cdots\cdots(14)$$

$$
\left.\begin{aligned}
&\text{where} \quad \beta_{N,n}=\beta_{N-n+1,1}\beta_{N,n-1}-\alpha^2\beta_{N,n-2} \\
&\qquad \text{specially} \\
&\qquad \beta_{N,0}=1 \\
&\qquad \beta_{N,1}=\alpha\left(2\cosh p_{10}-\frac{1-\nu}{N}\sinh p_{10}\right)
\end{aligned}\right\}\cdots\cdots\cdots\cdots(15)
$$

Table 1.

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
a_1	3.0074	1.4691	0.9404	0.6633	0.4861	0.3577	0.2556	0.1680	0.0863
a_2	1.1361	0.5569	0.3585	0.2551	0.1892	0.1415	0.1034	0.0701	0.0379
a_3	0.6229	0.3059	0.1976	0.1413	0.1056	0.0797	0.0591	0.0409	0.0228
a_4	0.4021	0.1977	0.1280	0.0918	0.0689	0.0524	0.0392	0.0275	0.0158
a_5	0.2847	0.1401	0.0909	0.0654	0.0492	0.0376	0.0283	0.0201	0.0117
a_6	0.2141	0.1055	0.0685	0.0493	0.0372	0.0285	0.0216	0.0155	0.0092
a_7	0.1680	0.0828	0.0538	0.0388	0.0293	0.0226	0.0172	0.0124	0.0074
a_8	0.1359	0.0670	0.0436	0.0315	0.0238	0.0184	0.0140	0.0102	0.0062
a_9	0.1127	0.0556	0.0362	0.0261	0.0198	0.0153	0.0117	0.0085	0.0052
a_{10}	0.0953	0.0470	0.0306	0.0221	0.0168	0.0130	0.0100	0.0073	0.0045
φ_m	6.702	3.667	2.826	2.577	2.633	2.958	3.674	5.257	10.196
φ_t	6.500	3.250	2.167	1.625	1.300	1.083	0.929	0.813	0.722



W−diagram $M_{(rr)}$−diagram $M_{(\theta\theta)}$−diagram $V_{(1)}$−diagram

Fig. 3. Case of a concentrated force ($\alpha=0.5$)

Calculations of values of a_n are easily carried out by use of the electronic digital computer. Table 1 shows values of a_n thus obtained. (assumed $\nu=0.3$) Deflections, bending moments and reactive forces in case $\alpha=0.5$ are illustrated in Fig. 3 as an example of the results in this case.

ii) Case of a bending couple (shown in Fig. 2 (a))

We regard the couple as a concentrated one at first. From the conditions of equilibrium, similarly as in the preceding case, we find

$$A_0=C_1=K \qquad \left(K=\frac{M}{4\pi D}\right) \qquad \cdots\cdots\cdots\cdots(16)$$

In this case, we can also derive the eqs. (8) and (9), but instead of (11) we must use

$$H = -\alpha^{-1}\left(1-\frac{1+\nu}{2}\alpha^{-1}\sinh p_{10}\right)K \qquad \cdots\cdots(17)$$

Therefore, we can write

$$\left.\begin{aligned} A_n &= (-1)^n \varphi_M(\alpha) a_n' K \\ \varphi_M(\alpha) &= \frac{1+\nu}{2}\alpha^{-1} - \frac{1}{\sinh p_{10}} \end{aligned}\right\} \qquad \cdots\cdots(18)$$

The values of modifying factor $\varphi_M(\alpha)$ are given in Table 1 numerically.

iii) Case of a twisting coupl (shown in Fig. 2 (b))

We find similarly

$$C_1 = K \qquad \left(K = \frac{T}{4\pi D}\right) \qquad \cdots\cdots(19)$$

This time the eq. (11) must be replaced by

$$H = \frac{1+\nu}{2}\alpha^{-2}\sinh p_{10} K \qquad \cdots\cdots(20)$$

It follows that

$$\left.\begin{aligned} A_n &= (-1)^n \varphi_T(\alpha) a_n'' K \\ \varphi_T(\alpha) &= \frac{1+\nu}{2}\alpha^{-1} \end{aligned}\right\} \qquad \cdots\cdots(21)$$

Values of $\varphi_T(\alpha)$ are also listed in Table 1.

In above discussions, we considered that the couples were concentrated at a point of the plate, but in consequence the slope of the deflection surface under the load becomes infinitely large and in order to obtain the finite values of the slope we must consider that the small circular portion of radius b to which the couple may be regarded as acting is absolutely rigid and have some constant slope angle θ. (Fig. 4) To determine the constants B_n and C_n, as eq. (5) does not hold in this time, equations obtained from the boundary conditions along the circle K ($p_1 = \log\varepsilon$) should be taken into account.

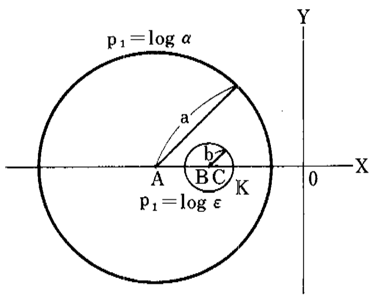


Fig. 4.

When we can regard $\varepsilon = \frac{1}{2\alpha k}\sqrt{(1-\alpha^2)^2+4\alpha^2k^2}-(1-\alpha^2)$ $\left(k=\frac{b}{a}\right)$ as quite small in comparison with 1, we obtain the following expressions

$$\left.\begin{aligned}
&A_0 = K\\
&B_0 = \frac{\varepsilon^2}{1+\nu}\left[\{(1-\nu)\alpha^2+(3+\nu)\}K+2(1-\nu)\alpha^2A_1\right]\\
&A_0^* = -\frac{1}{2(1+\nu)(1-\alpha^2)}\\
&\qquad\times\left[\{(1-\nu)\alpha^2+(3+\nu)\}K+4\alpha^2A_1\right]\\
&B_1 = \frac{\varepsilon^2}{2}K,\quad C_1=K\\
&B_n = C_n = 0 \qquad\qquad (n\geq 2)
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots(22)$$

(A_0, B_0 and A_0^* are unnecessary in case iii)) and as for the constants A_n, both eqs. (18) and (21) are also available in this time.

Using these constants, we can give the expressions for the deflection w_B of the center of the circular portion and the slope θ :

case ii)

$$\left.\begin{aligned}
&w_B = \frac{\alpha a}{2(1+\nu)}\left[\{(1-\nu)\alpha^2+(3+\nu)\}K+4\alpha^2A_1\right]\\
&\theta = \frac{2\alpha}{1-\alpha^2}\,\frac{w_B}{a}-\left(\frac{1}{2}+\log\frac{\varepsilon}{\alpha}\right)+\alpha^2A_1
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots(23)$$

case iii)

$$\left.\begin{aligned}
&w_B = 0\\
&\theta = -\left(\frac{1}{2}+\log\frac{\varepsilon}{\alpha}\right)+\alpha^2A_1
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots(24)$$

These relations are illustrated in Fig. 5 and Fig. 6 for cases k = 0.1 and k = 0.2.

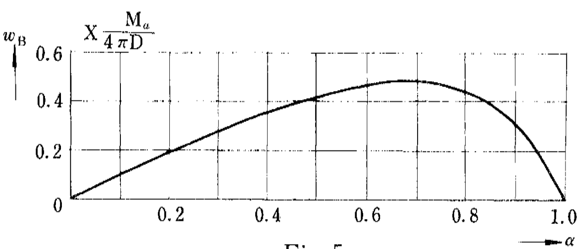


Fig. 5.

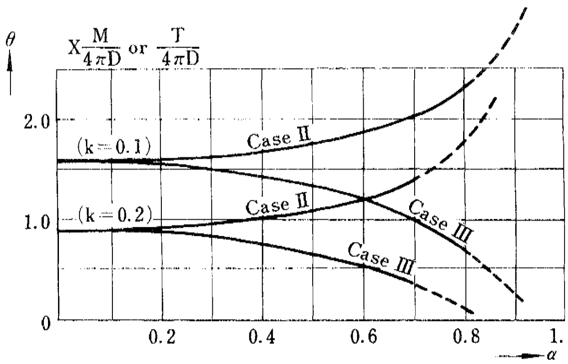


Fig. 6.

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