

THEORY OF CONSOLIDATION OF A POROUS MATERIAL WITH HEAT EFFECT BASED ON THE IRREVERSIBLE THERMODYNAMICS.

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Synopsis.

Based on the thermodynamics of linear irreversible process, an unified treatment is presented of the mechanics of deformation of porous elastic media through which compressible fluid and heat are flowing simultaneously. Coupling phenomenon between fluid flow and heat diffusion was investigated by applying the Onsager's principle to the possible irreversible changes within the solid-fluid system. By introducing the proper expression for the internal energy, physical meanings of the coefficients specifying the characteristics of the reversible changes of the system were examined in connection with experimental measurements. The equations of the coupled thermoelasticity and the poroelasticity developed by M.A. Biot are shown to be the special cases of the more general equation derived in this paper. By introducing the concepts of normalized coordinates used in the vibration problem of multi-degree of freedom, the derived set of basic equations is transformed in alternative form convenient for further analysis.

INTRODUCTION.

In recent years a considerable growth of thermodynamics has been achieved by introducing new and remarkably fruitful concepts and methods which lead to a phenomenological approach for irreversible phenomena. The major contribution by Onsager based on his famous reciprocity relations made it possible to broaden the scope and applicability of thermodynamics to a variety of phenomena from an unified point of view. This general thermodynamics was applied first by M.A. Biot^{(1), (2)} and J. Meixner⁽¹⁾ successfully to the fields of linear mechanics of solids such as viscoelasticity, thermoelasticity

and poroelasticity. As the results of their works, various fields of mechanics which had been developed separately on the less general basis were renewed in the frame work of general principle and brought within the scope of the single general theory.

The poroelasticity which describes the process of deformations and flow of fluid in porous media was treated extensively by Biot^{(2), (3)} as one branch of his more general approach to the mechanics of solids. But he did not take into account explicitly the effects of other factors such as heat, electric current, and chemical species which might have some influence on the behaviors of poroelastic materials. It seems of significance both from academic and practical point of view to extend his theory of consolidation to the case where thermal effect is included during the consolidation process. In fact, in the regions where seasonal or daily change of temperature is predominant, several problems of engineering practices have been raised concerning the treatment of soils which is subject to the acute change in temperature. The aim of the present paper is to extend Biot's theory of consolidation to include the effect of temperature change in the consolidation equation.

If we follow the way of approach similar to Biot, a difficulty may be confronted in that the coupling effect between flows of fluid and heat could not be introduced as a natural consequence of thermodynamic dissipation phenomena. This is because the dissipation function for the fluid flow through porous media was not derived from the rational mechanical basis, but was considered to be derivable from the conventional law of Darcy. It is therefore necessary in the first place to reconsider the Darcy's law and to afford a relevant interpretation to it in the realm of thermodynamic knowledge. On the other hand, the technique of nonreversible

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thermodynamics was applied quite recently by R.G. Mokadam³⁾ to the suitably defined solid-fluid system to yield a general equation of motion of a fluid through porous media. The Darcy law could then be obtained as a special case of his more general dissipation equations. With the use of his concept, it becomes possible to treat the process of fluid flow through pores on the same basis as the phenomenon of heat flow. Therefore, the difficulty as mentioned above can be eliminated and it is made possible to derive constitutive equations of our problem in a natural way. In the model of the porous media considered by Mokadam, a hypothesis is employed that the solid part of the entire system is fixed in space and is not allowed to deform. We will eliminate this restriction and consider the solid part to behave like elastic materials. With such modification we will attempt to establish general equations of the problem by the use of the concept introduced by Mokadam. As the results of our study, various new features will be exposed concerning the physical situation involved in the problem. Specific aspects derived will serve to a better understanding of the consolidation phenomenon coupled with other factors.

1. THERMODYNAMIC SYSTEM, EQUATIONS OF MOTION AND EQUATIONS OF CONTINUITY OF MATTERS.

In the model of a deformable porous medium, a suitable hypothesis should be made concerning the nature of the system under consideration in order that the set of macroscopic equations of dynamics and thermodynamics might be applied to the system. For this purpose, it will be hypothesized that the system consists of elastic skeletons uniformly dispersed in space and of compressible liquid or gas filling the pores of the skeletons. Furthermore, since the porous medium under consideration is considered to be essentially a two-phase system, it is necessary to employ bulk properties of the both phases, in contrast with the intrinsic properties of each phase. A detailed discussion is presented elsewhere⁵⁾ with regard to the relation between bulk and intrinsic properties of each phase and

we will not refer to it herein. In what follows, it is to be understood that the equations of motion, continuity of matter, energy, and entropy are described in terms of the bulk quantities of each phase. In establishing the equations of motion for solid and fluid phases, it is necessary to introduce, besides stress tensors and body forces, an additional force which could be caused by interaction of both phases. This force has the nature of volume force, because it acts in every point of infinitesimal bulk volume of the system. If attention is drawn to the infinitesimal volume of the liquid, the liquid may be subjected to the restraint by the existence of solid phase and can not move freely as is usually the case in fluid mechanics. The solid phase is an obstacle for fluid to move freely. This restraint for fluid due to the solid might give rise a force which may be referred to as "interacting force." Conversely, the existence of the pore fluid may not allow the solid phase to deform like non porous medium, because there is a resistance due to the obstacle of the fluid. It is apparent that this resistance is of equal magnitude but of opposite direction of action to the interacting force defined above, because the resistance to the solid results from the reaction of the force which is applied from the solid to the fluid phase. This interacting force transfers momentum internally from one phase to the other. Let Y_i be the interacting force per unit volume of the system and F_i be the gravitational force per unit mass. Then, since each phase will be accelerated by the bulk stress tensors, gravitational force and the interacting force, equations of motions are, for the solid phase,

$$\rho^s \frac{d\dot{u}_i}{dt} = \sigma_{ij,j} + \rho^s F_i + Y_i \dots\dots\dots(1-1)$$

and for the fluid phase,

$$\rho^w \frac{d\dot{v}_i}{dt} = p_{ij,j} + \rho^w F_i - Y_i, \dots\dots\dots(1-2)$$

where, ρ^s and ρ^w designate the bulk density of the solid and the fluid, respectively and are given by

$$\rho^s = \frac{M^s}{V}, \quad \rho^w = \frac{M^w}{V} \dots\dots\dots(1-3)$$

with the infinitesimal bulk volume V , and the corresponding mass M^s and M^w of the solid

and the fluid respectively. σ_{ij} and p_{ij} in the above expressions are the bulk stress tensors acting on the solid phase and the fluid phase respectively. The displacement and velocity components of the solid are designated by u_i and \dot{u}_i respectively and those for the fluid by v_i and \dot{v}_i .

The subscripts i, j range over the values 1, 2, 3 and repeated subscripts will be summed over the entire range. Subscripts following a comma denote partial differentiation with respect to Cartesian coordinates.

Equations of continuity of matter are written down separately for both phases as follows.

$$\frac{\partial \rho^s}{\partial t} + (\rho^s \dot{u}_i)_{,i} = 0 \dots\dots\dots(1-4)$$

$$\frac{\partial \rho^w}{\partial t} + (\rho^w \dot{v}_i)_{,i} = 0 \dots\dots\dots(1-5)$$

or alternatively,

$$\frac{d\rho^s}{dt} + \rho^s \dot{u}_i = 0 \dots\dots\dots(1-6)$$

$$\frac{d\rho^w}{dt} + \rho^w \dot{v}_i = 0 \dots\dots\dots(1-7)$$

2. ENERGY EQUATIONS.

In the thermodynamic system of the solid-fluid mixture, the specific total energy e^s of the solid consists of the specific kinetic energy $\frac{1}{2} \dot{u}_i \dot{u}_i$, the specific potential energy ψ and the specific internal energy u^s :

$$e^s = \frac{1}{2} \dot{u}_i \dot{u}_i + \psi + u^s \dots\dots\dots(2-1)$$

Similarly, the specific values of kinetic, potential and internal energies $\frac{1}{2} \dot{v}_i \dot{v}_i$, ψ and u^w respectively, constitute the specific total energy e^w of the fluid phase:

$$e^w = \frac{1}{2} \dot{v}_i \dot{v}_i + \psi + u^w \dots\dots\dots(2-2)$$

From a macroscopic point of view, the relations (2-1) and (2-2) can be considered as the definition of the internal energy u^s and u^w of the solid and fluid phases, respectively. Microscopically, they represent the energy of thermal agitation as well as the energy due to the short-range molecular interaction.

According to the principle of conservation of energy, the rate of change of the total energy per unit bulk volume of the solid is equal to

the energy flowing into (or out of) this volume through its boundary in the absence of production of energy due to chemical reaction.

However, in our two-phase system in addition to these usual categories of energy, such a new kind of energy that is transferred from one phase to the other should be considered to establish general energy equation. Indeed, the interacting force Y_i which has been introduced in eqs. (1-1) and (1-2) may give rise to the transfer of mechanical energy between two phases as the result of relative deformation between them. The "give and take" of such energy can be considered to occur not through the boundary of the volume, but uniformly within every point of the infinitesimal volume considered. Since no energy could be supplied from outside by the interacting force, the energy supplied from one system is equal to the one received by the other system. In other words, the algebraic sum of the energy received by both phases is equal to zero. From these consideration, it follows necessarily that only one way of expressing the amount of this energy transfer is as follows; the rate of the energy transfer from the fluid to the solid is equal to $Y_i(\dot{v}_i - \dot{u}_i)$ and the rate of the energy delivered to the fluid from the solid can be represented by $-Y_i(\dot{v}_i - \dot{u}_i)$. If we denote the energy flux across the boundary of the solid by J_i^s , the principle illustrated above gives,

$$\frac{\partial \rho^s e^s}{\partial t} = -J_{i,i}^s + Y_i(\dot{v}_i - \dot{u}_i) \dots\dots\dots(2-3)$$

Similar relation holds true also for the fluid phase:

$$\frac{\partial \rho^w e^w}{\partial t} = -J_{i,i}^w - Y_i(\dot{v}_i - \dot{u}_i) \dots\dots\dots(2-4)$$

where J_i^w designates energy flux of the fluid phase.

The energy flux J_i^s consists of three parts in our simple case, the energy flux $-\dot{u}_j \sigma_{ij}$ due to the mechanical work performed on the solid part of the system, the energy flux J_i^{sq} due to the heat flow, and the energy transport $\rho^s e^s \dot{u}_i$ due to convection of materials. Similarly the energy flux J_i^w includes the terms $-\dot{v}_j p_{ij}$, J_i^{wq} and $\rho^w e^w \dot{v}_i$

Then we have,

$$J_i^s = -\dot{u}_j \sigma_{ij} + J_i^{sq} + \rho^s e^s \dot{u}_i \dots\dots\dots(2-5)$$

$$J_i^w = -\dot{v}_j p_{ij} + J_i^{wq} + \rho^w e^w \dot{v}_i \dots\dots\dots(2-6)$$

*Ref. 6) p. 17.

where, σ_{ij} and p_{ij} are taken positive when they are tensile stresses. Inserting the expressions (2-1), (2-2) and (2-5), (2-6) into (2-3) and (2-4), we obtain explicit forms of energy equation in tensoral notation

$$\rho^s \frac{d}{dt} \left(\frac{1}{2} \dot{u}_i \dot{u}_i + \psi + u^s \right) = \dot{u}_{j,i} \sigma_{ij} + \dot{u}_{j,i} p_{ij} - J_{i,i} \dot{q}^s + Y_i (\dot{v}_i - \dot{u}_i) \dots\dots\dots(2-7)$$

$$\rho^w \frac{d}{dt} \left(\frac{1}{2} \dot{v}_i \dot{v}_i + \psi + u^w \right) = \dot{v}_{j,i} p_{ij} + \dot{v}_{j,i} p_{ij} - J_{i,i} \dot{q}^w - Y_i (\dot{v}_i - \dot{u}_i) \dots\dots\dots(2-8)$$

In order to obtain the balance equations for internal energies u^s and u^w , it is needed to eliminate the terms due to the kinetic energy and the potential energy.

If the equations of motion (1-1) and (1-2) are multiplied by \dot{u}_i and \dot{v}_i respectively, and summed up over the index i , it follows that:

$$\frac{1}{2} \rho^s \frac{d \dot{u}_i \dot{u}_i}{dt} = \dot{u}_i \sigma_{ij,j} + \rho^s F_i \dot{u}_i + Y_i \dot{u}_i \dots\dots\dots(2-9)$$

$$\frac{1}{2} \rho^w \frac{d \dot{v}_i \dot{v}_i}{dt} = \dot{v}_i p_{ij,j} + \rho^w F_i \dot{v}_i - Y_i \dot{v}_i \dots\dots\dots(2-10)$$

If the body force F_i is assumed to be conservative and represented by a time-independent potential function ψ in (2-1) and (2-2), it is easily shown with the use of (1-4) and (1-5) that ψ satisfies the relations* ;

$$\rho^s \frac{d \psi}{dt} = -\rho^s \dot{u}_i F_i \dots\dots\dots(2-11)$$

$$\rho^w \frac{d \psi}{dt} = -\rho^w \dot{v}_i F_i \dots\dots\dots(2-12)$$

By subtracting (2-9), (2-10) and (2-11), (2-12) from the energy equations (2-7), (2-8) the equations for the internal energies u^s and u^w are obtained,

$$\rho^s \frac{d u^s}{dt} = \dot{u}_{j,i} \sigma_{ij} - J_{i,i} \dot{q}^s + Y_i (\dot{v}_i - \dot{u}_i) - Y_i \dot{u}_i \dots\dots\dots(2-13)$$

$$\rho^w \frac{d u^w}{dt} = \dot{v}_{j,i} p_{ij} - J_{i,i} \dot{q}^w + Y_i \dot{u}_i \dots\dots\dots(2-14)$$

The strain rates in the above equations can be desolved into symmetric and antisymmetric parts as follows,

$$\left. \begin{aligned} \dot{u}_{i,j} &= \frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i}) + \frac{1}{2} (\dot{u}_{i,j} - \dot{u}_{j,i}) = \dot{e}_{ij} + \dot{\omega}_{ij} \\ \dot{v}_{i,j} &= \frac{1}{2} (\dot{v}_{i,j} + \dot{v}_{j,i}) + \frac{1}{2} (\dot{v}_{i,j} - \dot{v}_{j,i}) = \dot{e}_{ij} + \dot{\pi}_{ij} \end{aligned} \right\} \dots\dots\dots(2-15)$$

where \dot{e}_{ij} and $\dot{\pi}_{ij}$ denote the strain rates and $\dot{\omega}_{ij}$, $\dot{\pi}_{ij}$, the rates of rotation of the solid and the fluid, respectively. By referring to the identities,

$$\sigma_{ij} \dot{\omega}_{ij} = 0, \quad p_{ij} \dot{\pi}_{ij} = 0 \dots\dots\dots(2-16)$$

which follow immediately from the symmetry property of the stress tensors and the antisymmetry property of the rotation tensors, the energy equation (2-13) and (2-14) can be modified as follows,

$$\rho^s \frac{d u^s}{dt} = \sigma_{ij} \dot{e}_{ij} - J_{i,i} \dot{q}^s + Y_i (\dot{v}_i - \dot{u}_i) - Y_i \dot{u}_i \dots\dots\dots(2-17)$$

$$\rho^w \frac{d u^w}{dt} = p_{ij} \dot{e}_{ij} - J_{i,i} \dot{q}^w + Y_i \dot{u}_i \dots\dots\dots(2-18)$$

Adding eqs. (2-17) and (2-18) may allow to give the equation for internal energy u^t of the entire system including both solid and fluid.

$$\rho \frac{d u^t}{dt} = \rho^s \frac{d u^s}{dt} + \rho^w \frac{d u^w}{dt} = \sigma_{ij} \dot{e}_{ij} + p_{ij} \dot{e}_{ij} - J^q_{i,i} + Y_i (\dot{v}_i - \dot{u}_i) \dots\dots\dots(2-19)$$

where

$$\rho = \rho^s + \rho^w, \quad J_i^q = J_i^{qs} + J_i^{qw} \dots\dots\dots(2-20)$$

In deriving eq. (2-19) the total heat flow is assumed to be given by the sum of the heat flows through solid and fluid phases.

3. ENTROPY EQUATIONS.

In order to proceed in the development of the theory, it is necessary to relate the variations in the properties of the system considered above, to the rate of change of the entropy. The rate of change of the entropy s^s of the solid phase is given in this case by the well-known formula of Gibbs,

$$T \rho^s \frac{d s^s}{dt} = \rho^s \frac{d u^s}{dt} - \sigma_{ij} \dot{u}_{i,j} \dots\dots\dots(3-1)$$

Similarly, the change of the entropy s^w of the fluid phase is expressed by,

$$T \rho^w \frac{d s^w}{dt} = \rho^w \frac{d u^w}{dt} - p_{ij} \dot{v}_{i,j} \dots\dots\dots(3-2)$$

Eqs. (3-1) and (3-2) were derived by Gibbs for a system in equilibrium. However it was shown by Prigogine by use of statistical mechanics that the Gibbs equation is also valid for a system undergoing irreversible changes not far from equilibrium. It should be remarked here that the stresses σ_{ij} and p_{ij} appearing in Gibbs equation are to be interpreted as reversible forces, where as the stresses entering in the energy eqs. (2-19) and (2-20) or the equation of motion (1-1) and (1-2) are considered to include both reversible and irreversible forces.

In this paper, we will confine ourselves to

*Ref. 6) p. 16.

the consideration of the case where σ_{ij} does not include irreversible forces. This means that irreversible process due to viscous stresses in not taken into consideration.

The total change of entropy of the system as a whole is obtained by summation of eqs. (3-1) and (3-2).

$$T\rho \frac{ds^t}{dt} = T\rho^s \frac{ds^s}{dt} + T\rho^w \frac{ds^w}{dt} = \rho \frac{du^t}{dt} - \sigma_{ij} \dot{\epsilon}_{ij} - p_{ij} \dot{\epsilon}_{ij} \dots\dots\dots(3-3)$$

In deriving equation (3-3) from (3-1) and (3-2), the quantities $\dot{u}_{i,j}$ and $\dot{v}_{i,j}$ are replaced by strain rate tensors $\dot{\epsilon}_{ij}$ and $\dot{\xi}_{ij}$ with reference to (2-16).

In order to find the explicit form of the entropy balance equation, the expression (2-19) for the internal energy has to be inserted into eq. (3-3). This gives:

$$T\rho \frac{ds^t}{dt} = -J^q_{i,i} + Y_i(\dot{v}_i - \dot{u}_i) \dots\dots\dots(3-4)$$

It is easy to cast eq. (3-4) into the form of a balance equation.

$$\rho \frac{ds^t}{dt} = -\left(\frac{J^q_i}{T}\right)_{,i} + \frac{Y_i}{T}(\dot{v}_i - \dot{u}_i) - J^q_i \frac{T_{,i}}{T} \dots\dots\dots(3-5)$$

Equation (3-5) can be interpreted as stating that the rate of change of the entropy of the system consists of *the entropy flow and the entropy production*. Hence the entropy generation σ is given by,

$$\sigma = Y_i \frac{\dot{v}_i - \dot{u}_i}{T} - J^q_i \frac{T_{,i}}{T^2} \dots\dots\dots(3-6)$$

Two irreversible processes contribute to the generation of entropy. They are the heat flow in the presence of temperature gradient and the motion of the fluid relative to that of the solid under the action of the interacting force. The heat flow and the relative motion of the system between two phases are regarded as fluxes, and the temperature gradient and the interacting force are considered as corresponding forces in the thermodynamic sense. It is known empirically that for a large class of irreversible phenomena, the irreversible flows are linear functions of the thermodynamic forces, as expressed by the phenomenological laws. Therefore we may assume that:

$$J^q_i = -l_{ti} \frac{T_{,i}}{T^2} + l_{tv} Y_i \dots\dots\dots(3-7)$$

$$\frac{\dot{v}_i - \dot{u}_i}{T} = -l_{vt} \frac{T_{,i}}{T^2} + l_{vv} Y_i \dots\dots\dots(3-8)$$

where, l_{tt} , l_{tv} , l_{vt} and l_{vv} are called phenomenological coefficients since they are to be determined experimentally. According to the Onsager's reciprocal relation, the cross coefficients are assumed to be equal,

$$l_{tv} = l_{vt} \dots\dots\dots(3-9)$$

Eq. (3-9) indicates a restriction for the cross phenomena to occur. From (3-7), it is seen that the heat flow is caused by both the temperature gradient and the interacting force. The heat flow due to the latter can be considered as a kind of heat convection. Eq. (3-8) indicates the dependence of the relative velocity on both the temperature gradient and the interacting force. The novel aspect which came to existence in (3-7) and (3-8) is the presence of the interference phenomena—heat flow due to the interacting force and the relative velocity caused by the temperature gradient. Eq. (3-9) indicates how these cross phenomena are interconnected. Substituting the phenomenological relations (3-7) and (3-8) into (3-6), we obtain the entropy production expressed in terms of the forces.

$$\sigma = l_{vv} Y_i Y_i - 2 l_{vt} Y_i \frac{T_{,i}}{T} + l_{tt} \frac{T_{,i} T_{,i}}{T} \dots\dots\dots(3-10)$$

According to the second law of thermodynamics, the entropy production inside the system is never negative. This requirement places an important restriction on the phenomenological coefficients which follows immediately from (3-10),

$$l_{vv} \geq 0, \quad l_{tt} \geq 0, \quad l_{vt}^2 - l_{vv} l_{tt} \leq 0 \dots\dots\dots(3-11)$$

4. DARCY LAW.

The Darcy law which had been extensively used to describe the flow of fluids through porous media was proposed merely on the basis of experimental fact. A rational systematic interpretation for this law was given quite recently by Mokadam⁵⁾ on the basis of nonequilibrium thermodynamics. This approach seems to yield a general equation of motion of a fluid through porous medium. A generalization of Mokadam's approach to include the effect of deformation of the solid is immediately done from the preceding discussion. Generally, the inertia effect is neglected in the treatment of flow through porous medium, because the velo-

city is considered to be slow. Eq. (1-2) is therefore peduced to,

$$Y_i = -p_{,i} + \rho^w F_i \dots\dots\dots(4-1)$$

where the stress tensor p_{ij} is replaced by $-p\delta_{ij}$ because in the absence of viscous stresses, merely the normal pressure $-p$ is assumed to act on the fluid.

Eq. (4-1) is considered to be *the definition of the interacting force* Y_i , which has not been given any explicit physical meanings.

If eq. (4-1) is inserted in eqs. (3-7) and (3-8) by neglecting the effect of the body force for the time being, the following equations are obtained.

$$J_i^q = -l_{tt} \frac{T_{,i}}{T} - l_{tv} p_{,i} \dots\dots\dots(4-2)$$

$$\frac{\dot{v}_i - \dot{u}_i}{T} = -l_{tv} \frac{T_{,i}}{T} - l_{vv} p_{,i} \dots\dots\dots(4-3)$$

In the isothermal condition, eq. (4-3) become

$$\frac{\dot{v}_i - \dot{u}_i}{T} = -l_{vv} p_{,i} \dots\dots\dots(4-4)$$

This is equivalent to the conventional definition of the Darcy law. The proportionality constant l_{vv} is called coefficient of the permeability.

Let us consider the situation in which the pressure gradient of the flow vanishes. In this case, (4-2) is reduced to the well-known Fourier's law of heat conduction.

$$J_i^q = -l_{tt} \frac{T_{,i}}{T^2} \dots\dots\dots(4-5)$$

The coefficient l_{tt}/T^2 of the gradient T is called the thermal conductivity.

5. EQUATION OF STATE.

In order to describe completely the behavior of a thermodynamical system, it is needed further to establish a relation called equation of state which relates a set of thermodynamic variables to another conjugate set of variables. Usually, the equation of state is derived on the basis of the existence of the thermodynamic potential. Hence, we will begin with the consideration of the potential. From the extended form of the Gibbs equation for the solid-fluid system as given by eq. (3-3), we have

$$\rho du^t = T \rho ds^t + \sigma_{ij} de_{ij} - p d\epsilon_{ii} \dots\dots\dots(5-1)$$

where $p_{ij} = -p\delta_{ij}$ is used for the fluid. From eq. (5-1), it is seen that the internal energy of the whole system consists of the heat energy and the mechanical energies stored seperately

in the solid and the fluid phases. Eq. (5-1) can be modified so that the terms of the mechanical energies can be expressed in terms of deviator and volumetric components of stress and strain tensors. Hence,

$$d\bar{u} = T d\bar{s} + \sigma de - p d\epsilon + \tau_{ij} d\tau_{ij} \dots\dots\dots(5-2)$$

where,

$$\left. \begin{aligned} e &= e_{ii} = u_{i,i} = \text{div } \mathbf{u}, & \epsilon &= \epsilon_{ii} = v_{i,i} = \text{div } \mathbf{v}, \\ \sigma &= \frac{1}{3} \sigma_{ii}, \\ \tau_{ij} &= \sigma_{ij} - \delta_{ij} \sigma, \\ \tau_{ij} &= e_{ij} - \delta_{ij} e \end{aligned} \right\} \dots\dots\dots(5-3)$$

The quantities \bar{u} and \bar{s} measured per unit bulk volume are used for convenience in eq. (5-2) instead of u^t and u^s in eq. (5-1) expressed on "per unit mass basis." From (5-2), it is easy to obtain the differential expression for the free energy \bar{F} which is equal to $\bar{F} = \bar{u} - \bar{s}T$,

$$d\bar{F} = -\bar{s}dT + \sigma de - p d\epsilon + \tau_{ij} d\tau_{ij} \dots\dots\dots(5-4)$$

Denoting the reference temperature and entropy by T_0 and s_0 , respectively and designating the variations of temperature and entropy around the reference state by $\theta = T - T_0$ and $s = \bar{s} - s_0$, respectively, we can modify the differential relations (5-2) and (5-4) as follows,

$$du' = \theta ds + \sigma de - p d\epsilon + \tau_{ij} d\tau_{ij} \dots\dots\dots(5-5)$$

$$dF' = -s d\theta + \sigma de - p d\epsilon + \tau_{ij} d\tau_{ij} \dots\dots\dots(5-6)$$

where

$$\left. \begin{aligned} u' &= u - T_0 s & u &= \bar{u} - \bar{u}_0 \\ F' &= F + s_0 \theta & F &= \bar{F} - \bar{F}_0 \end{aligned} \right\} \dots\dots\dots(5-7)$$

and \bar{u}_0 and \bar{F}_0 denote the internal and free energies at the reference state. The quantities u and F express the increases of the internal and the free energies respectively above their values at the reference state. The quantity u' defined by $u' = u - T_0 s$ has the similar nature to that introduced first by M.A. Biot¹⁾ as "*the generalized free energy*" in his thermoelastic treatment. In the similar sense, the quantity F' which is defined by $F' = F + s_0 \theta$ will be called "*the generalized internal energy.*" The term generalized internal energy is introduced here to emphasize the fact that in contrast to the classical definition, it is an indications of such an internal energy where the entropy distribution departs from uniformity by an amount locally defined by s .

Since eq. (5-5) possesses the form of a total

differential, there must exist an equation of state of the form

$$u' = u'(s, e, \epsilon, \tau_{ij}) \dots \dots \dots (5-8)$$

through which θ , σ , p and τ_{ij} are defined as

$$\left. \begin{aligned} \theta &= \left(\frac{\partial u'}{\partial s} \right)_{e, \epsilon, \tau_{ij}}, & p &= - \left(\frac{\partial u'}{\partial \epsilon} \right)_{s, e, \tau_{ij}}, \\ \sigma &= \left(\frac{\partial u'}{\partial e} \right)_{s, \epsilon, \tau_{ij}}, & \tau_{ij} &= \left(\frac{\partial u'}{\partial \tau_{ij}} \right)_{s, e, \epsilon}, \end{aligned} \right\} \dots \dots \dots (5-9)$$

where subscript after bracket indicate that they are retained unchanged during differentiation.

To obtain θ , p , σ and τ_{ij} for small strains and temperature rise, the quantities u' is expanded in a power series on the assumption of isotropy of the medium under consideration,

$$u' = [\alpha'e + \beta's + k'\epsilon] + [\alpha_c e \epsilon + \beta_c s e + k_c s \epsilon] + \frac{1}{2} [\alpha e^2 + \beta s^2 + k \epsilon^2 + 2 \mu \tau_{ij} \tau_{ij}] \dots \dots (5-10)$$

where α' , β' , k' ; α_c , β_c , k_c ; α , β , k , μ are constants which depend on the physical properties of materials.

In the above expression, cross terms such as $s \sigma_{ij}$, $e \tau_{ij}$, or $\epsilon \tau_{ij}$ can not be permitted to enter, because deviatoric and volumetric factors are of different tensoral character in an isotropic body.

By introducing the expression (5-10) into (5-9) the general forms of linear stress-strain relations are obtained as follows.

$$\left. \begin{aligned} \sigma &= \alpha' e + \alpha_c e + \alpha_c \epsilon + \beta_c s \\ -p &= k' + \alpha_c e + k \epsilon + k_0 s \\ \theta &= \beta' + \beta_c e + k_c \epsilon + \beta s \\ \tau_{ij} &= 2 \mu \tau_{ij}. \end{aligned} \right\} \dots \dots \dots (5-11)$$

If the reference state is stress-free at temperature T_0 , we find

$$\alpha' = \beta' = k' = 0 \dots \dots \dots (5-12)$$

Then relations (5-10) and (5-11) become

$$u' = [\alpha_c e \epsilon + \beta_c s \epsilon + k_c s \epsilon] + \frac{1}{2} [\alpha e^2 + \beta s^2 + k \epsilon^2 + 2 \mu \tau_{ij} \tau_{ij}] \dots \dots (5-13)$$

and

$$\left. \begin{aligned} \sigma &= \alpha e + \alpha_c \epsilon + \beta_c s \\ -p &= \alpha_c e + k \epsilon + k_c s \\ \theta &= \beta_c e + k_c \epsilon + \beta s \\ \tau_{ij} &= 2 \mu \tau_{ij}. \end{aligned} \right\} \dots \dots \dots (5-14)$$

Now, let us consider to derive the explicit expression for the generalized internal energy F' in terms of the coefficients in (5-13). It is easy to transform (5-14) and to express s , p , σ and τ_{ij} in terms of θ , e , ϵ and τ_{ij} ,

$$\left. \begin{aligned} s &= \frac{1}{\beta} \theta - \frac{\beta_c}{\beta} e - \frac{k_c}{\beta} \epsilon \\ -p &= \frac{k_c}{\beta} \theta + \left(\alpha_c - \frac{\beta_c k_c}{\beta} \right) e + \left(k - \frac{k_c^2}{\beta} \right) \epsilon \\ \sigma &= \frac{\beta_c}{\beta} \theta + \left(\alpha - \frac{\beta_c^2}{\beta} \right) e + \left(\alpha_c - \frac{\beta_c k_c}{\beta} \right) \epsilon \\ \tau_{ij} &= 2 \mu \tau_{ij}. \end{aligned} \right\} (5-15)$$

If the relations (5-15) are substituted into (5-6), the differential $\alpha F'$ thus obtained allows direct integration and gives,

$$F' = F + s_0 \theta = \left[-\frac{1}{2} \theta^2 + \frac{1}{2} \left(\alpha - \frac{\beta_c^2}{\beta} \right) e^2 + \frac{1}{2} \left(k - \frac{k_c^2}{\beta} \right) \epsilon^2 + \mu \tau_{ij} \tau_{ij} \right] + \left[\frac{\beta_c}{\beta} \theta e + \frac{k_c}{\beta} \theta \epsilon + \left(\alpha_c - \frac{\beta_c k_c}{\beta} \right) e \epsilon \right] \dots \dots \dots (5-16)$$

If the third relations of (5-14) is inserted in (5-16), another simple form for F' is obtained,

$$F' = F + s_0 \theta = \frac{1}{2} \alpha e^2 + \frac{1}{2} k \epsilon^2 - \frac{1}{2} \beta s^2 + \alpha_c e \epsilon + \mu \tau_{ij} \tau_{ij} \dots \dots \dots (5-17)$$

Inserting the first relation of (5-15) into (5-13), we obtain the expression for u' written in terms of θ , e , ϵ and τ_{ij} ,

$$u' = u - T_0 s = \frac{1}{2} \theta^2 + \frac{1}{2} \left(\alpha - \frac{\beta_c^2}{\beta} \right) e^2 + \frac{1}{2} \left(k - \frac{k_c^2}{\beta} \right) \epsilon^2 + \left(\alpha_c - \frac{\beta_c k_c}{\beta} \right) e \epsilon + \mu \tau_{ij} \tau_{ij} \dots \dots \dots (5-18)$$

The above expression has the analogous form to that of the generalized free energy due to Biot in that there are no coupling terms between θ and e or ϵ .

It is taken for granted that the internal energy at adiabatic process ($s=0$) and the free energy at isothermal process ($\theta=0$) must be non-negative for any choice of e , ϵ and τ_{ij} .

Even if we confine ourselves to the consideration of the process in which shear deformation does not take place ($\tau_{ij}=0$), the above statement should be considered to be still valid.

Therefore, from (5-13) (5-18),

$$\left. \begin{aligned} \frac{1}{2} \alpha e^2 + \frac{1}{2} k \epsilon^2 + \alpha_c e \epsilon &\geq 0 \\ \frac{1}{2} \left(\alpha - \frac{\beta_c^2}{\beta} \right) e^2 + \frac{1}{2} \left(k - \frac{k_c^2}{\beta} \right) \epsilon^2 + \left(\alpha_c - \frac{\beta_c k_c}{\beta} \right) e \epsilon &\geq 0 \end{aligned} \right\} \dots \dots \dots (5-19)$$

In order that these inequalities are valid for any values of e and ϵ , the following restrictions must be imposed on the coefficients.

$$\left. \begin{aligned} \alpha &\geq 0, k \geq 0, \alpha_c^2 - \alpha k \leq 0 \dots \dots \dots (5-20) \\ \beta_c^2 - \alpha \beta &\leq 0, k_c^2 - k \beta \leq 0 \\ (\alpha_c \beta - \beta_c k_c)^2 - (\alpha \beta - \beta_c^2) (k \beta - k_c^2) &\leq 0. \end{aligned} \right\} (5-21)$$

It is more convenient for further development of the present theory to introduce into (5-14) alternative dependent variable defined by

$$\zeta = e - \epsilon \dots \dots \dots (5-22)$$

It is clearly understood that the quantity ζ multiplied by the porosity f of the system represents the volume of fluid which enters the pores of an unit volume of bulk material.

Inserting the relation (5-22) into (5-14), we have

$$\left. \begin{aligned} \sigma &= (\alpha + \alpha_c) e - \alpha_c \zeta + \beta_c s \\ p &= -(\alpha_c + k) e + k \zeta - k_c s \\ \theta &= (k_c + \beta_c) e - k_c \zeta + \beta_c s \\ \tau_{ij} &= 2 \mu \tau_{ij} \end{aligned} \right\} \dots \dots \dots (5-23)$$

After some rearrangements, eq. (5-23) are transformed as follows to express σ , ζ , s and τ_{ij} in terms of e , p , θ and τ_{ij}

$$\left. \begin{aligned} \sigma &= c_{\sigma\sigma} e - c_{\sigma\zeta} p - c_{\sigma s} \theta \\ \zeta &= (1 + c_{\sigma\zeta}) e + c_{\zeta\zeta} p - c_{\zeta s} \theta \\ s &= c_{s\sigma} e - c_{s\zeta} p + c_{ss} \theta \\ \tau_{ij} &= 2 \mu \tau_{ij} \end{aligned} \right\} \dots \dots \dots (5-24)$$

where

$$\left. \begin{aligned} c_{\sigma\sigma} &= \left(\alpha - \frac{\beta_c^2}{\beta} \right) - \frac{(\alpha_c \beta - k_c \beta_c)^2}{\beta (\beta k - k_c^2)} \\ c_{\sigma\zeta} &= \frac{\alpha_c \beta - \beta_c k_c}{\beta k - k_c^2} \\ c_{\sigma s} &= \frac{\alpha_c k_c - \beta_c k}{\beta k - k_c^2} \\ c_{\zeta\zeta} &= \frac{\beta}{\beta k - k_c^2} \\ c_{\zeta s} &= \frac{-k_c}{\beta k - k_c^2} \\ c_{ss} &= \frac{k}{\beta k - k_c^2} \end{aligned} \right\} \dots \dots \dots (5-25)$$

6. PHYSICAL IMPLICATIONS OF THE COEFFICIENTS.

The general forms of stress-strain relations as given by (5-23) involve six material constants which should be determined by some sets of six independent measurements. In alternative forms of stress-strain relations (5-24), six constants are also included which can be calculated through relations (5-25). We will demonstrate herein some of the methods of conceptual tests to illustrate the physical implications of the material constants appearing in (5-23) and (5-24).

(i) Let us consider a cubic of soil specimen to which normal pressure p_0 is applied from three mutually perpendicular directions. The

fluid and heat are allowed to flow in or out of this cubic freely. If the volume change of the skeleton is to be measured at the immediate instance of load application, we must put $\zeta = s = 0$ in the relation (5-23), because there is no time for fluid and heat flow to occur. Then we have,

$$-p_0 = (\alpha + 2\alpha_c + k) e \dots \dots \dots (6-1)$$

because

$$-p_0 = \sigma - p.$$

The coefficient $\alpha + 2\alpha_c + k$ is a measure of instantaneous volume change caused by the load application and corresponds to the modulus of compression under isentropic equi-fluid-content condition. The inverse of $\alpha + 2\alpha_c + k$ may be termed initial compressibility. Similarly, from the second relation of (5-23), we obtain,

$$p = -(\alpha_c + k) e \dots \dots \dots (6-2)$$

$$\theta = (k_c + \beta_c) e \dots \dots \dots (6-3)$$

The coefficients $\alpha_c + k$ and $k_c + \beta_c$ signify the change of pore pressure and temperature respectively due to the volume change e which is determined from (6-1). It is to be noted that, owing to incapability of occurrence of fluid and heat flow some amounts of pore pressure and temperature as given by (6-2) and (6-3) is induced at initial time in the system to balance the change in the volume of the skeleton. Since the volume decrease induces rises in pore pressure and temperature, the coefficients $\alpha_c + k$ should be positive and $k_c + \beta_c$ be negative.

Next, let us consider the equilibrium state which may be reached after a long time of loading in the same experimental condition as above. This final state is characterized by vanishing of the pore pressure and temperature rise which has been caused at the beginning of load application. Putting $p = \theta = 0$ and $\sigma = -p_0$ in the first relation of (5-24), we obtain

$$-p_0 = c_{\sigma\sigma} e \dots \dots \dots (6-4)$$

The coefficient $c_{\sigma\sigma}$ characterizes the volume change of the skeleton which is attained at the final stage. The quantity $c_{\sigma\sigma}$ is the modulus of compression under isothermal equi-pore-pressure condition. The name "final compressibility" may be given to the inverse of $c_{\sigma\sigma}$. It is apparant from the first relation of (5-25) that the volumetric modulus $\alpha + 2\alpha_c + k$ is larger than the modulus $c_{\sigma\sigma}$. During the consolida-

ting process, in which the rise of temperature and pore pressure at the beginning are being cancelled out with time, irreversible processes are taking place in which certain amounts of heat and fluid are flowing in or out of the cubic specimen under consideration. The change of the fluid content in this process is inferred from the relation,

$$\zeta = (1 + c_{\sigma\zeta}) e \dots\dots\dots(6-5)$$

which follows from the second of (5-24). Therefore, the coefficient $1 + c_{\sigma\zeta}$ is indicative of the fluid content that is squeezed in or out of the specimen during this process. In the similar way, the amount of heat to be taken in or out can be estimated by the relation,

$$s = c_{\sigma s} e \dots\dots\dots(6-6)$$

The change of entropy resulting from the volume change e which is caused by p_0 after a long time of loading is therefore characterized by the coefficient $c_{\sigma s}$.

(ii) Let us consider a sample of soil which is incased in a rigid porous box, so that the volume change of the skeleton is prohibited to occur while the fluid and heat are allowed to flow in or out freely. Suppose that this box is immersed under the water level at the certain depth to apply a water pressure p_0 . After a long time, an equilibrium state may be reached in which a certain amount of water is forced into the sample and heat is expelled from this sample to cancel the temperature rise at the beginning. At this stage, we must have the condition $e = \theta = 0$ and $-p_0 = \sigma - p$.

Then we have from (5-24),

$$p_0 = (1 + c_{\sigma\zeta}) p \dots\dots\dots(6-7)$$

$$\zeta = c_{\zeta\zeta} p \dots\dots\dots(6-8)$$

$$s = -c_{\zeta s} p \dots\dots\dots(6-9)$$

An alternative interpretation of the coefficient $1 + c_{\sigma\zeta}$ is afforded also by (6-7) which indicate the amount of pore pressure which might be induced in a long process to maintain the volume of skeleton unchanged. Similarly, the coefficient $c_{\zeta\zeta}$ is indicative of the fluid content forced into the soil and the coefficient $c_{\zeta s}$ signifies the entropy supplied and generated in the soil during this process.

(iii) Assume that a soil sample is put in a rigid porous box and is not allowed for the change of the volume of skeleton to take place.

The heat is permitted to go through the wall of this box freely. If this box is placed in a large reservoir of heat where temperature differs by θ from that of the sample, some amounts of heat and fluid may flow in or out of this sample until an equilibrium state will be attained. If attention is drawn to this final state, in which $e = p = 0$ hold, we obtain from (5-23)

$$\sigma = -c_{\sigma s} \theta \dots\dots\dots(6-10)$$

$$\zeta = -c_{\zeta s} \theta \dots\dots\dots(6-11)$$

$$s = c_{s s} \theta \dots\dots\dots(6-12)$$

These relations may serve to understand the physical implication of the constants $c_{\sigma s}$, $c_{\zeta s}$ and $c_{s s}$. When the temperature of the sample is raised by θ , for example, the soil tends to expand in its volume. To prohibit this volume expansion, the stress given by (6-10) should be applied to the skeleton. Hence, the coefficient $c_{\sigma s}$ is interpreted as indicating the stress induced by temperature change under the condition of $e = p = 0$. Another meanings of $c_{\sigma s}$ was illustrated in (6-6). The fact that the coefficient specifying the physical condition in (6-6) is equal to the one which signifies the situation just mentioned above is a natural consequence resulting from the existence of the potential energy.

When the temperature rises by an amount θ , the fluid in the sample tends to expand and causes the rise of pore pressure at the initial stage. Hence, a certain amount of the fluid is to be squeezed out of the box, until the pore pressure gradient due to this will vanish at the final stage. Total amount of the fluid to be squeezed out is given by the relation (6-11). Finally, the entropy to be taken into the sample in this case is given by the relation of (6-12). The coefficient $c_{s s}$ multiplied by the reference temperature T_0 is the usual definition of heat capacity.

7. THERMOELASTIC CONSOLIDATION EQUATIONS.

The equations so far derived for the solid-fluid system in which elastic deformation is taking place in parallel with heat condition and fluid flow are sufficient to describe the behaviors of the system under given boundary conditions and initial conditions. We will rearrange these equations in more sophisticated form to be

relevant to interpretation in a physical context. Some background information will be obtained that might assist the basic understanding of the phenomena.

If the effect of inertia is assumed to be sufficiently small, the equations of motion (1-1) and (1-2) are equivalent to the following set,

$$\sigma_{ij,j} - p_{,i} = 0 \dots\dots\dots(7-1)$$

$$p_{,i} - Y_i = 0, \dots\dots\dots(7-2)$$

where the effect of body force is neglected. Eq. (7-1) is a requirement that the total stresses should be in equilibrium. Eq. (7-2) may be regarded as the *definition of the interacting force*. An alternative interpretation for eq. (7-2) is that it is disequilibrium equation in the same sense as proposed by Biot. Eq. (7-2) will have close connection with the diffusion type equation as referred to later. Introducing eqs. (5-24) into (7-1), we have the equilibrium equation expressed in terms of displacements u_i , temperature θ and pore pressure p ,

$$\begin{aligned} \mu \Delta u_i + (\lambda_{\sigma\sigma} + \mu) \text{grad } e - (1 + c_{\sigma\zeta}) \text{grad } p \\ - c_{\sigma s} \text{grad } \theta = 0, \dots\dots\dots(7-3) \end{aligned}$$

where Δ denotes the Laplacian operator. In deducing the above expression, the modulus of compression $c_{\sigma\sigma}$ is replaced by

$$c_{\sigma\sigma} = \lambda_{\sigma\sigma} + \frac{2}{3} \mu, \dots\dots\dots(7-4)$$

which corresponds to the expression given by Lamé for the modulus of compression in the theory of elasticity.

It is evidently seen that eq. (7-3) is reduced to the usual form of thermoelasticity equation in the case $\text{grad } p = 0$ or $1 + c_{\sigma\zeta} = 0$ and reduced as well to the poroelasticity equation in the case $\text{grad } \theta = 0$ or $c_{\sigma\zeta} = 0$.

If the relations (5-24) are inserted into eq. (7-1), a different form of the basic equations is obtained which connect the displacements u_i , entropy s and fluid content ζ ,

$$\begin{aligned} \mu \Delta u_i + (\lambda_{cc} + \mu + \alpha_c + k) \text{grad } e + (\beta_c + k_c) \text{grad } s \\ - (\alpha_c + k) \text{grad } \zeta = 0, \dots\dots\dots(7-5) \end{aligned}$$

where the modulus of compression $\alpha + \alpha_c$ is replaced by,

$$\alpha + \alpha_c = \lambda_{cc} + \frac{2}{3} \mu, \dots\dots\dots(7-6)$$

which corresponds to the modulus of compression due to Lamé.

Now, let us turn our attention to the esta-

blishment of the equations which describe the nonequilibrium process of heat conduction and pore fluid flow which occur simultaneously.

The terms of entropy production in eq. (3-4) which have the form of the product of two variables may have the second order effect in the range of linear treatment because of its nonlinearity. Then, neglecting these terms, eq. (3-4) becomes

$$\frac{\partial s}{\partial t} = - \frac{J^q_{i,i}}{T} \dots\dots\dots(7-7)$$

Insertion of (4-2) into (7-7) yields the equation of heat conduction with simultaneous pore fluid flow,

$$\frac{\partial s}{\partial t} = l_s \Delta \theta + l_a \Delta p \dots\dots\dots(7-8)$$

where

$$l_s = \frac{l_{tt}}{T^3}, \quad l_a = \frac{l_{tv}}{T} \dots\dots\dots(7-9)$$

Performing divergence operation to eq. (4-3), we obtain the equation of fluid flow through porous medium coupled with heat conduction,

$$\frac{\partial \zeta}{\partial t} = l_a \Delta \theta + l_\zeta \Delta p \dots\dots\dots(7-10)$$

where

$$l_\zeta = T l_{vv} \dots\dots\dots(7-11)$$

It is to be noticed that the substantial derivative in eq. (3-4) and (4-3) is replaced in this case by the partial derivative owing to the postulate of linearity of the variables.

Let us now proceed to transform eq. (7-8) and (7-10), so that they are expressed in terms of the entropy s and the fluid content ζ .

Operating Laplacian to eq. (5-23) and taking the divergence of eq. (7-5), we obtain,

$$\Delta p = -(\alpha_c + k) \Delta e + k_c \Delta \zeta - k_c \Delta s \dots\dots\dots(7-12)$$

$$\Delta \theta = (\beta_c + k_c) \Delta e - k_c \Delta \zeta + \beta \Delta s \dots\dots\dots(7-13)$$

$$\begin{aligned} (\lambda_{cc} + 2\mu + \alpha_c + k) \Delta e = -(\beta_c + k_c) \Delta s \\ + (\alpha_c + k) \Delta \zeta. \dots\dots\dots(7-14) \end{aligned}$$

Elimination of Δe from these equations yields

$$\left. \begin{aligned} \Delta p = a \Delta s + b \Delta \zeta \\ \Delta \theta = c \Delta s + a \Delta \zeta \end{aligned} \right\} \dots\dots\dots(7-15)$$

where

$$\left. \begin{aligned} a &= \frac{\beta_c(\alpha_c + k) - k_c(\lambda_{cc} + 2\mu)}{\lambda_{cc} + 2\mu + \alpha_c + k} \\ b &= \frac{k(\lambda_{cc} + 2\mu) - \alpha_c(\alpha_c + k)}{\lambda_{cc} + 2\mu + \alpha_c + k} \\ c &= \frac{-(\beta_c + k_c)^2 + \beta(\lambda_{cc} + 2\mu + \alpha_c + k)}{\lambda_{cc} + 2\mu + \alpha_c + k} \end{aligned} \right\} (7-16)$$

Introducing the relations (7-15) into (7-8) and (7-10), we obtain the following two equations.

$$\frac{\partial \zeta}{\partial t} = A \Delta s + B \Delta \zeta \dots\dots\dots(7-17)$$

$$\frac{\partial s}{\partial t} = G \Delta s + H \Delta \zeta \dots\dots\dots(7-18)$$

where

$$\left. \begin{aligned} A &= cl_a + al_\zeta, & B &= bl_\zeta + al_a \\ G &= lc_s + al_a, & H &= bl_a + al_s. \end{aligned} \right\} \dots\dots\dots(7-19)$$

Two equations (7-17) and (7-18) can be used to determine the time history of the fields of the entropy s and the fluid content ζ . By using the entropy and the fluid content thus determined, it is possible from eq. (7-5) to find the time process of the distribution of the displacements u_i .

It is also possible to express eqs. (7-8) and (7-10) in terms of θ and p by using the relation (7-15). By taking the Laplacian of eqs. (7-8) and (7-10) and inserting the relation (7-15), the following set will be obtained.

$$\Delta \frac{\partial p}{\partial t} = H \Delta^2 \theta + B \Delta^2 p \dots\dots\dots(7-20)$$

$$\Delta \frac{\partial \theta}{\partial t} = G \Delta^2 \theta + A \Delta^2 p. \dots\dots\dots(7-21)$$

The above form of the equations may also be used to determine the fields of p and θ . Together with these values of p and θ , the displacement fields u_i can be determined from eq. (7-3). The alternative equations equivalent to the set (7-20) and (7-21) are deduced directly, if substitution is made from the relations (5-24),

$$c_{ss} \frac{\partial \theta}{\partial t} - c_{\zeta s} \frac{\partial p}{\partial t} + c_{s\zeta} \frac{\partial e}{\partial t} = l_s \Delta \theta + l_a \Delta p \dots\dots(7-22)$$

$$c_{\zeta\zeta} \frac{\partial p}{\partial t} - c_{\zeta s} \frac{\partial \theta}{\partial t} + (1 + c_{s\zeta}) \frac{\partial e}{\partial t} = l_a \Delta \theta + l_\zeta \Delta p. \dots\dots\dots(7-23)$$

The above equations, together with three equations in (7-3), determine the time history of the deformation of the solid phase as well as the thermal and the pore pressure fields. Because of the change in fluid pressure and in temperature associated with a change of volume of the solid part, the three fields are coupled through the coefficient $c_{\zeta s}$, $c_{s\zeta}$ and $1 + c_{s\zeta}$, the physical meanings of which were discussed in Sec. 6.

In solving the given problem, either set of equations (7-17), (7-18) and (7-5); (7-20), (7-21) and (7-3); or (7-22), (7-23), and (7-3) may be utilized, according to whether boundary and initial conditions may be given in terms of either pair s and ζ or p and θ . It is of considerable interest to recognize that the set

of the equations (7-17) and (7-18) including s and ζ as the variables has the form of the diffusion type equation, where as the set (7-20) and (7-21) expressed in terms of θ and p , does not satisfy the diffusion equation. This fact indicates a fundamental physical aspect that it is neither the temperature change θ nor the fluid pressure p but the entropy s and the fluid content ζ that have a close connection with the propagation of disorder within the system. This fact was found by Biot¹⁾ for the cases of thermoelasticity and poroelasticity, but he treated these two cases separately and found an analogy between them. Eqs. (7-17) and (7-18) which are derived on the basis of the irreversible thermodynamics are the generalization of Biot's work.

It is seen from (7-17) and (7-18) that the variables s and ζ do not satisfy a usual type of diffusion equation. Eq. (7-17) states that the time rate of change of fluid content is caused both by the transfer of the fluid and the conduction of heat. A cross phenomenon where the fluid flow is caused by the heat flow contributes to the increase of the fluid content per unit time within unit bulk volume of the system. Likewise, the time rate of change of entropy within unit volume may be caused by the entropy flow as well as the fluid flow which carries the heat therewith.

Let us next consider in more detail the several cases of occurrence of the cross phenomenon. It is seen from (7-17) and (7-18) that the coupling may appear when the coefficients A and H are different from zero. From inspection of the definition of the coefficients A and H as indicated in (7-19), it is recognized, that the coupling between ζ and s stems from *two sources of different physical phenomena*, that is, *the coupling in the reversible process and that related to the irreversible phenomenon*. The coupling appearing in the course of reversible change of the variables is represented through the coefficient a given by (7-16). The coefficient l_a characterizing the coupling between heat and fluid belongs to the category of irreversible coupling. Therefore, in order that the eqs. (7-17) and (7-18) could be reduced to two single

independent equations, both coupling coefficients a and l_a , one from the reversible change and the other from the irreversible change, should vanish concurrently. The physical situation in which the coefficient l_a vanishes is easily understood from the discussion in Sec. 4. However, the implication of vanishing of the coefficient a is somewhat complicated and may require further a closer scrutiny. From the definition of a as represented in (7-16), the condition for a to vanish is seen to be

$$\beta_c(\alpha_c+k) - k_c(\lambda_{cc}+2\mu) = 0 \dots\dots\dots(7-24)$$

The quantity $\lambda_{cc}+2\mu = \alpha + \alpha_c + \frac{4}{3}\mu$ could never be equal to zero because $\alpha + \alpha_c > 0$ from (6-1) and the shearing modulus μ is always positive. Consequently, the situation where the condition (7-24) is satisfied occurs in the following two ways,

$$\beta_c = k_c = 0 \dots\dots\dots(7-25)$$

or

$$\alpha_c + k = k_c = 0 \dots\dots\dots(7-26)$$

Let us investigate these two cases separately.

(i) the case $k_c = 0, \beta_c = 0$.

From the second relation of (5-23), the vanishing of k_c is understood to be equivalent to saying that the fluid pressure is independent of temperature change. Furthermore, it is seen from (7-14) that the deformation of the solid phase is not affected by temperature, if the condition $\beta_c + k_c = 0$ is satisfied. Accordingly, the conditions $k_c = \beta_c = 0$ corresponds to the case where the temperature change does not couple with the deformation of the solid and fluid phases. This process coincides with the usual problem of consolidation.

By putting $k_c = \beta_c = 0$ and $l_a = 0$ in (7-17) (7-18) and (7-5), we obtain the basic equations for consolidation problem proposed by Biot,

$$\mu \Delta u_i + (\lambda_{cc} + \mu + \alpha_c + k) \text{grad } e - (\alpha_c + k) \text{grad } \zeta = 0 \dots\dots\dots(7-27)$$

$$\frac{\partial \zeta}{\partial t} = B_1 \Delta \zeta \dots\dots\dots(7-28)$$

$$\left(\frac{\partial s}{\partial t} = G_1 \Delta s \right) \dots\dots\dots(7-29)$$

where $B_1 = b l_\zeta, G_1 = \beta l_s$.

The eq. (7-29) is, in this case, of supplementary meanings because the entropy change dose not influence the deformation of the solid and the fluid. The set of equations expressed in terms

of u_i, θ and p instead of u_i, s and ζ are deduced immediately from (7-22), (7-23) and (7-3) by referring to the relations (5-25). The conditions $k_c = 0$ and $\beta_c = 0$ are equivalent to $c_{\zeta s} = 0$ and $c_{\sigma s} = 0$.

Thus we have,

$$\mu \Delta u_i + (\lambda_{\sigma\sigma} + \mu) \text{grad } e - \frac{\alpha_c + k}{k} \text{grad } p = 0 \dots\dots\dots(7-30)$$

$$\frac{1}{k} \frac{\partial p}{\partial t} + \frac{k + \alpha_c}{k} \frac{\partial e}{\partial t} = l_\zeta \Delta p \dots\dots\dots(7-31)$$

$$\left(\frac{1}{\beta} \frac{\partial \theta}{\partial t} = l_s \Delta \theta \right) \dots\dots\dots(7-32)$$

Since the heat conduction is going on alone independently of the deformation of the solid and the fluid flow through the pores, there is no need for the usual consolidation problem to take into account the effect of thermal field.

(ii) the case $k_c = 0, k + \alpha_c = 0$.

As easily seen from the third relation of (5-23), the vanishing of k_c means the independence of the temperature variation on the change of fluid content. It is also known from (7-14) that under the condition of $\alpha_c + k = 0$, the deformation of the solid takes place without being influenced by the change of fluid content. In other words, the coupling except between the deformation of the solid and the change of temperature is excluded from the consideration. In this situation, the equations are attributed to the usual form of coupled thermoelasticity. By putting $k_c = 0, k + \alpha_c = 0$ and $l_a = 0$ in (7-17), (7-18), we obtain,

$$\mu \Delta u_i + (\lambda_{cc} + \mu) \text{grad } e + \beta_c \text{grad } s = 0 \dots\dots\dots(7-33)$$

$$\frac{\partial s}{\partial t} = G_2 \Delta s \dots\dots\dots(7-34)$$

$$\left(\frac{\partial \zeta}{\partial t} = B_2 \Delta \zeta \right) \dots\dots\dots(7-35)$$

where

$$B_2 = k l_\zeta, G_2 = \left(\beta - \frac{\beta_c^2}{\lambda_{cc} + 2\mu} \right) l_\zeta$$

It is also possible to express the above set of equations in terms of p and θ instead of ζ and s . With reference to (5-25), the conditions $k_c = 0$ and $\alpha_c + k = 0$ are equivalent to $c_{\zeta s} = 0$ and $1 + c_{\sigma \zeta} = 0$, and we obtain,

$$\mu \Delta u_i + (\lambda_{\sigma\sigma} + \mu) \text{grad } e + \frac{\beta_c}{\beta} \text{grad } \theta = 0 \dots\dots\dots(7-36)$$

$$\frac{1}{\beta} \frac{\partial \theta}{\partial t} - \frac{\beta_c}{\beta} \frac{\partial e}{\partial t} = l_s \Delta \theta \dots\dots\dots(7-37)$$

$$\left(\frac{1}{k} \frac{\partial p}{\partial t} = l_\zeta \Delta p \right) \dots\dots\dots(7-38)$$

As seen from (7-35) or (7-38), the movement of the fluid through the pores of the system is taking place independently of the process of the deformation of the solid and heat conduction. In the absence of the fluid of the pores, the problem is reduced exactly to the usual one of the coupled thermoelasticity.

8. TRANSFORMATION OF THE BASIC EQUATIONS.

The fundamental characteristics of the basic equations derived previously will be examined in detail in this section. It seems of considerable interest to draw attention to a similarity which exists between the eqs. (7-17), (7-18) and the equations of vibration problem with multi-degree of freedom. To facilitate the systematic solution of vibration problem in which there are many independent variables, the concept of normalized coordinates has been frequently used so far. This concept consists in transforming the given variables into the other set in such a way that the original coupled equations may be reduced to the set of uncoupled equations in which individual equation is written down only through the corresponding single coordinate. It will now be attempted to modify the eqs. (7-17) and (7-18) by introducing the new variables properly selected.⁷⁾ Multiplying certain constants *l* and *m* on (7-17) and (7-18) respectively and adding them, we obtain

$$\frac{\partial}{\partial t}(l\zeta + ms) = 4[(lB + mH)\zeta + (lA + mG)s]. \dots\dots\dots(8-1)$$

In order that this equation is reduced to the one which contains only single variable, the following relation should be satisfied for the values of *l/m*.

$$G + \frac{l}{m}A = B + \frac{m}{l}H = \nu \dots\dots\dots(8-2)$$

or

$$\frac{\nu - G}{A} = \frac{H}{\nu - B} = \frac{l}{m} \dots\dots\dots(8-3)$$

With proper choice of the ratio *l/m*, the equations (7-17) and (7-18) expressed in terms of the new coordinates, give rise to two single diffusion equations. The quantities represented by this new variables diffuse, independently of each other, according to the usual law of

diffusion. The equation (8-3) quadratic with respect to ν determines the two values of ν which may be denoted by ν_s and ν_ζ ,

$$(\nu_s, \nu_\zeta) = \frac{B+G}{2} \pm \sqrt{\frac{(B-G)^2 + 4AH}{4}} \dots\dots(8-4)$$

It should be recalled here that, in order that each of the transformed differential equations are of parabolic type—characteristics of diffusion equation, the values of ν_ζ and ν_s should be real and positive. The restrictions imposed on the coefficients *A*, *B*, *G* and *H* by this condition of real positiveness of ν are represented directly by the knowledge of elementary algebra as follows.

$$B + G \geq 0, \quad (B - G)^2 + 4AH \geq 0. \dots\dots\dots(8-5)$$

It can be proved easily on the basis of the fundamental thermodynamic relations (3-11), (5-20) and (5-21), that the conditions (8-5) are always satisfied for any choice of the parameters specifying physical properties of the material.

Let us consider to determine normal coordinates. With the values ν_ζ and ν_s determined from (8-4), the following relations are obtained from (8-2),

$$H + \left(\frac{l}{m}\right)_s B = \frac{\nu_s H}{\nu_s - B}, \quad G + \left(\frac{l}{m}\right)_s A = \nu_s \dots\dots\dots(8-6)$$

$$B + \left(\frac{m}{l}\right)_\zeta H = \nu_\zeta, \quad A + \left(\frac{m}{l}\right)_\zeta G = \frac{\nu_\zeta A}{\nu_\zeta - G} \dots\dots\dots(8-7)$$

where the quantity denoted by $(l/m)_s$ or $(m/l)_\zeta$ is the value of *l/m* or *m/l* in (8-3) corresponding to ν_s or ν_ζ respectively. With the use of the relations (8-3), (8-6) and (8-7) the eq. (8-1) can now be written,

$$\frac{\partial}{\partial t} \left(s + \frac{H}{\nu_s - B} \zeta \right) = \nu_s \Delta \left(s + \frac{H}{\nu_s - B} \zeta \right) \dots\dots(8-8)$$

$$\frac{\partial}{\partial t} \left(\zeta + \frac{A}{\nu_\zeta - G} s \right) = \nu_\zeta \Delta \left(\zeta + \frac{A}{\nu_\zeta - G} s \right) \dots\dots(8-9)$$

By putting

$$s + \frac{H}{\nu_s - B} \zeta = \psi, \quad \zeta + \frac{A}{\nu_\zeta - G} s = \varphi, \dots\dots(8-10)$$

eqs. (8-8) and (8-9) become

$$\frac{\partial \psi}{\partial t} = \nu_s \Delta \psi \dots\dots\dots(8-11)$$

$$\frac{\partial \varphi}{\partial t} = \nu_\zeta \Delta \varphi \dots\dots\dots(8-12)$$

which are two simple diffusion equations for the normal coordinates ψ and φ with corresponding diffusion coefficients ν_s and ν_ζ respectively. Denoting the solutions of (8-11) and (8-12)

by ψ_0 and φ_0 respectively which may be obtained subject to appropriate boundary and initial conditions, we will be able to determine the fields of entropy and fluid content from (8-10),

$$\left. \begin{aligned} s &= \frac{1}{D} \left(\psi_0 - \frac{H}{\nu_s - B} \varphi_0 \right) \\ \zeta &= \frac{1}{D} \left(\varphi_0 - \frac{A}{\nu_\zeta - G} \psi_0 \right) \end{aligned} \right\} \dots\dots\dots(8-13)$$

where

$$D = 1 - \frac{AH}{(\nu_\zeta - G)(\nu_s - B)}.$$

The equations (8-13) show that the most general solutions of s and ζ are made up of the superposition of two modes of pure diffusion. In each of these diffusion modes, two quantities s and ζ diffuse at the same progression speed, but with their amplitudes in a definite ratio. Speaking more physically, the main advancement of ζ due to the mode φ_0 gives rise to propagation of s proceeding at the same rate with the amplitude proportional to that of ζ . Likewise, the main entropy progression is accompanied by a subsidiary diffusion of fluid content.

Let us next examine the characteristics of the diffusion constants ν_ζ and ν_s in (8-4). If the value AH is small in comparison with $(B-G)^2/4$, i.e. if either the coupling between the two diffusions is weak, or B and G are of widely different magnitudes, the expressions (8-4) are reduced to,

$$\left. \begin{aligned} \nu_\zeta &\doteq B + \frac{AH}{B-G} \\ \nu_s &\doteq G - \frac{AH}{B-G} \end{aligned} \right\} \dots\dots\dots(8-14)$$

From (8-4), it is easily inferred that, as the coupling becomes weaker, ν_ζ tends to B and ν_s to G . Hence ν_ζ is considered to be concerned mainly with the diffusion of the fluid content and similarly ν_s is associated chiefly with the diffusion of the entropy. The diffusion coefficients ν_ζ and ν_s are concerned with both diffusion processes to a greater or less extent, according as the coupling is strong or weak. When the coupling vanishes, ν_ζ and ν_s become B and G respectively.

With the use of the relation eqs. (8-10), (7-5) are modified to yield the form expressed in terms of φ and ψ ,

$$\mu \Delta u_i + (\lambda_{cc} + \mu + \alpha_c + k) \text{grad } e$$

$$\begin{aligned} & - \frac{1}{\nu_\zeta - \nu_s} \{ (\alpha_c + k)(\nu_\zeta - G) - (\beta_c + k_c)H \} \text{grad } \varphi \\ & - \frac{1}{\nu_s - \nu_\zeta} \{ -(\beta_c + k_c)(\nu_s - B) + (\alpha_c + k) \} \text{grad } \psi \\ & = 0, \dots\dots\dots(8-15) \end{aligned}$$

In the similar fashion, it is also possible to transform eqs. (7-20), (7-21) and (7-3) into the normal coordinates. Merely the resulting equations will be presented herein.

With the transformation of the variables p and θ into the normal coordinates as given by

$$\left. \begin{aligned} p + \frac{H}{\nu_\zeta - G} \theta &= \Phi \\ \theta + \frac{A}{\nu_s - B} p &= \Psi \end{aligned} \right\} \dots\dots\dots(8-16)$$

eqs. (7-20) and (7-21) may be simplified as follows,

$$\left. \begin{aligned} \Delta \frac{\partial \Phi}{\partial t} &= \nu_\zeta \Delta^2 \Phi, \\ \Delta \frac{\partial \Psi}{\partial t} &= \nu_s \Delta^2 \Psi. \end{aligned} \right\} \dots\dots\dots(8-17)$$

Eq. (7-3) is written as,

$$\begin{aligned} & \mu \Delta u_i + (\lambda_{\sigma\sigma} + \mu) \text{grad } e \\ & - \frac{1}{\nu_\zeta - \nu_s} \{ c_\zeta(\nu_\zeta - G) + c_{\sigma s} A \} \text{grad } \Phi \\ & - \frac{1}{\nu_s - \nu_\zeta} \{ c_{\sigma s}(\nu_s - B) + c_\zeta H \} \text{grad } \Psi \\ & = 0 \dots\dots\dots(8-18) \end{aligned}$$

where $c_\zeta = 1 + c_{\sigma\zeta}$.

The application of the above equations to the practical problems will be made in other papers.

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Nomenclature

- e^s ; specific total energy of the solid phase.
- e^w ; specific total energy of the fluid phase.

e_{ij} ; strain tensor of the solid phase.
 $e=e_{ii}$; volume change of the solid phase.
 F_i ; component of body force.
 \bar{F} ; free energy per unit volume of the mixture.
 \bar{F}_0 ; free energy at the reference state.
 F' ; generalized free energy.
 $F=\bar{F}-\bar{F}_0$; change of free energy around reference state.
 J_i^{sq} ; heat flow through the solid phase.
 J_i^{wq} ; heat flow through the fluid phase.
 $J_i^q=J_i^{sq}+J_i^{wq}$; heat flow through the mixture.
 J_i^s ; energy flux through the solid phase.
 J_i^w ; energy flux through the fluid phase.
 M^s ; mass of the solid.
 M^w ; mass of the fluid.
 p_{ij} ; bulk stress tensor of the fluid phase.
 p ; bulk pressure of the fluid phase.
 s^s ; specific entropy of the solid phase.
 s^w ; specific entropy of the fluid phase.
 s^t ; specific entropy of the mixture.
 \bar{s} ; entropy per unit volume of the mixture.
 s_0 ; reference entropy per unit volume of the mixture.
 $s=\bar{s}-s_0$; change of entropy around reference state.
 t ; time.
 T ; temperature.
 T_0 ; temperature at reference state.
 u^s ; specific internal energy of the solid phase.
 u^w ; specific internal energy of the fluid phase.
 u^t ; specific internal energy of the mixture.
 \bar{u} ; internal energy per unit volume of the mixture.
 \bar{u}_0 ; internal energy at the reference state.
 u' ; generalized internal energy defined in this paper.
 $u=\bar{u}-\bar{u}_0$; change of internal energy around reference state.

u ; displacement vector of the solid phase.
 u_i ; displacement component of the solid phase.
 V ; bulk volume of the medium.
 v ; displacement vector of the fluid phase.
 v_i ; displacement component of the fluid phase.
 Y_i ; component of interacting force.
 δ_{ij} ; kronecker delta.
 ϵ_{ij} ; strain tensor of the fluid phase.
 $\epsilon=e_{ii}$; volume change of the fluid phase.
 $\tau_{ij}=e_{ij}-\delta_{ij}e$; deviator of strain of the solid phase.
 ρ^s ; bulk density of the solid phase.
 ρ^w ; bulk density of the fluid phase.
 $\rho=\rho^s+\rho^w$; density of the mixture.
 $\zeta=e-\epsilon$; volume change of the solid phase relative to the fluid phase.
 σ_{ij} ; bulk stress tensor of the solid phase.
 $\sigma=\frac{1}{3}\sigma_{ii}$; mean stress of the solid phase.
 $\theta=T-T_0$; change of temperature around reference state.
 $\tau_{ij}=\sigma_{ij}-\delta_{ij}\tau$; stress deviator of the solid phase.
 π_{ij} ; rotation tensor of the fluid phase.
 w_{ij} ; rotation tensor of the solid phase.
 Material constants characterizing the reversible process;

$$\begin{pmatrix} a, b, c, c_{\sigma\sigma}, c_{\sigma\zeta}, c_{\sigma s}, c_{\zeta\zeta}, c_{\zeta s}, c_{ss}, c_{\zeta} \\ \alpha, \beta, k, \mu, \alpha_c, \beta_c, k_c, \alpha', \beta', k' \\ \lambda_{\sigma\sigma}, \lambda_{cc} \end{pmatrix}$$

Material constants characterizing the irreversible process;

$$(l_{tt}, l_{tv}, l_{vt}, l_{vv}, l_a, l_s, l_{\zeta}).$$

Constants expressible by the material constants cited above;

$$(A, B, G, H, B_1, G_1, B_2, G_2, D)$$

Normalized coordinates;

$$(\psi, \varphi, \Psi, \Phi)$$

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