MATHEMATICAL STUDY OF THE MOTION OF INTUMESCENCES IN OPEN CHANNELS OF UNIFORM SLOPE

*By Taizo Hayashi, C.E. Member**

SYNOPSIS This paper is the first of a series dealing with the motion of translation waves in open channels. The case treated is that of an arbitrary intumescence with a small height in a uniform channel with a rectangular cross section containing water originally moving with a uniform velocity. Approximate formulas are derived which give the deformation and motion of the intumescence. The case of the intumescence of the form of a rectangular type is also dealt with and for that the form of the rigorous solution is shown. Special emphasis is laid on the dispersive property of intumescences. A method of illustration of the dispersion is presented and both the propagation speed and the dispersion of intumescences of any arbitrary shape are illustrated. The results are all derived with the method of operational calculus on the plane of the complex variables.

1. INTRODUCTION

The growing importance of long waves and flood waves for many practical purposes has in recent years led to a marked increase in the literature on these subjects. As pointed out by Messrs. Keulegan and Patterson[1]**, perhaps a simple classification of translation waves is obtained by considering the effect of friction. Then the long waves and the flood waves may be consideted two extreme cases. In long waves the weight and inertia are the factors controlling the motion and the effect of friction is less important. On the other hand, in flood waves friction is all-important factor, the effects of weight and inertia being negligible. In the former case the propagation speed of waves c_0 is approximately given by the Lagrange velocity law $c_0=\sqrt{gH}$, where g and H being the acceleration of gravity and the channel depth, respectively. In the latter case, the propagation speed of waves c_q is given, according to the Kleitz-Seddon law [2], by $c_q=\mathrm{d}q/\mathrm{d}H$, where q being the discharge per unit width. In a wide rectangular channel $c_q=3U/2$, if the Chezy resistance law is used and $c_q=5U/3$, if the Manning resistance law is applied.

However, as suggested in the literature [1], it should be understood that in reality there exists a continuous transition between the two extreme types of waves described above. The wave problems in such a transition region between the two extreme types were dealt with mathematically by Mr. Massé and later by Messrs. Keulegan and Patterson. Mr. Massé applied the Chezy resistance law and assumed wave amplitude infinitesimally small [3]. He worked out an approximate solution of the wave problems with the method of saddle-point integration. Messrs. Keulegan and Patterson analysed the effect of channel slope on the motion of translation waves [4]. In their

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** Figures in square brackets indicate the literature references at the end of this paper.

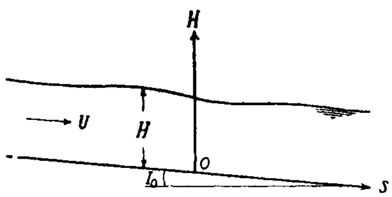
study no assumption was made for the amplitudes of waves, the theory being useful particularly for the study of waves of finite amplitude. As for the propagation speed of waves, however, they assumed as Boussinesq had done that the velocity of wave-volume elements differs little from the theoretical velocity of waves without friction. The last assumption might be necessary to overcome the mathematical difficulty of dealing with waves of finite amplitude. However, as far as waves of infinitesimally small amplitude are concerned, the asumption seems unnecessary for solving the wave problems. The main purpose of this paper is concerned to treat mathematically with intumescences of infinitesimally small amplitude in the transition zone which Mr. Massé dealt with. In considering Mr. Massé's solution the author explains the dispersive property of intumescences in the transition region with a new method of illustration and derives rigorous expressions of solutions which correspond to Mr. Massé's approximate solutions.

In giving Mr. Massé's equations in Section **II**, slight changes have been made, since it was desired to adopt Manning's law of friction in open channel [5] and to introduce the factor of the channel width. In giving Mr. Massé's method of approximation in Section **III.** alterations here, too, have been given, a straight line being used in place of a parabola as the line of the steepest descent. However, although the formulas presented in Section **II.** and **III.** are considerably different in form from those of Mr. Massé, they are due essentially to him.

A new method of illustration of dispersive property of waves is presented and the dispersion of intumescences is explained in Section **IV**. The rigorous solution of the movement of the intumescence of a rectangular type is shown in Section **V**.

A brief presentation of the theory of the wave height is given in Section **VII**, the theory being compared with that of the wave discharge described in the previous sections.

Fig. 1 System of coordinates for rectangular channels.



II. FORMULATION OF THE WAVE PROBLEM

Consider a uniform open channel of rectangular cross section and take the s-axis along the bottom of the channel and the H-axis vertically upwards (see **Fig 1**). H represents the depth of liquid in the channel and I_0 is the channel slope. Let U and Q respectively denote the velocity of the particles in, and the discharge per unit width through, a cross section at s. We consider phenomena in the channel as one-dimensional. The equation of continuity is

$$\frac{\partial H}{\partial t}=-\frac{\partial Q}{\partial s} \quad \cdots\cdots(2.1)$$

For the equation of motion, we use as the resistance formula Manning's law

$$U=(1/n)R^{2/3}I^{1/2}\ (\mathrm{m,\ sec}),\quad =(1\ 49/n)R^{2/3}I^{1/2}\ (\mathrm{ft,\ sec}), \quad \cdots\cdots(2.2)$$

where R is the hydraulic radius, n Kutter's roughness. Then the equation of motion is

$$\frac{\mathrm{d}U}{\mathrm{d}t}=-g\frac{\partial H}{\partial s}+g\left(I_0-\frac{\gamma^2U^2}{R^{3/4}}\right), \quad \cdots\cdots(2.3)$$

where g is the intensity of gravity, $\gamma=n$ in metric system or $\gamma=n/1.49$ in ft-lb system. The inertia term of the above equation is written $\frac{\mathrm{d}U}{\mathrm{d}t}=\beta\frac{\partial U}{\partial t}+\alpha U\frac{\partial U}{\partial s}$, $\cdots\cdots(2.4)$

where α and β are velocity coefficients and their values are about 1.096 and 1.040

respectively in wide rectangular channels. Substituting (2.4) in (2.3), we have

$$\beta\frac{\partial U}{\partial t}+\alpha U\frac{\partial U}{\partial s}=-g\frac{\partial H}{\partial s}+g\left(I_0-\frac{r^2U^2}{R^{4/3}}\right) \quad \cdots\cdots(2.5)$$

Let us consider an intumescence of small height is produced in a liquid which is initially in uniform motion in a rectangular channel. Let us put

$$Q=Q_0+q,\quad H=H_0+h,\quad U=U_0+u, \quad \cdots\cdots(2.6)$$

where Q_0, H_0 and U_0 are the quantities for the uniform flow and q, h and u for the intumescence. As assumed above, $(Q_0,H_0,U_0) \gg (q,h,u)$. $\cdots\cdots(2.7)$

Substituting (2.6) in (2.1) and(2.5) and linearizing (2.5) by the use of (2.7), we obtain

$$\frac{\partial h}{\partial t}=-\frac{\partial q}{\partial s} \quad \cdots\cdots(2.1')$$

and $$\left(1-\frac{\alpha Q_0^2}{gH_0^3}\right)\frac{\partial h}{\partial s}+\frac{\alpha+\beta}{gH_0^2}Q_0\frac{\partial q}{\partial s}+\frac{\beta}{gH_0}\frac{\partial q}{\partial t}=\frac{2I_0}{Q_0}\left\{-q+\frac{U_0}{3}(5-4\varepsilon)h\right\}, \quad \cdots\cdots(2.5')$$

where ε is the undisturbed depth of the liquid in the channel divided by the wetted perimeter, i. e. $\varepsilon=H_0/(2H_0+B)$ $\cdots\cdots(2.8)$

and B the width of the channel. Eliminating h between (2.1') and (2.5'),

$$-\frac{gH_0-\alpha U_0^2}{\beta}\frac{\partial^2 q}{\partial s^2}+\frac{\alpha+\beta}{\beta}U_0\frac{\partial^2 q}{\partial s\partial t}+\frac{\partial^2 q}{\partial t^2}+\frac{2I_0 g}{\beta U_0}\left\{\frac{\partial q}{\partial t}+\frac{U_0}{3}(5-4\varepsilon)\frac{\partial q}{\partial s}\right\}=0. \quad \cdots\cdots(2.9)$$

The above equation is hyperbolic, if $\Delta\equiv\left(\frac{\alpha+\beta}{2\beta}U_0\right)^2+\frac{gH_0-\alpha U_0^2}{\beta}>0$ $\cdots\cdots(2.10)$

[6]. (2.10) is written $\Delta\equiv-\left\{\frac{\alpha}{\beta}-\left(\frac{\alpha+\beta}{2\beta}\right)^2\right\}U_0^2+\frac{gH_0}{\beta}>0,$

or $$\frac{U_0^2}{gH_0}<\frac{1}{\beta\left\{\frac{\alpha}{\beta}-\left(\frac{\alpha+\beta}{2\beta}\right)^2\right\}}. \quad \cdots\cdots(2.10.a)$$

As $\alpha\doteqdot\beta$, for practical purposes the relation (2.10.a) always holds. Therefore we may justifiably think that (2.9) is always hyperbolic.

Next, in order to reduce the number of parameters in the equation, we non-dimensionalize it by the following linear transformations: $\frac{gI_0}{\beta U_0}\sqrt{\frac{\beta}{gH_0-\alpha U_0^2}}s=s'$ $\cdots\cdots(2.11)$

and $$\frac{gI_0}{\beta U_0}t=t'. \quad \cdots\cdots(2.12)$$

Substituting (2.11) and (2.12) in (2.9) and dropping the primes on s and t for the simplicity's sake, we obtain

$$-\frac{\partial^2 q}{\partial s^2}+\frac{\alpha+\beta}{\beta}a\frac{\partial^2 q}{\partial s\partial t}+\frac{\partial^2 q}{\partial t^2}+2\left\{\frac{\partial q}{\partial t}+\frac{a}{3}(5-4\varepsilon)\frac{\partial q}{\partial s}\right\}=0, \quad \cdots\cdots(2.13)$$

where $$a=U_0\Big/\sqrt{\frac{gH_0-\alpha U_0^2}{\beta}}. \quad \cdots\cdots(2.14)$$

The a of (2.14) corresponds to the average velocity U_0 and is considered the non-dimensionalized form of the average velocity. If we write $\frac{\alpha+\beta}{\beta}a=2b,$ $\cdots\cdots(2.15)$

(2.13) is written $$-\frac{\partial^2 q}{\partial s^2}+2b\frac{\partial^2 q}{\partial s\partial t}+\frac{\partial^2 q}{\partial t^2}+2\left\{\frac{\partial q}{\partial t}+\frac{a}{3}(5-4\varepsilon)\frac{\partial q}{\partial s}\right\}=0 \quad \cdots\cdots(2.16)$$

which is the basic equation of this paper. Writing (2.16) in the operational form,

$$\frac{d^2q}{ds^2}-\left\{2bp+\frac{2}{3}(5-4\varepsilon)a\right\}\frac{dq}{ds}-p(p+2)q=0 \quad \cdots\cdots(2.17)$$

Since (2.17) is an ordinary differential equation of the second order, the solution of it in its operational form is easily obtained as following:

$$q=A\exp\left[bp+\frac{a}{3}(5-4\varepsilon)-\sqrt{\left\{bp+\frac{a}{3}(5-4\varepsilon)\right\}^2+p(p+2)}\right]s$$
$$+B\exp\left[bp+\frac{a}{3}(5-4\varepsilon)+\sqrt{\cdots\cdots}\right]s \quad \cdots\cdots(2.18)$$

where A, B are integration constants to be determined by boundary conditions. If we assume that the channel is semi-infinitely long ($s \geq 0$) and a boundary condition is given by $(q)_{s=0}=g_1(p)$, the values of A and B are determined as

$$B=0 \quad \text{and} \quad A=g_1(p). \qquad \cdots\cdots(2.18.\text{a})$$

Let p_1 and p_2 denote the two roots of the polinominal of the second order under the root of (2.18), then

$$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \frac{-(1+a'b)\pm\sqrt{1+2a'b-a'^2}}{1+b^2}, \qquad \cdots\cdots(2.19)$$

where $a'=(5-4\varepsilon)a/3.$ $\cdots\cdots(2.20)$

Insertion of (2.18.a), (2.19) and(2.20) into (2.18) yields

$$q=g_1(p)\exp[bp+a'-\sqrt{1+b^2}\sqrt{(p-p_1)(p-p_2)}]s \qquad \cdots\cdots(2.21)$$

Writing (2.21) in the form of Bromwich-integral, we obtain

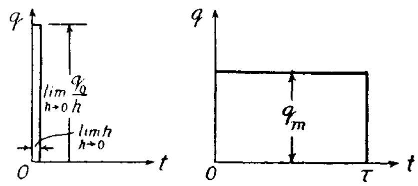
$$q=\frac{1}{2\pi i}\int_{\text{Br}}\frac{g_1(p)}{p}e^{a's+p(t+bs)-s\sqrt{1+b^2}\sqrt{(p-p_1)(p-p_2)}}\mathrm{d}p \qquad \cdots\cdots(2.22)$$

which is the expression for q in Bromwich-integral form. As the examples for the forms of $g_1(p)$ the following two formulas are given for the boundary conditions shown in **Fig. 2**:

For the δ-function type: $g_1(p)=q_0 p \cdots\cdots(2.23)$

and for the rectangular type: $g_1(p)=q_m(1-e^{-p\tau})$ $\cdots\cdots(2.24)$

Fig 2. Two examples of the types of intumescences.



(a)δ-function type (b)rectangular type

III. APPROXIMATE SOLUTION OF DISCHARGE WITH THE METHOD OF SADDLE POINT INTEGRATION

We will work out the integration (2.22). For abbreviation we write the power of e in (2.22) as $f(p)$, i.e. $f(p)=a's+p(t+bs)-s\sqrt{1+b^2}\sqrt{(p-p_1)(p-p_2)}$ $\cdots\cdots(3.1)$

For the same purpose we put $G(p)=g_1(p)/p$ $\cdots\cdots(3.1.\text{a})$

Then (2.22) is written $q=\frac{1}{2\pi i}\int G(p)\exp[f(p)]\mathrm{d}p$ $\cdots\cdots(3.2)$

For the integration of the type of the above expression, the method of saddle-point integration is considered as one of the most useful methods of approximation [7]. Hence let us use the method here.

At saddle points $f'(p)=0$ $\cdots\cdots(3.3)$

Therefore at the saddle points (3.1) yields

$$(t+bs)-s\sqrt{1+b^2}\frac{p-C}{\sqrt{(p-p_1)(p-p_2)}}=0, \qquad \cdots\cdots(3.4)$$

where $C=(p_1+p_2)/2$ $\cdots\cdots(3.5)$

Solving (3.4) with respect to p, we obtain as the co-ordinate of the saddle-point

$$p_s=C+\frac{p_1-p_2}{2}\frac{1+bK}{\sqrt{1+2bK-K^2}}, \qquad \cdots\cdots(3.6)$$

where $K=s/t.$ $\cdots\cdots(3.7)$

Expansion of $f(p)$ into Taylor's series around the saddle-point gives

$$f(p)=f(p_s)+\frac{f''(p_s)}{2!}(p-p_s)^2+\cdots\cdots \qquad \cdots\cdots(3.8)$$

In the above equation the term $f'(p_s)$ vanishes due to (3.3). Insertion of (3.8) and (3.6) in (3.2) and integration of (3.2) along the line of the steepest descent give an approximate solution. For that let us find the line of the steepest descent.

On the line the imaginary part of $f(p)$ or $\mathrm{I}f(p)$ is constant [7]. Therefore on the line $\quad \mathrm{I}f(p)-\mathrm{I}f(p_s)=0$ ························(3.9)

Furthermore, the insertion of (3.6) into (3.1) is readily seen to yield

$$\mathrm{I}f(p_s)=0 \qquad (3.10)$$

Hence, (3.9) and (3.10) give $\quad \mathrm{I}f(p)=0$ ························(3.11)

on the line of the steepest descent. Substituting (3.8) in (3.11), we obtain

$$\mathrm{I}\left[\frac{f''(p_s)}{2}(p-p_s)^2\right]=0 \qquad (3.12)$$

on the line. Since $f''(p_s)$, too, is readily seen to be a real quantity, we find on the line the relation $\quad p-p_s=|p-p_s|\exp(i\pi/2)$. ························(3.13)

(3.13) offers the equation of the line of the steepest descent which is parallel to the imaginary axis. Substituting (3.8) in (3.2) and integrating (3.2) along the line of the steepest descent, we obtain

$$q=\frac{1}{2\pi i}\int_{p-p_s=-i\infty}^{+i\infty} G(p_s)\cdot e^{f(p_s)+\frac{f''(p_s)}{2}(p-p_s)^2+\cdots\cdots}\mathrm{d}p$$

or

$$q=\frac{1}{2\pi i}G(p_s)e^{f(p_s)}\int_{p-p_s=-i\infty}^{+i\infty} e^{\frac{f''(p_s)}{2}(p-p_s)^2}\mathrm{d}p \qquad (3.14)$$

In order to transform the above complex integral into a real integral, we put

$$p-p_s=iy.$$

Then (3.14) becomes

$$q=\frac{G(p_s)}{2\pi}e^{f(p_s)}\cdot\int_{-\infty}^{+\infty}\exp\left[-\frac{f''(p_s)}{2}y^2\right]\mathrm{d}y, \qquad (3.15)$$

which essentially is the Gauss' integral. Therefore, as the desired integral we obtain from (3.15)

$$q=\frac{G(p_s)}{\sqrt{2\pi f''(p_s)}}\cdot e^{f(p_s)}. \qquad (3.16)$$

Substitution of (3.6) in (3.16) yields the desired solution

$$q=G(p_s)\frac{\sqrt{(1+b^2)(p_1-p_2)}\cdot K^{3/2}}{2\sqrt{\pi s}\{1+2bK-K^2\}^{3/4}}\cdot\exp\left[-\frac{s}{K}\left\{C+(a'+Cb)K+\frac{p_1-p_2}{2}\sqrt{1+2bK-K^2}\right\}\right] \qquad (3.17)$$

$G(p_s)$ is determined by a boundary condition, and for the two types shown in **Fig. 2** it is as following:

For the δ-function type, $\quad G(p_s)=q_0,$ ························(3.18)

and for the rectangular type,

$$G(p_s)=\frac{q_m}{C+\frac{p_1-p_2}{2}\cdot\frac{1+bK}{\sqrt{1+2bK-K^2}}}\cdot\left[1-\exp\left\{-\tau\left(C+\frac{p_1-p_2}{2}\cdot\frac{1+bK}{\sqrt{1+2bK-K^2}}\right)\right\}\right]. \qquad (3.19)$$

IV. DISPERSION OF INTUMESCENCES

We now consider on the dispersive property of intumescences at our case.

Putting $\quad f(p)=pt-\sigma(p)s$ ························(4.1)

in (3.16), (3.16) is written

$$q=\frac{G(p_s)}{\sqrt{2\pi\sigma''(p_s)}}\cdot e^{p_st-\sigma(p_s)s}, \qquad (4.2)$$

with which we consider the dispersive property of an arbitrary intumescence.

If the values of t and s are considerably large, the wave velocity of the intumescence c is approximately given by $\quad c=p_s/\sigma(p_s)$. ························(4.3)

Substitution of (3.6) in the above equation yields

$$c=\frac{C\sqrt{1+2bK-K^2}+\frac{p_1-p_2}{2}(1+bK)}{-(a'+bC)\sqrt{1+2bK-K^2}+\frac{p_1-p_2}{2}(K-b)} \quad \cdots\cdots(4.4)$$

Since the wave velocity is expressed as a function of $K(=s/t)$, we can find by (4.4) the wave velocity of the intumescence at any section and at any instant.

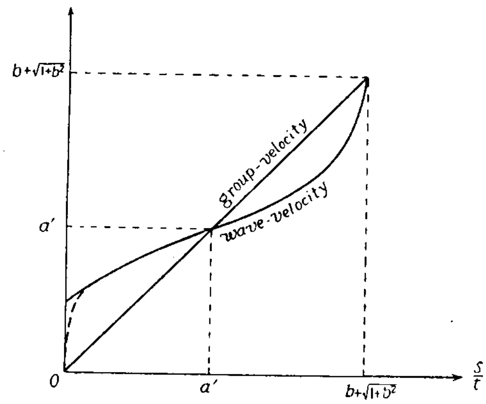
On the other hand, at the saddle-point (3.3)and (4.1) give

$$t-\sigma'(p_s)s=0, \quad \cdots\cdots(4.5)$$

by which p_s has been defined. Hence, a given wave-length and period occur when s/t has a particular value; they seem to travel out with velocity $\sigma'(p_s)$, which is called group-velocity.

A suggestive figure is obtained by plotting the wave-velocity and the group-velocity against s/t, the tendency of the figure becomes like **Fig. 3**. By the figure wave-velocity is compared with group-velocity. The tendency of the interrelation in magnitude between the two velocities shown in **Fig. 3** is independent to all parameters (b and ε) in our case. In fact, after a laborious curve-tracing of the formula (4.4), the following results are obtained:

Fig. 3 Wave-velocity and group-velocity of intumescences when the value of time is sufficiently large.



In the range $0\leqq s/t<a'$; wave-velocity>group-velocity,
at the point $s/t=a'$; wave-velocity=group-velocity,
in the range $a'<s/t<b+\sqrt{b^2+1}$; wave-velocity<group-velocity,
and at the point $s/t=b+\sqrt{b^2+1}$; wave-velocity=group-velocity.

Fig. 3 shows the tendency described just above.

If the quantity $b+\sqrt{b^2+1}$ is retransformed to that in the original units, it is $\{(\alpha+\beta)U_0/2\beta\}+\sqrt{(gH_0/\beta)+[U_0(\alpha-\beta)/2\beta]^2}\ (\fallingdotseq U_0+\sqrt{gH_0})$ and is easily seen to be the propagation speed of a long wave produced in a liquid which is initially in uniform flow of depth H_0 and velocity U_0. Similarly, if a' is retransformed to that in the original units, it is easily seen to be the propagation speed given by the Kleitz-Seddon law of the flood wave in the flow.

It is obviously seen that in **Fig. 3** wave-velocity is not identically equal to group-velocity. When t becomes infinitely large, no trace of the intumescence will remain at any section of finite distance from the origin. Therefore wave-velocity should equal to zero for $s/t=0$. In **Fig. 3**, wave-velocity is not zero for $s/t=0$. That is because the accuracy of the approximate integration with the method of saddle-point integration has not been sufficient near $s/t=0$. Hence, we should modify the wave-velocity as the broken-line shown in the figure.

As for group-velocity it is given by s/t as described before. Hence, if we fix time, i.e. at any instant, the group-velocity is proportional to the distance s from the origin and the velocity corresponds to the speed of a linear elongation. Therefore, if wave-velocity were identically equal to group-velocity, an intumescence OPF would deform to OP_1F_1, OP_2F_2, ……(see **Fig. 4**), which would essentially be the linearly elongated figures of the original intumescence OPF. However, wave-velocity, in fact, is not identically equal to group-velocity, and the intumescence travels with wave-velocity different from group-velocity. As we have seen by **Fig. 3**

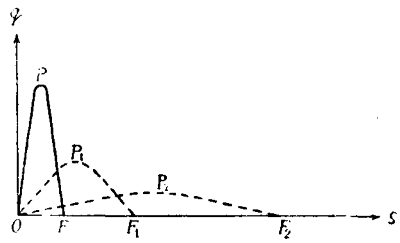
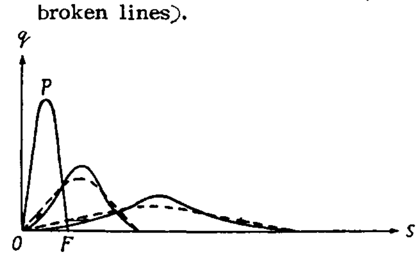


Fig. 4 The shapes which an intumescence OPF would successively deform to, if wave-velocity were identically equal to group-velocity s/t.

at $s/t=0$;	wave-velocity = group-velocity,
in the range $0<s/t<a'$;	wave-velocity > group-velocity,
at $s/t=a'$;	wave-velocity = group-velocity,
in the range $a'<s/t<b+\sqrt{b^2+1}$;	wave-velocity < group-velocity,
and at $s/t=b+\sqrt{b^2+1}$;	wave-velocity = group-velocity.

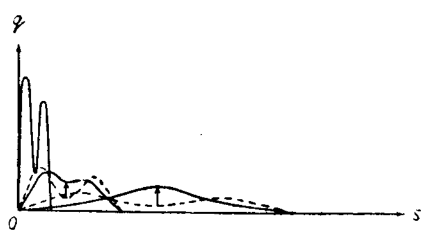
As described before, $s/t=0$ and $s/t=b+\sqrt{b^2+1}$ represent the rear and the front of the intumescence, respectively. Let us suppose that P, P_1, P_2, ……in **Fig. 4** are the points where the equation $s/t=a'$ is satisfied. Then, in fact, every point on the parts $\overset{\frown}{OP}$, $\overset{\frown}{OP_1}$, $\overset{\frown}{OP_2}$, ……of **Fig. 4** travels with faster velocity than the group-velocity for the parts and every point on the parts $\overset{\frown}{PF}$, $\overset{\frown}{P_1F_1}$, $\overset{\frown}{P_2F_2}$, ……travels with slower velocity than the group-velocity for the parts. In consequence, the shape of the initial intumescence is distorted to the direction as indicated with the arrows in **Fig. 5** and the shapes will successively be as shown in the figure by full lines. If, however, wave-velocity were identically equal to group-velocity s/t, the intumescence would successively deform to the shapes shown in the same figure by broken lines.

Fig. 5 The shapes which an intumescence OPF will successively deform to (shown by full lines) and those which it would successively deform to were wave-velocity identically equal to group-velocity (shown by broken lines).



Therefore we find that an intumescence does not uniformly disperse, but it disperses concentrating around the section of $s(=a't)$ at an instant $t=t$. Therefore, what shape of an intumescence may originally have been produced, it disperses concentrating around the point $s/t=a'$, and will have the maximum at t of nearly equal to s/a' at a section $s=s$, if s is sufficiently large. That is to say, intumescences do "concentrative dispersion" around the point $s/t=a'$, if s or t is sufficiently large. It is to be noted here that wave fronts always do not become steeper as they move downstream in a flowing water due to the dispersive property, as far as intumescences of infinitesimally small amplitude are concerned. The last deduction is somewhat different from a description given in the previous literature [4, p.500].

Fig. 6 The shapes which an intumescence originally with two peaks will successively deform to (shown by full lines) and those which it would successively take were wave-velocity identically equal to group-velocity (shown by broken lines).

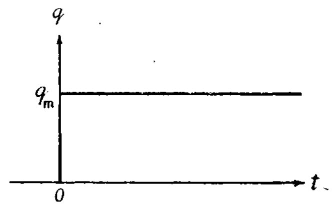


We can similarly find the deformation of an intumescence whose shape is more complicated than that in **Fig. 5**; see **Fig. 6**. By the figure two peaks of an intumescence are seen to have the tendency to combine into one as they move downstream.

V. RIGOROUS SOLUTION OF THE DISCHARGE OF INTUMESCENCES

As we have seen, the solutions obtained in Section **III.** are sufficiently accurate for large values of t, but not accurate for small values of t. Hence, the rigorous solution is desired, and we will work out it in this section.

Fig. 7 The disturbance of discharge of the unit step function type.



First, let us consider the disturbance of the type of a unit step function shown in **Fig. 7**. The rigorous expression for q of the type is given by (2.22) as following:

$$q=\frac{q_m}{2\pi i}\int_{Br}\frac{1}{p}e^{a's+p(t+bs)-s\sqrt{1+b^2}\sqrt{(p-p_1)(p-p_2)}}\mathrm{d}p \qquad (5.1)$$

Putting $\quad l=(p_1-p_2)/2,\quad (p-p_1)(p-p_2)=(p-C)^2-l^2. \qquad (5.2)$

Substitution of (5.2) in (5.1) yields

$$q=\frac{q_m}{2\pi i}\int_{Br}\frac{1}{p}e^{a's+p(t+bs)-s\sqrt{1+b^2}\sqrt{(p-C)^2-l^2}}\mathrm{d}p \qquad (5.3)$$

Let us change the variable p to a new variable μ by the transformation

$$p-C=\mu, \qquad (5.4)$$

then (5.3) reduces to

$$q=q_m e^{a's+C(t+bs)}\cdot\frac{1}{2\pi i}\int_{Br}\frac{1}{\mu+C}\cdot e^{\mu(t+bs)-s\sqrt{1+b^2}\sqrt{\mu^2-l^2}}\mathrm{d}\mu. \qquad (5.5)$$

The integration in the above equation is performed following Jeffreys' calculation [8]. Finally (5.5), after integration, becomes

$$q=q_m e^{s\{(a'+bC)+C/K\}}\cdot\Bigg[I_0\left\{\frac{p_1-p_2}{2}\frac{s}{K}\sqrt{1+2bK-K^2}\right\}$$
$$+\sum_{n=1}^{\infty}\left\{\frac{4}{p_1-p_2}\sqrt{\frac{1+(b-\sqrt{1+b^2})K}{1+(b+\sqrt{1+b^2})K}}\right\}^n\cdot(\nu_1^n+\nu_2^n)I_n\left\{\frac{p_1-p_2}{2}\frac{s}{K}\sqrt{1+2bK-K^2}\right\}\Bigg], \qquad (5.6)$$

where $\quad \left.\begin{matrix}\nu_1\\ \nu_2\end{matrix}\right\}=\frac{-C\pm\sqrt{C^2-l^2}}{2}. \qquad (5.7)$

(5.6) gives the desired rigorous solution of the discharge for the boundary condition of the unit step function type.

The solution (5.6) may be generalized with regard to the boundary condition. We may construct the effect of any arbitrary boundary condition given at the origin, say

$$(q)_{s=0}=F(t), \qquad (5.8)$$

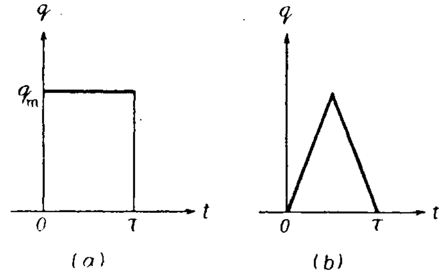
by the use of Duhamel's theorem [9]. Let the solution (5.6) be denoted by $q_m\cdot q_1(t, s)$. Then the solution for the generalized boundary condition is expressed by

$$q=F(0)q_1+\int_0^t F'(\tau)q_1(t-\tau)\mathrm{d}\tau, \qquad (5.9)$$

which is the desired form of solution. However, in the actual calculation it usually depends on numerical or graphical integration. If the form of $F(t)$ is either of the forms shown in **Fig.** 8, the actual calculation is simpler than for other forms of $F(t)$. For the case of the rectangular type of **Fig. 8** (a) the expression of the rigorous solution (5.9) becomes

$$q=q_m\{q_1(t,\,s)-q_1(t-\tau,\,s)\}\cdots\cdots(5.10)$$

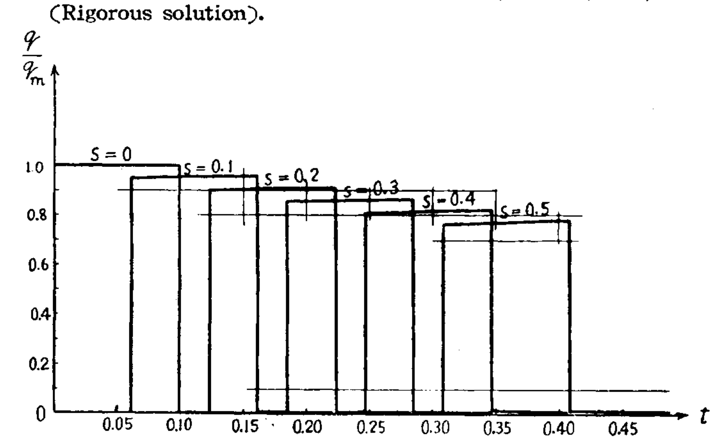
Fig. 8 Disturbances of a rectangular type and a triangular type.



VI. NUMERICAL EXAMPLES

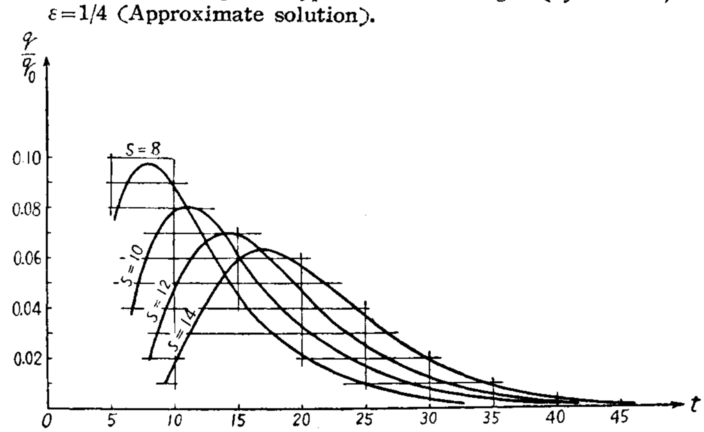
In order to illustrate the rigorous solution for the boundary condition of the rectangular type shown in **Fig. 8** (a) the discharge-time curves given by (5.10), (5.6) and (5.7) are plotted in **Fig. 9** for the system defined by $b=0.5$ and $\varepsilon=1/4$. For a definite value of s the convergency of the infinite series in (5.6) becomes worse as K decreases and for a definite value of K it becomes worse as s increases.

Fig. 9 Discharge-time curves for the boundary condition of the rectangular type shown in Fig.8 (a). $b=0.5$, $\varepsilon=1/4$. (Rigorous solution).



Another numerical example is given in **Fig. 10** to illustrate the approximate solution for the boundary condition of the unit impulse type by the use of eqs (3.17) and (3.18). The numerical values of the two parameters b and ε are the same used in the above example. The accuracy of the solution is poor near the front of the intumescence, where method of Goldschmidt's approximation should have been used.

Fig. 10 Discharge-time curves for the boundary condition of the unit impulse type shown in Fig. 2(a). $b=0.5$, $\varepsilon=1/4$ (Approximate solution).



VII. EQUATIONS FOR THE HEIGHT OF DISTURBANCES

Up to now we have considered intumescences only through the quantity of discharge. In this section let us refer briefly to intumscences through the quantity of the height of the disturbances. The fundamental equation for the height of any arbitrary intumescence h becomes

$$-\frac{\partial^2 h}{\partial s^2}+2b\frac{\partial^2 h}{\partial s\partial t}+\frac{\partial^2 h}{\partial t^2}+2\left\{\frac{\partial h}{\partial t}+a'\frac{\partial h}{\partial s}\right\}=0, \quad \cdots\cdots(7.1)$$

which is derived in just the same way as in Section **II**. The equation is just the same form as (2.16) which has been the fundamental equation for q. Therefore, if an intumescence is given as a boundary condition, say

$$(h)_{s=0}=F(t) \text{ or } g_1(p), \quad \cdots\cdots(7.2)$$

every result obtained hitherto samely applies to the height of the intumescence. Thus we can derive just the same equation as (3.17) in terms of h as an approximate solution and just the same expression as (5.9) in terms of h as the rigorous expression. We can also deduce the property of "concentrative dispersion" of the height of any arbitrary intumescence when t is sufficiently large.

VIII. CONCLUSION

In this paper the author dealt with mathematical study of the motion of intumescences with infinitesimal amplitude in open channels of uniform slope. Main conclusions derived are as following:

(a) The propagation speeds of both the discharge and the water level of any intumescence distribute between 0 and $\{(\alpha+\beta)U_0/2\beta\}+\sqrt{(gH_0/\beta)+\{U_0(\alpha-\beta)/(2\beta)\}^2}\doteqdot U_0+\sqrt{gH_0}$. For this reason intumescences disperse as they travel downstream. The former speed forms the rear of intumescences and the latter speed forms the front of them.

(b) Both the discharge and the water level of any intumescence have the tendency to do "concentrative dispersion" around the point $s/t=(5-4\varepsilon)U_0/3$ when t is sufficiently large.

(c) Wave fronts of intumescences always do not become steeper as they move downstream in a flowing water, as far as intumescences of infinitesimally small amplitude are concerned.

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