PROGRAMMING FOR DIGITAL COMPUTATION OF SUSPENSION BRIDGES UNDER VERTICAL, HORIZONTAL AND TORSIONAL LOADINGS

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1. Introduction

A treatment is introduced over "how-to" utilize the electronic digital computers for the calculation of suspension bridges under horizontal and torsional forces as well as vertical loadings.

Because of non-linearity of suspension bridges on the spatial deformation, statical values such as bending moments of stiffening girder are not easily determined that we may operate manual calculs.

Equilibrium equations for the structure are three differential equations on the deflection v, the horizontal displacement u, and the torsional deformations φ of the stiffning girder, and they are not independent each other, in other words, each equation shows non-linear differential one including every other variable such as u, v, φ and various static values. Some elastic equations are also to be considered in order to accomodate elastic conditions for cables.

Problems are to solve these equations and to obtain the numerical values for given suspension bridge. When we use the electronic digital computers, it is the most important task that how to compose the programme of calculation.

Since the most of the engineers who are concerned with the structural mechanique have few chances really to operate the digital computers by themselves, and operaters or programmers of the computers also know few knowledge about the real meaning of the equations, so previous treatment of equations is necessary before they are undertaken by the programmers.

The previous treatment is, in short, to compose the calculating flow chart and to give the practical calculating formulas instead of analytical display. When we are to utilize the digital computers, usual theoretical and analytical results and methods of calculs are not always suitable, we must therefore serch somewhat different sort of calculating methods.

The author introduces in the following articles how to compose the flow chart for the calculation of the vertical, horizontal and torsional behaviours of suspension bridge. In the second article, theoretical and analytical presentation is dealed. The third article shows the modification of such theoretical equations to practical formulas to fit in the practical solutions. The last article gives the conclusive results of the flow chart. This article be sent to any operators or programmers, we would obtain the values of given suspension bridge without any explanation.

2. Fundamental equations

Let a suspension bridge be considered as are shown in Figs. 1 and 2 with definitions of

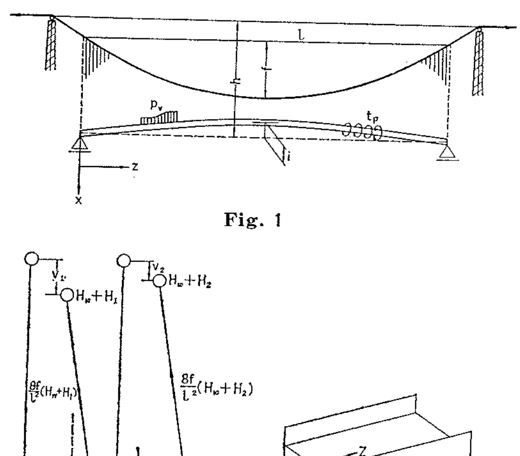

Fig. 1

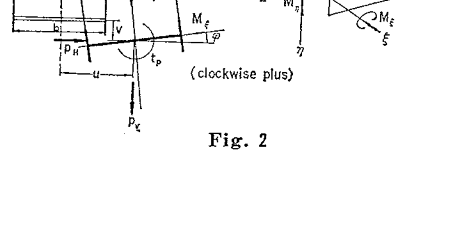

Fig. 2

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spatial co-ordinates, and of directions of deformations and forces. Two parallel cables are to carry total weight of structures by the infinitely close spaced curtain-like suspenders causing no stresses in the stiffning girder under the standard state, that is, there are no loadings and under usual temperature.

The shapes of the cables and the stiffning girder are to be defined as eqs. (1) and (2), wherein the cables take a symmetric curve of parabola and the stiffning girder is to have the initially logitudinal parabolic camber.

$$Y_c=\frac{4f}{l^2}z(l-z) \quad\cdots\cdots(1)$$

$$Y_J=h-\frac{4i}{l^2}z(l-z) \quad\cdots\cdots(2)$$

The deflection v and torsional angle φ of the stiffning girder cause the deflection v_1, and v_2 of the left and right cables, respectively.

$$\left.\begin{aligned} v_1&=v-b\varphi/2\\ v_2&=v+b\varphi/2\end{aligned}\right\}\quad\cdots\cdots(3)$$

Each cable then varies its tension. Usually we consider the horizontal reaction of tensile forces that varies from H_w to (H_w+H_1) and (H_w+H_2) for each cable. By the close spaced suspenders, each cable gives the distributing oad p_1 and p_2 to the stiffning girder.

$$\left.\begin{aligned} p_1&=(H_w+H_1)\frac{d^2}{dz^2}(Y_c+v_1)\\ p_2&=(H_w+H_2)\frac{d^2}{dz^2}(Y_c+v_2)\end{aligned}\right\}\quad\cdots\cdots(4)$$

As p_1 and p_2 work along the left and right side of the stiffning girder, they are composed so as to cause the vertical and horizontal forces along the sectional gravity center and, the twisting moment around the shear center or twisting center of the stiffning girder.

External forces or loadings against the stiffning girder are, the vertical loadings p_v like traffic weight, also the dead weight p_d of stiffning girder itself, the horizontal forces p_H like wind thrust, and the torque t_p which would occure when the traffic loads do not lie just on the center axis of bridge width or when wind thrust works so as to make a torque around the shear center.

The stiffning girder is then affected under the three component of forces as shown eqs. (5)

$$\left.\begin{aligned} &\text{vertical} && p_v+p_d-(p_1+p_2)\\ &\text{horizontal} && p_H-\theta(p_1+p_2)\\ &\text{torque} && t_p+\frac{b}{2}(p_1-p_2)\end{aligned}\right\}\quad\cdots\cdots(5)$$

where θ is the slope angle of the suspenders, and it may be allowed to assume nearly equal to the ratio u/h. The dead weight p_d balances to the cable reaction as eq. (6)

$$p_d=-2H_w\frac{d^2Y_c}{dz^2}=2\cdot\frac{8f}{l^2}\cdot H_w \quad\cdots\cdots(6)$$

Succeedingly, we consider the deformation and static values of the stiffning girder itself. Sectional bending moments around the two principal axis ξ,η and torsional moment around the longitudinal axis are;

$$\left.\begin{aligned} M_\xi&=-EJ_\xi\left[\frac{d^2v}{dz^2}-\frac{d^2u}{dz^2}\varphi\right]\\ M_\eta&=-EJ_\eta\left[\frac{d^2u}{dz^2}+\frac{d^2v}{dz^2}\varphi\right]\\ M_z&=-EC\frac{d^3\varphi}{dz^3}+GK\frac{d\varphi}{dz}\end{aligned}\right\}\quad\cdots\cdots(7)$$

As for the reduced components inside the vertical and horizontal plain, we obtain from M_ξ and M_η,

$$\left.\begin{aligned} M_x&=-EJ_\xi\frac{d^2v}{dz^2}+(EJ_\xi-EJ_\eta)\varphi\frac{d^2u}{dz^2}\\ M_y&=-EJ_\eta\frac{d^2u}{dz^2}+(EJ_\xi-EJ_\eta)\varphi\frac{d^2u}{dz^2}\end{aligned}\right\}(8)$$

As the axis of the stiffning girder runs with the spatial curvature, bending moments M_x and M_y cause the additional torque around the axis;

$$t(z)=\frac{d^2u}{dz^2}M_x-\left(\frac{d^2Y_J}{dz^2}+\frac{d^2v}{dz^2}\right)M_y \quad\cdots(9)$$

Differentiate the eqs. (8) twice by z, and the M_z of eq. (7) once, and we obtain three fundamental differential equations upon u, v and φ. Considering eqs. (5) and (9), and assuming the shear center of the section is not so deviated from the gravity center, we get;

$$EJ_\xi\frac{d^4v}{dz^4}-(EJ_\xi-EJ_\eta)\frac{d^2}{dz^2}\left[\varphi\frac{d^2u}{dz^2}\right]=p_{V_1}$$

$$+(2H_W+H_1+H_2)\frac{d^2v}{dz^2}+\frac{b}{2}(H_2-H_1)$$

$$\times\frac{d^2\varphi}{dz^2}-(H_1+H_2)\frac{8f}{l^2} \quad\cdots\cdots(10\text{-}1)$$

$$EJ_\eta\frac{d^4u}{dz^4}-(EJ_\xi-EJ_\eta)\frac{d^2}{dz^2}\left[\varphi\frac{d^2v}{dz^2}\right]=p_H$$

$$+\frac{u}{h}\Big[(2H_W+H_1+H_2)\frac{d^2v}{dz^2}+\frac{b}{2}(H_2-H_1)\times\frac{d^2\varphi}{dz^2}-(2H_W+H_1+H_2)\frac{8f}{l^2}\Big] \cdots(10\text{-}2)$$

$$EC\frac{d^4\varphi}{dz^4}-GK\frac{d^2\varphi}{dz^2}=t_p-\frac{d^2u}{dz^2}M_x+\left(\frac{d^2v}{dz^2}+\frac{4i}{l^2}\right)M_y+\frac{b^2}{4}(2H_W+H_1+H_2)\times\frac{d^2\varphi}{dz^2}+\frac{b}{2}(H_2-H_1)\left(\frac{d^2v}{dz^2}-\frac{8f}{l^2}\right) \cdots\cdots(10\text{-}3)$$

The above equations contain two unknown constants H_1 and H_2. These are decided by the elastic elongation of each cable under the conditions that the elastic elongation is equal to the one induced by the deflection of cable, plus the one of spanlength between top of towers, excluding thermal elongation of whole cable length.

$$\frac{L_s}{E_cA_c}H_1=\frac{8f}{l^2}\int_0^l v_1dz+\Delta l_1 \cdots\cdots(10\text{-}4)$$

$$\frac{L_s}{E_cA_c}H_2=\frac{8f}{l^2}\int_1^l v_2dz+\Delta l_2 \cdots\cdots(10\text{-}5)$$

These five equations are the fundamental.

3. Solution and practical formulation

In order to solve the equations (10), there would not be any suitable method than "try and error iteration method", that is, we try calculs and find error, repeat again after correcting some values, and try………

The first step starts in solving the eq. (10-1) taking no regards but v.

$$EJ_\xi\frac{d^4v}{dz^4}-(2H_W+H_1+H_2)\frac{d^2v}{dz^2}=P_V(z) \cdots\cdots(11)$$

Integrating twice by z and Considering the boundary conditions that the stiffning girder is simply supported at both ends, we get.

$$-EJ_\xi\frac{d^2v}{dz^2}+(2H_W+H_1+H_2)v=\mathfrak{M}_V(z) \cdots\cdots(12)$$

$\mathfrak{M}_V(z)$ is the bending moment of simple supported beam under the load $P_V(z)$. We use the Green's function to obtain the values.

$$\mathfrak{M}_V(z)=l\int_0^l P_V(k)G_1(z,k)dk \cdots\cdots(13)$$

where

$$G_1(z,k)=\begin{cases}\dfrac{l-k}{l}\cdot\dfrac{z}{l} & z\leq k\\ \dfrac{l-z}{l}\cdot\dfrac{k}{l} & z\geq k\end{cases} \cdots\cdots(14)$$

It is more convinient for $\mathfrak{M}_V(z)$ to be written as below, referring eqs. (10-1) and (13)

$$\mathfrak{M}_V(z)=l\int_0^l p_V(k)G_1(z,k)dk-(H_1+H_2)\times\frac{8f}{l^2}\frac{z(l-z)}{2}-\frac{b}{2}(H_2-H_1)\varphi-(EJ_\xi-EJ_\eta)\varphi\frac{d^2u}{dz^2} \cdots\cdots(15)$$

The equation (12) can be solved also by Green's formula,

$$v(z)=\frac{l}{EJ_\xi}\int_0^l G_2(z,k)\cdot\mathfrak{M}_V(k)dk \cdots\cdots(16)$$

where $G_2(z,k)$ is an exponential function,

$$G_2(z,k)=\begin{cases}\dfrac{\sinh c(l-k)\sinh cz}{cl\sinh cl} & z\leq k\\ \dfrac{\sinh c(l-z)\sinh ck}{cl\sinh cl} & z\geq k\end{cases} \cdots\cdots(17)$$

$$c=\sqrt{(2H_W+H_1+H_2)/EJ_\xi} \cdots\cdots(18)$$

Operations of Green's integral in eqs. (13) and (16), are, practically, the same as the calculations of matrices in the computer. It must be specially taken care in eq. (17) so that the values may remain within few errors even when cl varies from infinitely small to some value about 20. Usual suspension bridges have the value cl within about 20, but if the bending stiffness EJ_ξ is very small, cl will exceeds more than 20. $G_2(z,k)$ grows, in such a case, to infinitely large values as to exceed the register of computer. In this programme the author does not care the treatment when cl is large, but if necessary, the same process may give good references as are shown in eqs. (24) (25)'.

Secondly, we consider the eq. (10-2) in regard to the horizontal displacement u. There are two ways to modify the eq. (10-2). They are,

$$EJ_\eta\frac{d^4u}{dz^4}=P_H(z) \cdots\cdots(19\text{-}1)$$

or

$$EJ_\eta\frac{d^4u}{dz^4}+\frac{8f}{l^2}(2H_W+H_1+H_2)u=P_H(z) \cdots\cdots(19\text{-}2)$$

The former equation is that of a simple supported beam and the latter that of a beam on an elastic bed. In order to shorten the programme steps and to save the memories of computers, the former presentation prefers to the latter. Because, we can utilize the same Green's function $G_1(z,k)$, on the contrary, the latter needs new function. The solutions for the former are,

$$-EJ_\eta\frac{d^2u}{dz^2}=\mathfrak{M}_H(z) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots(20)$$

$$u=\frac{l}{EJ_\eta}\int_0^l G_1(z,k)\mathfrak{M}_H(k)dk \quad\cdots\cdots\cdots(21)$$

$$\mathfrak{M}_H(z)=l\int_0^l p_H(k)G_1(z,k)dk$$
$$-(EJ_\xi-EJ_\eta)\varphi\cdot\frac{d^2v}{dz^2}+l\int_0^l G_1(z,k)$$
$$\times\left\{\left[(2H_W+H_1+H_2)\left(\frac{d^2v}{dz^2}-\frac{8f}{l^2}\right)\right.\right.$$
$$\left.\left.+\frac{b}{2}(H_2-H_1)\frac{d^2\varphi}{dz^2}\right]\frac{u}{h}\right\}_{z=k}dk \quad\cdots(22)$$

In the third step, we find the solutions upon φ, we let

$$EC\frac{d^4\varphi}{dz^4}-\left[GK+\frac{b^2}{4}(2H_W+H_1+H_2)\right]$$
$$\times\frac{d^2\varphi}{dz^2}=\tau(z) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(23)$$

The solutions are presented using Green's function as well.

$$-\frac{d^2\varphi}{dz^2}=\frac{l}{EC}\int_0^l G_3(z,k)\tau(k)dk\cdots\cdots\cdots(24)$$

$$\varphi=\frac{l^3}{EC}\int_0^l G_4(z,k)\tau(k)dk\cdots\cdots\cdots(25)$$

where $G_3(z,k)$ is given as the same expression as eq.(17) using the value c_1 instead of c

$$c_1=\sqrt{\overline{GK}/EC} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(26)$$

$$\overline{GK}=GK+\frac{b^2}{4}(2H_W+H_1+H_2) \quad\cdots\cdots(27)$$

$G_4(z,k)$ is obtained as below

$$G_4(z,k)=\frac{1}{c_1^2l^2}\{G_1(z,k)-G_3(z,k)\} \quad (28)$$

As mentioned before, practical effectivity of eqs. $G_3(z,k)$ and $G_4(z,k)$ does not guarantee up to the large values of c_1l. To save such a fault, we consider the branch routine when c_1l exceeds more than $c_1l=10$.

$$-\frac{d^2\varphi}{dz^2}=\frac{1}{\overline{GK}}\tau(z) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots(24)'$$

$$\varphi=\frac{l}{\overline{GK}}\int_0^l G_1(z,k)\tau(k)dk \quad\cdots\cdots(25)'$$

Now, above all equations are the practical formulations of the fundamental equations (10-1), (10-2) and (10-3). Most important task is then how to combinate these and to project the flow of digital computations. The composer of flow chart must have at least so much experiences that he is engaged himself on a great deal of numerical calculs and knows what errors cause by.

Analytical solutions often fall in the fault that they can not be available for digital computation even they are correct themselves. Iteration methods seems on the contrary, less analytical. If the technique of reduction of errors, however, be fairly projected, try and error iterations have a merit to be able to repeat up to reach the value having few errors.

The author's process is shown in next article introducing the iteration methods. Following instruction and notes were really trusted to Mr. Omori, programming engineer of Fuji calculating center, using FACOM 2203 digital computer. The author is utilizing this programm for the calculation of a suspension bridge under static wind thrust. The results of calculation will be shown in the progress report.

4. Flow chart

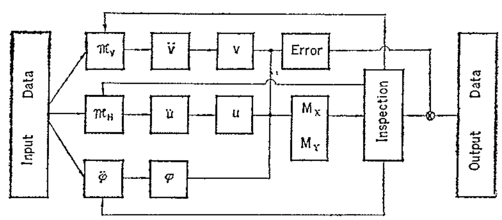

Fig. 3

Table 1 Input Constants

Symbols	Dimensions	Descriptions
l	m	Span length of stiffning girder
f	m	Sag of cable between length l
b	m	Distance between two cables, also width of stiffning girder
h	m	Height of towers from gravity center of stiffning girder at supports
i	m	Center height of camber of stiffening girder
H_W	t	Horizontal reaction of each cable by whole dead weight of structure
EJ_ξ	tm²	Bending rigidity of whole stiffning girder
EJ_η	tm²	Bending rigidity against horizontal bending of stiffning girder
GK	tm²	Torsional rigidity of stiffning girder
EC	tm⁴	Torsional rigidity that cause bending in stiffning girder. about $b^2 EJ_\xi/4$
E_cA_c/L_s	t/m	Ratio of tension increace of each cable caused by unit elastic elongation of whole cable length

Table 2 Input variables

N		Number of division of whole length l
e		Desirable relative error
μ_1,μ_2		Coefficient to check the reduction of Iteration
$p_V(z)$	t/m	Vertical loadings
$p_H(z)$	t/m	Horizontal forces
t_p	tm/m	Torque loadings
Δl_1	m	Distance growth between towers minus thermal elongations of each cable
Δl_2	m	

Notes

1) N may be arbitral number about more than 20.
2) p_V, p_H and t_p are given by the intensity per unit length at every N-devidened panel point. The concentrated load p is then given as PN/l at the panel point where P rests.
3) e is decided as 10^{-2m} if relative error is needed for 10^{-m}.
4) Reset μ_1,μ_2 so that iteration may converge rapidly.

Table 3 Main Calculation Routine

①	$cl=l\sqrt{(2H_W+H_1+H_2)/EJ_\xi}$	cl
②	$v=\frac{l}{EJ_\xi}\int_0^l G_2(z,k)\mathfrak{M}_V(k)dk$ $G_2(z,k)=\begin{cases}\frac{\sinh c(l-k)\sinh cz}{cl\sinh cl} & z\leq k\\ \frac{\sinh c(l-z)\sinh ck}{cl\sinh cl} & z\geq k\end{cases}$	$v(z)$
3	$\Delta L_v=\int_0^l vdz$	ΔL_v
4	$\frac{d^2v}{dz^2}=c^2v-\frac{1}{EJ_\xi}\mathfrak{M}_V$	$\frac{d^2v(z)}{dz^2}$
⑤	$u=\frac{l}{EJ_\eta}\int_0^l G_1(z,k)\mathfrak{M}_H(k)dk$ $G_1(z,k)=\begin{cases}\frac{l-k}{l}\cdot\frac{z}{l}\\ \frac{l-z}{l}\cdot\frac{k}{l}\end{cases}$	$u(z)$
⑥	$c_1l=l\sqrt{\overline{GK}/EC}$ $\overline{GK}=GK+\frac{b^2}{4}(2H_W+H_1+H_2)$	c_1l
⑦	when $c_1l\leq 10$ (if $c_1l>10$, select step No. ⑦′) $\frac{d^2\varphi}{dz^2}=-\frac{l}{EC}\int_0^l G_3(z,k)\tau(k)dk$ $G_3(z,k)$ is the same expression as $G_2(z,k)$ only using c_1l instead of cl Refer step No. ②	$\frac{d^2\varphi(z)}{dz^2}$
⑦′	when $c_1l>10$ $\frac{d^2\varphi}{dz^2}=-\frac{1}{\overline{GK}}\tau(z)$	$\frac{d^2\varphi(z)}{dz^2}$
⑧	$\varphi=\frac{l^3}{EC}\int_0^l G_4(z,k)\tau(k)dk$ $G_4(z,k)=\frac{1}{c_1^2l^2}[G_1(z,k)-G_2(z,k)]$	$\varphi(z)$
⑧′	$\varphi=\frac{l}{\overline{GK}}\int_0^l G_1(z,k)\tau(k)dk$	$\varphi(z)$
9	$\Delta L_\varphi=\int_0^l \varphi\, dz$	ΔL_φ
10	$H_1'=\frac{E_cA_c}{L_s}\left[\Delta l_1+\frac{8f}{l^2}\left(\Delta L_v-\frac{b}{2}\Delta L_\varphi\right)\right]$	H_1'
11	$H_2'=\frac{E_cA_c}{L_s}\left[\Delta l_2+\frac{8f}{l^2}\left(\Delta L_v+\frac{b}{2}\Delta L_\varphi\right)\right]$	H_2'
⑫ ⑬	$(H_1)_{r+1}=(H_1)_r\mu_1+H_1'(1-\mu_1)$ $(H_2)_{r+1}=(H_2)_r\mu_1+H_2'(1-\mu_1)$ Reset more correct values of H_1,H_2	H_1 H_2
14	$\Delta\mathfrak{M}_V=-(H_1+H_2)\frac{8f}{l^2}\cdot\frac{z(l-z)}{2}-\frac{b}{2}$ $\times(H_2-H_1)\varphi-(EJ_\xi-EJ_\eta)\varphi\frac{d^2u}{dz^2}$	$\Delta\mathfrak{M}_V$
⑮	$\mathfrak{M}_V=\mathfrak{M}_{V_0}+\Delta\mathfrak{M}_V$	$\mathfrak{M}_V$
16	$\Delta p_H(z)=\left[\frac{b}{2}(H_2-H_1)\frac{d^2\varphi}{dz^2}+(2H_W\right.$ $\left.+H_1+H_2)\left(\frac{d^2v}{dz^2}-\frac{8f}{l^2}\right)\right]\frac{u}{h}$	Δp_H
17	$\mathfrak{M}_H'=\mathfrak{M}_{H_0}+\int_0^l l\Delta p_H(k)G_1(z,k)dk$ $-(EJ_\xi-EJ_\eta)\varphi\frac{d^2v}{dz^2}$	$\mathfrak{M}_H'$
⑱	$(\mathfrak{M}_H)_{r+1}=(\mathfrak{M}_H)_r\mu_2+\mathfrak{M}_H'(1-\mu_2)$ Reset more correct function of $\mathfrak{M}_H(z)$	$\mathfrak{M}_H$
19	$\frac{d^2u}{dz^2}=-\frac{1}{EJ_\eta}\mathfrak{M}_H$	$\frac{d^2u}{dz^2}$
⑳	$M_x=-(2H_W+H_1+H_2)v+\mathfrak{M}_V$ $-\frac{(EJ_\xi-EJ_\eta)}{EJ_\eta}\varphi\,\mathfrak{M}_H$	M_x
㉑	$M_y=\mathfrak{M}_H-\frac{(EJ_\xi-EJ_\eta)}{EJ_\xi}$ $\times\varphi[(\mathfrak{M}_V-(2H_W+H_1+H_2)v]$	M_y
22	$\Delta t(z)=\frac{b}{2}(H_2-H_1)\frac{d^2v}{dz^2}$ $-\frac{b}{2}(H_2-H_1)\frac{8f}{l^2}$ $-\frac{d^2u}{dz^2}M_x+\left(\frac{d^2v}{dz^2}+\frac{8i}{l^2}\right)M_y$	$\Delta t(z)$
23	$\tau(z)=t_p(z)+\Delta t(z)$	$\tau(z)$
㉔	$E=\frac{\int_0^l(v_{r+1}-v_r)^2dz}{\int_0^l v_{r+1}^2dz}$	E
25	when $E>e$, start again from step No.1. $E<e$, stop the Iteration and put out the data	

Table 4 Preliminary Calculation Routine

⑫′ ⑬′	$H_1=H_2=0.9\frac{3\,l}{8\,f}\int_0^l p_V(k)\frac{k(l-k)}{l^2}dk$	H_1 H_2
	$\mathfrak{M}_{V_0}=l\int_0^l p_V(k)G_1(z,k)dk$	$\mathfrak{M}_{V_0}$
	$\mathfrak{M}_{H_0}=l\int_0^l p_H(k)G_1(z,k)dk$	$\mathfrak{M}_{H_0}$
⑮′	$\mathfrak{M}_V=\mathfrak{M}_{V_0}-4f(H_1+H_2)\frac{z(l-z)}{l^2}$	$\mathfrak{M}_V$
⑱′	$\mathfrak{M}_H=(1-\mu_2)\mathfrak{M}_{H_0}$	$\mathfrak{M}_H$

Instructions and Notes

1) Step Nos. with a circle such as ①, ②, ⑤,…show that the values must be put out.
2) Preliminary Calculation give the first approximate values before the main iteration routine in order to save repeat times.
3) $\mathfrak{M}_{V0}$ and $\mathfrak{M}_{H0}$ are stored so that they may be utilized in every iteration.
4) Every value or every function must be stored till it is corrested in $(r+1)$th iteration calculs.

References

1) Hawranek-Steinhardt : "Theorie und Berechnung der Stahlbücken", Springer 1958

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