APPROXIMATE SOLUTION FOR FREE SURFACE PROFILE OF UNDULAR HYDRAULIC JUMP

Kyoto University, Student Member ○Thin Thwe Thwe
Kyoto University, Fellow Member Takashi Hosoda
Kyoto University, Member Shinichiro Onda
University of Cordoba, Non-Member Oscar Castro-Orgaz

1. INTRODUCTION

The mathematical formulation to derive an approximate solution for an undular hydraulic jump is described using the linearized Boussinesq equation with the eddy viscosity term. An approximate solution for a discontinuous hydraulic jump which is composed of two distributions of the upstream and downstream parts is firstly shown neglecting the vertical acceleration term in Boussinesq equation. Then, substituting the small deviation from a discontinuous jump into Boussinesq equation leads to a linearized equation which is applied to obtain the approximate solution of an undular jump. Extending the mathematical expression of a discontinuous jump solution, the simple equation is assumed as an approximate solution for an undular jump. And then the coefficients in the equation are fixed considering the coupling conditions of the upstream and downstream distributions. Finally, it is shown that the comparison of the approximate solution and the simulated result with Runge Kutter method is in good agreement.

2. BASIC EQUATION

Referring to Fig. 1, the fundamental equation used for deriving the analytical solution of undular hydraulic jump is Boussinesq equation [1, 2] which is given by Eq. 1.

$$\frac{q^2}{h}+\frac{gh^2}{2}+D_m\frac{q}{h}\frac{dh}{dx}+\frac{1}{3}q^2\frac{d^2h}{dx^2}-\frac{1}{3}\frac{q^2}{h}\left(\frac{dh}{dx}\right)^2=M_0 \tag{1}$$

where, x = special coordinate, q = unit width discharge, h = water depth, g = gravitational acceleration, D_m = eddy diffusivity coefficient, M_0 = momentum flux. Eq. 1 is non-dimentionalized by introducing the non-dimensional variables, $x'=x/h_1$ and $h'=h/h_1$, and then the following equation is obtained.

$$1+\frac{1}{2}\frac{1}{Fr_1^2}h'^3+\alpha\frac{dh'}{dx'}+\frac{1}{3}h'\frac{d^2h'}{dx'^2}-\frac{1}{3}\left(\frac{dh}{dx'}\right)^2=X_1h' \tag{2}$$

where, $\alpha=\dfrac{D_m}{q}$, $X_1=1+\dfrac{1}{2}\dfrac{1}{Fr_1^2}$

3. APPROXIMATE SOLUTION OF DISCONTINUOUS HYDRAULIC JUMP

Hosoda, Thin et al [3] derived the approximate solutions for a discontinuous hydraulic jump of Eq. 3 as follows:

$$1+\frac{1}{2}\frac{1}{Fr_1^2}h_J'^3+\alpha\frac{dh'}{dx'}=X_1h_J' \tag{3}$$

$$x\le 0:\ h_J=h_c+P_0+P_1\exp(\beta_1x)+P_2\exp(2\beta_1x) \tag{4a}$$

$$x>0:\ h_J=h_c-R_0-R_1\exp(\gamma_1x)-R_2\exp(2\gamma_1x) \tag{4b}$$

h_J at x = 0 is set to be the critical depth $h_c=Fr_1^{2/3}$.

4. LINEARIZED BOUSSINESQ EQUATION FROM h_J AND THE APPROXIMATE SOLUTIONS

Substituting $h'=h_J'+\delta h'$ into Eq. 2 we obtain the following Eq. 5. Hereafter ' is omitted.

$$\frac{3}{2}\frac{1}{Fr_1^2}h_J^2\delta h+\alpha\frac{d\delta h}{dx}+\frac{1}{3}h_J\frac{d^2h_J}{dx^2}+\frac{1}{3}h_J\frac{d^2\delta h}{dx^2}+\frac{1}{3}\delta h\frac{d^2h_J}{dx^2}-\frac{1}{3}\left(\frac{dh_J}{dx}\right)^2-\frac{2}{3}\frac{dh_J}{dx}\frac{d\delta h}{dx}=X_1\delta h \tag{5}$$

where, δh = the deviation of depth from h_J.

In evaluating the water surface profile of undular jump, we consider negative and positive regions separately and find out the solution as follows. For negative region $x\le 0$,

$$h=h_J+\delta h_m,\ \delta h_m=F_{m1}(x)\exp(\beta_1x)+F_{m2}(x)\exp(2\beta_1x) \tag{6a}$$

$$F_{m1}=C_{1m}\exp(\theta_{1m}x)+C_{2m}\exp(\theta_{2m}x)+AM\ ,\ F_{m2}=C_{3m}\exp(\theta_{3m}x)+C_{4m}\exp(\theta_{4m}x)+BM \tag{6b}$$

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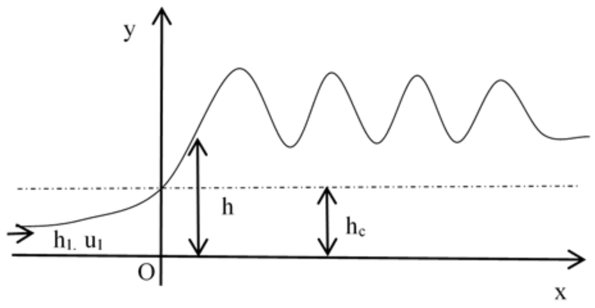
Contact address: River Engineering Lab., Dept. of Urban Management., Kyoto University (C1-3-265, Kyodai Katsura, Nishikyo-ku, Kyoto 615-8540, Tel. 075-383-3266)



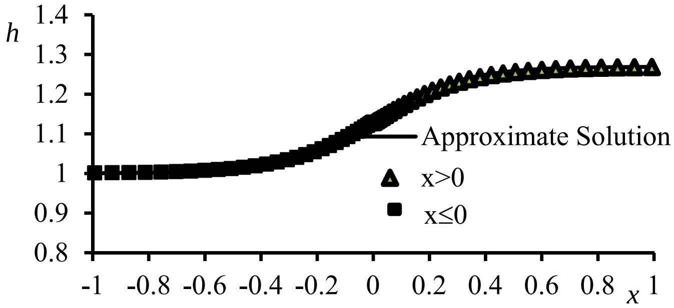


Figure 1 Sketch diagram of undular jump

Figure 2 Comparison of analytical and approximate solution

where, $C_{1m}, C_{2m}, C_{3m}, C_{4m}$, are constant terms and $\theta_{1m}, \theta_{2m}, \theta_{3m}, \theta_{4m}$ are calculated by using the following quadratic equations. $Am_1\theta^2 + Am_2\theta + Am_3 = 0$, $Bm_1\theta^2 + Bm_2\theta + Bm_3 = 0$

For the positive region $x > 0$,

$$h = h_J + \delta h_p, \quad \delta h_p = F_{p1}(x)\exp(\gamma_1 x) + F_{p2}(x)\exp(2\gamma_1 x) \tag{7a}$$

$$F_{p1} = C_{1p}\exp(\theta_{pr}x)\cos(\theta_{pi}x) + C_{2p}\exp(\theta_{pr}x)\sin(\theta_{pi}x) + AP \tag{7b}$$

$$F_{p2} = C_{3p}\exp(\theta_{pr'}x)\cos(\theta_{pi'}x) + C_{4p}\exp(\theta_{pr'}x)\sin(\theta_{pi'}x) + BP \tag{7c}$$

where, $C_{1p}, C_{2p}, C_{3p}, C_{4p}$ are constant terms and $\theta_{1p}, \theta_{1p}, \theta_{3p}, \theta_{4p}$ are calculated by using the following quadratic equations. $Ap_1\theta^2 + Ap_2\theta + Ap_3 = 0$, $Bp_1\theta^2 + Bp_2\theta + Bp_3 = 0$

5. CALCULATED RESULTS

In finding out the analytical solution, the coefficients included in the equations were fixed considering the coupling conditions of the upstream and downstream distributions at the origin. Numerical simulation was carried out with changing the coefficient parameters of C_{1m} and C_{2m} in the negative region and other parameters, C_{3m} and C_{4m} were calculated by setting the second order term of the fluctuating water depth and its first derivative to be zero at the inflow boundary condition ($x = -10$). Then, the coefficients C_{1p}, C_{2p}, C_{3p} and C_{4p} for positive region were calculated depending on that of negative region by enforcing two constraints $\delta h_m = \delta h_p$ and $d\delta h_m/dx = d\delta h_p/dx$ at the origin ($x = 0$) and also taking the second order term of the fluctuating water depth and its derivative to be zero. The Fourth order Runge Kutter method was employed in order to get numerical solution. The water depth and the free surface slope obtained from the analytical solution were taken as the initial boundary conditions for the numerical solution. The comparisons of the computed free surface profiles of the analytical and numerical solutions are displayed in Fig. 3.

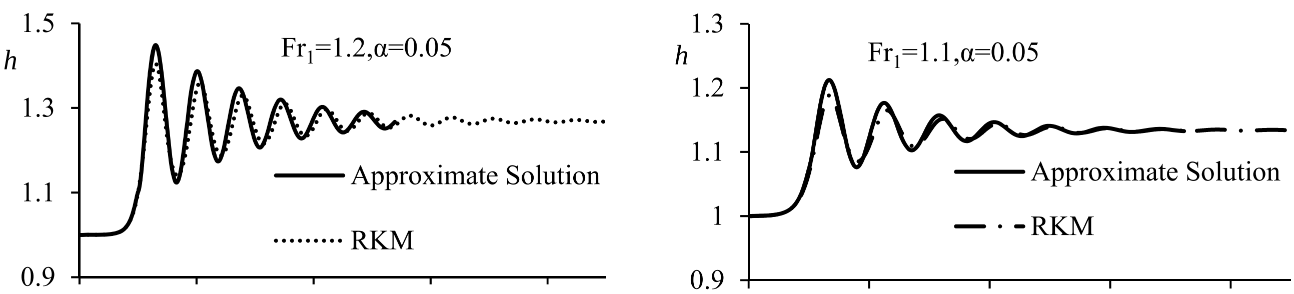


Figure 3 Comparison of analytical and numerical simulation results

6. CONCLUTIONS

It can be pointed out that the approximate solutions show good agreement with the computational results.

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