# Stability Analysis of Liquefaction for Plane Wave Propagation

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## 1. INTRODUCTION

Liquefaction refers to the phenomenon of soil behavior changing from solid-like to fluid-like. Earthquake induced liquefactions have caused severe damages. Theoretical studies of liquefaction are mainly based on Biot's formulation, regarding soil as a two-phase mixture. A great number of constitutive models for soil have been proposed and the numerical results can usually reproduce corresponding experimental results to some extent. However the triggering condition of liquefaction remains unknown. Through stability analysis of the governing equations, we will show that anisotropy of material property will lead to instability of the solutions, which implies the onset condition for liquefaction. The governing equations and stability analysis are explained in the second section. The stability of perturbation in form of plane wave is analyzed in the third section for isotropic and anisotropic cases, followed by concluding remarks and comments on further research.

## 2. FORMULATION

### 2.1 Governing equation

In this paper, we employ continuum mechanics based modeling, assuming elasto-plasticity for soil and slow flow for water. Denoting by $\dot{\mathbf{u}}, \dot{\sigma}, \dot{p}$, the rate of displacement and stress of soil and pressure of water, we have

$$\begin{aligned} \rho \mathrm{D}^2\dot{\mathbf{u}} - \nabla\cdot\mathbf{c}^{ep} : (\nabla\dot{\mathbf{u}}) + \nabla\dot{p} &= \mathbf{0}, \\ \nabla\cdot\dot{\mathbf{u}} - \nabla\cdot(\mathbf{k}\cdot\nabla p) &= 0, \end{aligned} \tag{1}$$

where $\rho, \mathbf{c}^{ep}, \mathbf{k}$ are soil density, elasto-plasticity and permeability; $\nabla$ and $\mathrm{D}$ stand for spatial and temporal derivatives; and $\cdot$, $:$ stands for the first and second-order contraction. In the first equation of Eq.(1), the fluid and the solid phase are assumed have the same acceleration while the second one is the continuity equation. On the mathematical viewpoint, Eq.(1) is a coupling equation of $\mathbf{u}$ and $p$. For simplicity, dropping the dot on $\mathbf{u}$ and $p$ in the first equation, we use the following set of the differential equations for stability analysis:

$$\begin{aligned} \rho \mathrm{D}^2\mathbf{u} - \nabla\cdot(\mathbf{c} : (\nabla\mathbf{u})) + \nabla p &= \mathbf{0}, \\ \nabla\cdot\mathrm{D}\mathbf{u} - \nabla\cdot(\mathbf{k}\cdot\nabla p) &= 0. \end{aligned} \tag{2}$$

Here, superscript *ep* is omitted for $\mathbf{c}^{ep}$.

### 2.2 Stability analysis

We consider the stability of a solution $\mathbf{u}$ and $p$ which satisfy the governing equation, Eq.(2), by introducing small perturbations $\mathbf{u}+\delta\mathbf{u}$ and $p+\delta p$. Substitute $\mathbf{u}+\delta\mathbf{u}$ and $p+\delta p$ into Eq.(2) and linearize the equation with respect to $\delta\mathbf{u}$ and $\delta p$. It turned out that the linearized equation for $\delta\mathbf{u}$ and $\delta p$ is the same as for $\mathbf{u}$ and $p$ in Eq.(2). Hereafter, Eq.(2) is used for the discussion of stability analysis of $\delta\mathbf{u}$ and $\delta p$ ($\delta$ is omitted hereafter). Exponential increase of the perturbation terms indicates instability while decay of the perturbation terms implies stability.

## 3. PLANE WAVE

We consider the stability of a disturbance as a *plane wave* which propagates in a fixed direction without changing its form normal to that direction. The plane wave is considered for a pair of $\mathbf{u}$ and $p$. Apply Fourier transform of Eq.(2) using $\exp(i(\xi\cdot\mathbf{x}-\omega t))$, we obtain

$$\begin{aligned} -\rho\omega^2\mathbf{u} + (\boldsymbol{\xi}\cdot\mathbf{c}\cdot\boldsymbol{\xi})\cdot\mathbf{u} + \imath\boldsymbol{\xi}p &= \mathbf{0}, \\ \omega\boldsymbol{\xi}\cdot\mathbf{u} + (\boldsymbol{\xi}\cdot\mathbf{k}\cdot\boldsymbol{\xi})p &= 0. \end{aligned} \tag{3}$$

Here, for simplicity, the same symbols $\mathbf{u}$ and $p$ are used for the transformed functions. This set is the target equations of the stability analysis for small perturbation in form of plane wave.

### 3.1 Isotropic case

We assume isotropy for the material property, i.e. $\mathbf{c} = \lambda\delta\otimes\delta + 2\mu\mathbf{I}, \mathbf{k} = k\delta$, where $\delta, \mathbf{I}$ are the second- and (symmetric) fourth-order identity tensor, $\lambda$ and $\mu$ are Lame parameters. The following equation can be derived from

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Eq.(3) for the isotropic case with wave vector $\boldsymbol{\xi} = (\xi, 0, 0)$,

$$\begin{bmatrix} -\rho\omega^2 + (\lambda+2\mu)\xi^2 & 0 & 0 & \imath\xi \\ 0 & -\rho\omega^2 + \mu\xi^2 & 0 & 0 \\ 0 & 0 & -\rho\omega^2 + \mu\xi^2 & 0 \\ \omega\xi & 0 & 0 & k\xi^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

The determinant of the matrix at the right-hand side of Eq.(4) must vanish for non-trivial solution of $\mathbf{u}$ and $p$, which gives an equation of $\omega$

$$k\xi^2(\rho\omega^2 - \mu\xi^2)^2(\rho\omega^2 - (\lambda+2\mu)\xi^2 + \frac{\imath\omega}{k}) = 0, \tag{5}$$

from which we can obtain two roots $\omega = \pm\xi c_s$ ($c_s = \sqrt{\mu/\rho}$) which correspond to the non-coupling shear wave and the other two roots for the coupling wave

$$\omega = -\frac{\imath}{2\rho k} \pm c_p\xi\sqrt{1 - \frac{1}{(2\rho k c_p \xi)^2}} \tag{6}$$

with $c_p = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$. It is found that $\omega$ of Eq.(6) has a negative imaginary value, which implies that the coupling wave always decays as $t$ increases, and cannot be unstable.

### 3.3 Anisotropic case

A key characteristic of soil is dilatancy, i.e., shear deformation leads to volumetric deformation. In the present framework of the stability analysis, dilatancy could be presented as non-zero components of the elastic tensor $\mathbf{c}$. For simplicity, we set $c_{1112} = c_{1113} = \alpha$ for stability analysis. The equation of $\omega$ for such anisotropic case is

$$k\xi^2(-\rho\omega^2 + \mu\xi^2)(k(2\alpha^2 + \mu(\lambda+2\mu))\xi^4 + \imath\mu\xi^2\omega + k(\lambda+3\mu)\xi^2\rho\omega^2 - \imath\rho\omega^3 - k\rho\omega^4) = 0. \tag{7}$$

It can be further simplified as

$$r^4 + \imath\, r_k\, r^3 - (1 + r_s^2)\, r^2 - \imath\, r_k\, r_s^2\, r + r_s^2 - 2r_a^4 = 0, \tag{8}$$

where $r = \dfrac{\omega}{\omega_p}$, $r_{s,a,k} = \dfrac{\omega_{s,a,k}}{\omega_p}$, $\omega_{p,s,a} = \xi c_{p,s,a}$, with $\omega_k = \dfrac{1}{\rho k}$ and $c_a = \sqrt{\alpha/\rho}$. An example of numerical results evaluated from the analytical solution of $r$ is given in Fig. 1. Positive imaginary part of $r$ indicates the exponential increase of $\mathbf{u}$ and $p$. This implies that liquefaction takes place when the degree of dilatancy exceeds a limit.

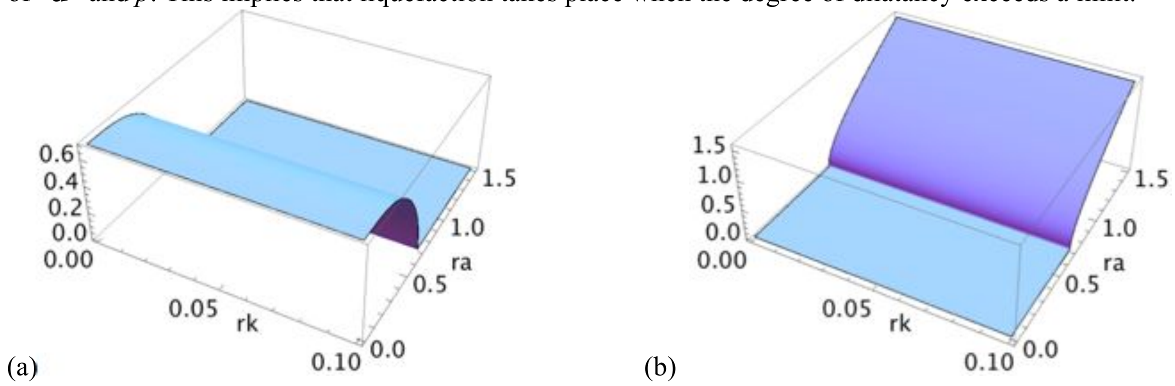


Fig. 1. A root of $\omega$ corresponds to unstable solution of the governing equation: (a) real part; (b) imaginary part.

## 4. CONCLUDING REMARKS

We carried out stability analysis for the governing equations of liquefaction. Small perturbation from the equilibrium solution in the form of plane wave is stable for the isotropic case, indicating that liquefaction does not happen. The unstable solution found for the anisotropic case implies that liquefaction could happen if the degree of dilatancy exceeds a certain critical value. Further investigations for spherical waves will be conducted.

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