Consistent integration for Iwan spring model under multisurface hyperplasticity framework

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Modeling of hysteretic behaviors undergone loading-unloading reversals and cyclic loading conditions requires the advantage of kinematic hardening responses in computational techniques. Nested yield surface models proposed by Iwan (1967) give a rise to the advanced models known as multisurface plasticity. Under a new framework of hyperplasticity initiated by Houlsby & Puzrin (2000), kinematic hardening multisurface plasticity can be described within the unified energy principle.

1. Objectives: In order to investigate the applicability of the consistent integration scheme under hyperplasticity, a preliminary attempt is conducted in one-dimensional model. The numerical procedures and results obtained by forward-Euler scheme are compared with those of backward-Euler scheme. The results of this study can motivate a basic understanding as well as indicating some difficulties on numerical implementation. Further research development for complicated material models in generalized stress space is encouraged in this subject.

2. Background: Formulations of a conceptual one-dimensional model (see Fig.1) in accordance with kinematic hardening multisurface plasticity can be found in Houlsby and Puzrin (2006) for details. Gibbs free energy potential function g for a total system and $g^{(n)}$ for each element defined in Eqs.(1)-(2) are employed as functions of stress σ and internal plastic strains $\alpha^{(n)}$, in which E is elastic stiffness and $H^{(n)}$ are the corresponding hardening stiffness for the n^{th}-yield function amongst a total N number. The Iwan model in series case is described in Eq.(3) in terms of multiple yield functions $y^{(n)}$ which are the similar function but different internal state variables and material properties. The relative stresses $\chi^{(n)}$ are defined as the center-shifted stress from the origin of stress space. $\chi^{(n)}$ cannot be greater in absolute value than yield stresses $k^{(n)} > 0$. Therefore the set of admissible $\chi^{(n)}$ are constrained in the closed interval $[-k^{(n)}, k^{(n)}]\subset\mathbb{R}$. Yield functions can judge state of stress, i.e. $y^{(n)} < 0$ under elastic state while $y^{(n)} = 0$ under elasto-plastic state. The superscript (n) represents the dummy index of the concerned variables. For an arbitrary variable •, the element-n^{th} is represented by $\bullet^{(n)}$. $\Sigma\bullet^{(n)}$ represents a summation over the range of element. Δ• represents incremental form. The numbering of the internal variables are sorted in order of $k^{(n+1)} \geq k^{(n)}$. Moreover, $H^{(n+1)} \leq H^{(n)}$ is defined where $H^{(N)} \approx 0$ indicates no hardening stiffness at the last yield surface. Therefore, failure is marked when all yield functions are entirely yielded. A total plastic strain ε^p is related to $\alpha^{(n)}$ by the constraint function defined in Eq.(4).

$$g=\frac{1}{N}\sum_{n=1}^{N}g^{(n)}\ \ldots(1),\quad g^{(n)}=-\frac{\sigma^2}{2E}-\sigma\alpha^{(n)}+\frac{1}{2}H^{(n)}\alpha^{(n)2}\ldots(2),\quad y^{(n)}=\left|\chi^{(n)}\right|-k^{(n)}\leq 0\ldots(3),\quad \varepsilon^p=\frac{1}{N}\sum_{n=1}^{N}\alpha^{(n)}\ \ldots(4)$$

In the framework of multisurface plasticity, stress variables σ and $\chi^{(m)}$ are evaluated from their conjugates as described in Eq.(5)-(6) respectively. Correspondingly, hardening stiffness $H^{(n)}$ and yield stress $k^{(n)}$ are calibrated and defined by Eqs.(7)-(8).

$$\varepsilon=-\frac{\partial g}{\partial\sigma}=\frac{\sigma}{E}+\varepsilon^p\ \ldots(5),\quad \chi^{(n)}=-\frac{\partial g^{(n)}}{\partial\alpha^{(n)}}=\sigma-H^{(n)}\alpha^{(n)}\ldots(6),\quad H^{(n)}=H_i w_H\left(\tfrac{n-0.5}{N}\right)\ldots(7),\quad k^{(n)}=k_f w_k\left(\tfrac{n-0.5}{N}\right)\ldots(8)$$

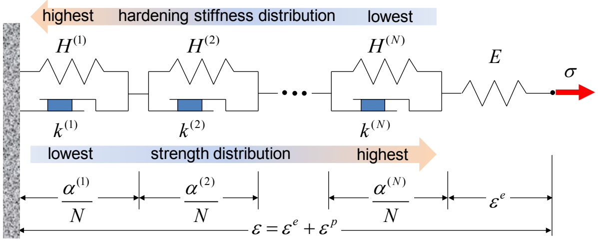

Figure 1: The Iwan model schematized by a serial combination of an elastic spring and friction slides

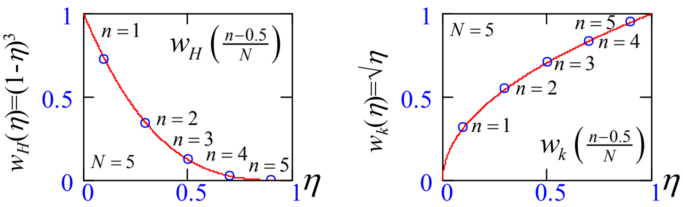

Figure 2: Continuous weight distributions of hardening stiffness (left) and yield stress (right) with mid-point linear piecewise

3. Consistent integration: Time continuity is discretized into a summation of equal time intervals. The previous and current stages are signified by subscripts i and i+1 respectively. Rate forms described in the previous section can be explicitly written by incremental forms in the way that $\Delta\bullet=\dot{\bullet}\Delta t=\bullet_{i+1}-\bullet_i$. Summation of incremental forms is a basic formulation of incremental integration scheme which is also known as forward-Euler method in strain-driven formulation for a given $\Delta\varepsilon$. An consistent integration (Simo & Taylor, 1985) which is known as backward-Euler method is regarded as the efficient algorithm. This algorithm is acceptable in the recent progress of finite element method due to high accuracy, robustness and stability despite of large increments. Updated processes based on two different choices of integration schemes are described by the following ways. Incremental and consistent integration schemes are summarized in Box 1 using the below relevant symbols.

$$\bullet_{i+1}=\begin{cases}\bullet_i+\dot{\bullet}_i\Delta t & \text{:forward-Euler}\\ \bullet_i+\dot{\bullet}_{i+1}\Delta t & \text{:backward-Euler}\end{cases},\quad sign(\bullet)=\begin{cases}\text{undefined} & \text{if }\bullet=0\\ \bullet/|\bullet| & \text{otherwise}\end{cases},\quad \langle\bullet\rangle=\begin{cases}0 & \text{if }\bullet<0\\ \bullet & \text{if }\bullet\geq 0\end{cases},\quad \Phi(\bullet)=\begin{cases}0 & \text{if }\bullet<0\\ 1 & \text{if }\bullet\geq 0\end{cases}$$

Keywords Hyperplasticity, Multisurface plasticity, Backward-Euler integration, Consistent integration, Elasto-plasticity

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4. Numerical demonstration: Responses of a closed-loop loading of the Iwan model are demonstrated. Two classes of chosen hyperbolic functions w_H and w_k in the normalized coordinate $\eta\in[0,1]$ were used to characterize the distribution of $H^{(n)}$ and $k^{(n)}$. The mid-point rule $\eta=(n-0.5)/N$ is employed to relate η with the n^{th} piecewise element in the way as shown in Fig. 2. Numerical analyses under incremental and consistent integration schemes were conducted using arbitrary material parameters H_i=20, k_f=5, E=100. The results in combinations of variation between number of mutisurface and sub-steps were shown in Fig. 3. It was found that number of multiple yield functions improves smoothness of stress-strain responses while number of sub-steps substantially affects accuracy in the incremental scheme but less in the consistent scheme.

Box 1: Integration schemes

Incremental integration scheme

For given $\Delta\varepsilon$, ε_i, $\alpha_i^{(m)}$; for $m\in\{1,2..N\}$

1. Initialize

$$\varepsilon_{i+1}=\varepsilon_i+\Delta\varepsilon\ ,\ \varepsilon_i^p=\frac{1}{N}\sum_{n=1}^{N}\alpha_i^{(n)}\ ,\ \sigma_i=E\left(\varepsilon_i-\varepsilon_i^p\right)$$

for $m\in\{1,2,..,N\}$

$$\chi_i^{(m)}=\sigma_i-H^{(m)}\alpha_i^{(m)}$$

2. Find the index number of the largest yield surface

$$N^*=\sum_{n=1}^{N}\Phi\left(y(\chi_i^{(n)})\right)$$

3. Determine plastic state variables

if $N^*=0$ then

for $m\in\{1,2..N\}$ $\Delta\gamma^{(m)}=0$, $\Delta\alpha^{(m)}=0$

else $E_{ep}^{-1}=\dfrac{N}{E}+\displaystyle\sum_{n=1}^{N^*}\frac{1}{H^{(n)}}$

for $m\in\{1,2..N\}$

$$\Delta\gamma^{(m)}=\frac{sign(\chi_i^{(m)})}{H^{(m)}}E_{ep}\Delta\varepsilon$$

$$\Delta\alpha^{(m)}=\left\langle\Delta\gamma^{(m)}\right\rangle sign(\chi_i^{(m)})$$

4. Compute incremental variables

$$\Delta\varepsilon^p=\frac{1}{N}\sum_{n=1}^{N^*}\Delta\alpha^{(n)}\ ,\ \Delta\sigma=E\left(\Delta\varepsilon-\Delta\varepsilon^p\right)$$

5. Update state variables

$\sigma_{i+1}=\sigma_i+\Delta\sigma$

for $m\in\{1,2,..,N\}$

$$\alpha_{i+1}^{(m)}=\alpha_i^{(m)}+\Delta\alpha^{(m)},\ \chi_{i+1}^{(m)}=\sigma_{i+1}-H^{(m)}\alpha_{i+1}^{(m)}$$

6. Set $i=i+1$ and go to step 1 for the next step

Consistent integration scheme

For given $\Delta\varepsilon$, ε_i, $\alpha_i^{(m)}$; for $m\in\{1,2..N\}$

1. Initialize

$$\varepsilon_{i+1}=\varepsilon_i+\Delta\varepsilon\ ,\ \varepsilon_i^p=\frac{1}{N}\sum_{n=1}^{N}\alpha_i^{(n)}\ ,\ \sigma_i=E\left(\varepsilon_i-\varepsilon_i^p\right)$$

2. Elastic trial stage $\sigma_{i+1}^{tr}=\sigma_i+E\Delta\varepsilon$

for $m\in\{1,2..N\}$

$$\chi_{i+1}^{tr(m)}=\sigma_{i+1}^{tr}-H^{(m)}\alpha_i^{(m)}\quad ,\quad y_{i+1}^{tr(m)}=y^{(m)}(\chi_{i+1}^{tr(m)})$$

$$\alpha_{i+1}^{(m)}=\alpha_i^{(m)},\quad \Delta\gamma^{(m)}=0$$

3. Find the largest yield surface $N^*=\displaystyle\sum_{n=1}^{N}\Phi\left(y_{i+1}^{tr(n)}\right)$

4. Compute state variables

If $N^*=0$ then set $(\bullet)_{i+1}=(\bullet)_{i+1}^{tr}$ and exit to step 7

$$\text{Else}\quad \sigma_{i+1}=\sigma_{i+1}^{tr}-\frac{\displaystyle\sum_{n=1}^{N^*}\frac{y_{i+1}^{tr(n)}}{H^{(n)}}sign\left(\chi_{i+1}^{tr(n)}\right)}{\dfrac{N}{E}+\displaystyle\sum_{n=1}^{N^*}\frac{1}{H^{(n)}}}$$

for $m\in\{1,2,..N^*\}$

$$\Delta\gamma^{(m)}=\frac{y_{i+1}^{tr(m)}-\left(\sigma_{i+1}^{tr}-\sigma_{i+1}\right)sign\left(\chi_{i+1}^{tr(m)}\right)}{H^{(m)}}$$

5. Check convergence

if $N^*<\displaystyle\sum_{n=1}^{N^*}\Phi\left(\Delta\gamma^{(n)}\right)$ then assign $N^*=\displaystyle\sum_{n=1}^{N^*}\Phi\left(\Delta\gamma^{(n)}\right)$

and repeat step 4, else step 6

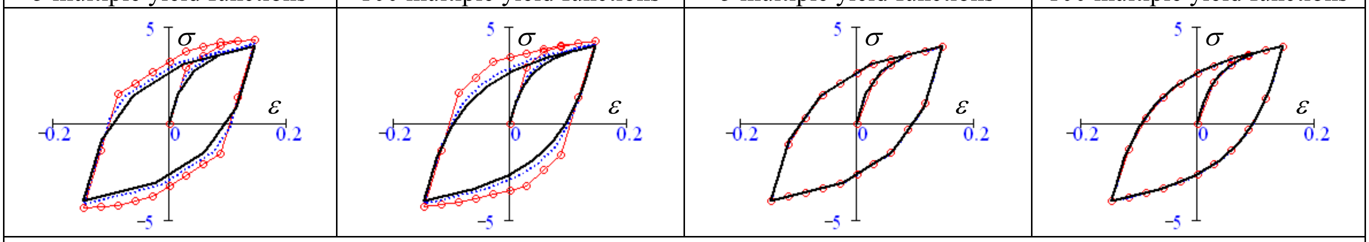
6. Update internal variables

$$\alpha_{i+1}^{(m)}=\alpha_i^{(m)}+\Delta\gamma^{(m)}sign\left(\chi_{i+1}^{tr(m)}\right),\ \chi_{i+1}^{(m)}=\sigma_{i+1}-H^{(m)}\alpha_{i+1}^{(m)}$$

7. Set $i=i+1$ and go to step 1 for the next step

Figure 3: Comparisons of stress-strain responses in a closed-loop loading

Incremental integration scheme		Consistent integration scheme	
5 multiple yield functions	100 multiple yield functions	5 multiple yield functions	100 multiple yield functions



-○-○- 25 intervals ($\Delta\varepsilon$=$0.5884k_f/E$) ------ 100 intervals ($\Delta\varepsilon$=$0.1471k_f/E$) —— 2000 intervals ($\Delta\varepsilon$=$0.0074k_f/E$)

5. Conclusion: The applicability of the consistent integration scheme to kinematic hardening multisurface hyperplasticity was demonstrated for one-dimensional Iwan spring model. Number of multiple yield functions slightly affects to the accuracy than the size of increments. Consistent integration can raise high accuracy despite of large increments. Though hyperplasticy framework provides a direct access to energy function, the accuracy is still relied on types of the numerical integration.

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