Bridge vibration characteristics estimation by Balanced Stochastic Realization (BSR) theory based on ambient vibration

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1. <u>Introduction</u>: This study is to estimate the dynamic characteristics (frequency, damping and vibration mode) of bridge by using balanced stochastic realization (BSR) theory[1) 2)] for ambient vibration. The numerical simulation for Langer bridges was performed using BSR method I and II. The dynamic characteristics estimation error was calculated for both methods. The performance of the methods based on the estimation accuracy was observed.

2. <u>Calculation process</u>: Let's $\mathbf{y}(\tau)\ (\tau = 0,1,\cdots N+2k-2)$ be the measured data in finite-time for executing the balanced stochastic realization. Block Teplitz and Hankel matrix are then found from the past and future response data block $\mathbf{Y}_p \in \mathbf{R}^{mk\times N}$ and $\mathbf{Y}_f \in \mathbf{R}^{mk\times N}$ respectively.

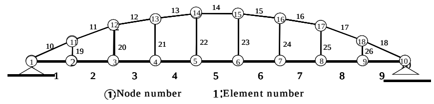

Fig. 1: Model bridge

$$\mathbf{Y}_p = \begin{bmatrix} \mathbf{y}(k-1) & \cdots & \cdots & \mathbf{y}(N+k-2) \\ \mathbf{y}(k-2) & & & \mathbf{y}(N+k-3) \\ \vdots & & & \vdots \\ \mathbf{y}(0) & \cdots & \cdots & \mathbf{y}(N-1) \end{bmatrix} \quad \mathbf{Y}_f = \begin{bmatrix} \mathbf{y}(k) & \mathbf{y}(k+1) & \cdots & \mathbf{y}(k+N-1) \\ \mathbf{y}(k+1) & & & \mathbf{y}(N+k) \\ \vdots & & & \vdots \\ \mathbf{y}(2k-1) & \cdots & \cdots & \mathbf{y}(N+2k-2) \end{bmatrix} \tag{1}$$

The covariance matrix of measured data

$$\begin{bmatrix} \boldsymbol{\Gamma}_{pp} & \boldsymbol{\Gamma}_{pf} \\ \boldsymbol{\Gamma}_{fp} & \boldsymbol{\Gamma}_{ff} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \mathbf{Y}_p \\ \mathbf{Y}_f \end{bmatrix}\begin{bmatrix} \mathbf{Y}_p{}^T & \mathbf{Y}_f{}^T \end{bmatrix} \tag{2}$$

Covariance matrix were obtained by the LQ orthogonal decomposition of data block matrix

$$\begin{bmatrix} \boldsymbol{\Gamma}_{pp} & \boldsymbol{\Gamma}_{pf} \\ \boldsymbol{\Gamma}_{fp} & \boldsymbol{\Gamma}_{ff} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{L}_{11}{}^T & \mathbf{L}_{21}{}^T \\ \mathbf{0} & \mathbf{L}_{22}{}^T \end{bmatrix} \tag{3}$$

The matrix representation of LQ decomposition yields the next relation

$$\boldsymbol{\Gamma}_{pp} = \mathbf{L}_{11}\mathbf{L}_{11}{}^T \quad , \quad \boldsymbol{\Gamma}_{fp} = \mathbf{L}_{21}\mathbf{L}_{11}{}^T \quad , \quad \boldsymbol{\Gamma}_{ff} = \mathbf{L}_{21}\mathbf{L}_{21}{}^T + \mathbf{L}_{22}\mathbf{L}_{22}{}^T \tag{4}$$

k is greater than system order and it is defined as $k > n$

(1) **<u>Method I</u>**: Full rank factorization of covariance matrixes $\boldsymbol{\Gamma}_{ff}$ and $\boldsymbol{\Gamma}_{pp}$ are

$$\boldsymbol{\Gamma}_{pp} = \mathbf{U}_p\mathbf{S}_p{}^{\frac{1}{2}}\mathbf{S}_p{}^{\frac{1}{2}}\mathbf{V}_p = \mathbf{L}\mathbf{L}^T \quad , \quad \boldsymbol{\Gamma}_{ff} = \mathbf{U}_f\mathbf{S}_f{}^{\frac{1}{2}}\mathbf{S}_f{}^{\frac{1}{2}}\mathbf{V}_f = \mathbf{M}\mathbf{M}^T \tag{5}$$

Thus the singular value decomposition of the covariance matrix $\boldsymbol{\Gamma}_{pf}$

$$\mathbf{L}^{-1}\boldsymbol{\Gamma}_{pf}\mathbf{M}^{-1T} = \mathbf{U}\boldsymbol{\Pi}\ \ \mathbf{V}^T \cong \mathbf{U}_s\boldsymbol{\Pi}_s\mathbf{V}_s{}^T \tag{6}$$

Here $\boldsymbol{\Pi}_s$ is determined by ignoring the relatively small value of $\boldsymbol{\Pi}$. The dimension of state vector is $n = \dim \boldsymbol{\Pi}_s$. Observability and controllability matrix are,

$$\mathbf{P}_k = \mathbf{L}\mathbf{U}_s\boldsymbol{\Pi}_s{}^{\frac{1}{2}} \quad \text{and} \quad \mathbf{Q}_k = \boldsymbol{\Pi}_s{}^{\frac{1}{2}}\mathbf{V}\mathbf{M}^T \tag{7}$$

From the definition of $\mathbf{P}_k$ and $\mathbf{Q}_k$ the state matrixes $\mathbf{A},\mathbf{C}$ can be calculated as following,

$$\mathbf{A} = \underline{\mathbf{P}}_k{}^{\downarrow m+}\mathbf{P}_k{}^{\uparrow m} \tag{8}$$

Where, $\underline{\mathbf{P}}_k{}^{\downarrow m}$, $\mathbf{P}_k{}^{\uparrow m}$ are respectively the last m rows and first m rows of $\mathbf{P}_k$ i.e.

$$\underline{\mathbf{P}}_k{}^{\downarrow m} = \mathbf{P}_k\begin{bmatrix} [1] \\ \vdots \\ [m(k-1)] \end{bmatrix}, \quad \mathbf{P}_k{}^{\uparrow m} = \mathbf{P}_k\begin{bmatrix} [m+1] \\ \\ [mk] \end{bmatrix} \tag{9}$$

$$\mathbf{C} = \mathbf{E}_m\mathbf{U}_s\mathbf{S}_s{}^{\frac{1}{2}} \ \text{where, educe matrix,} \ \mathbf{E}_m = \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m \end{bmatrix} \tag{10}$$

Table-1: Model properties

Bridge characteristics	Units	Numerical value
Effective span	L (m)	58.995
Ridge of the bridge	f(m)	9.36
Density of steel	D (t/m)	7.85
Young's Modulus	E (t/m^2)	2.1*10^7
Total mass of the bridge	Ton	19.3
Total weight of bridge	Ton	189.17
Sectional area	A1 (m^2)	0.0224
	A2 (m^2)	0.0123
	A3 (m^2)	0.0137
	A4 (m^2)	0.006015
Moment of inertia	I1 (m^4)	0.00066
	I2 (m^4)	0.00001
	I3 (m^4)	0.000001

Table-2: Natural Frequency(Hz)

Mode order	Frequency (Hz)
1st	1.749
2nd	2.893
3rd	5.69
4th	7.375
5th	9.416
6th	11.179
7th	14.236
8th	15.615

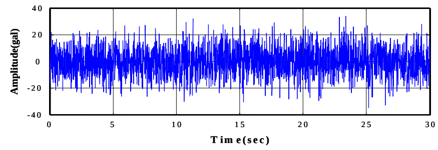

Fig. 2: White noise

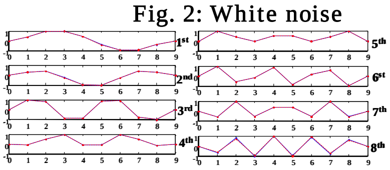
Fig. 3: Vibration mode

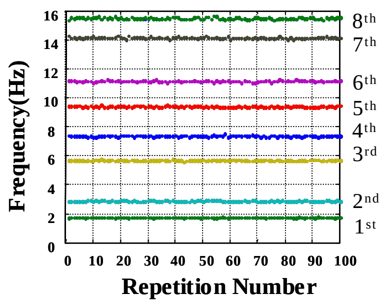

Fig.4: Frequencies (Hz)

Keywords: System Identification, Ambient vibration, Bridge, Realization theory, Dynamic characteristics
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(2) **<u>Method Ⅱ:</u>** State variable from the data block can be written as

$$\mathbf{X}_k = \mathbf{S}^{\frac{1}{2}}\mathbf{V}^T\mathbf{M}^{-1}\mathbf{Y}_p \in \mathbf{R}^{n\times N} \tag{11}$$

Here, the matrix formed by the last N－1 rows of $\mathbf{X}_k \in \mathbf{R}^{n\times N}$ be

$$\hat{\mathbf{X}}_k = X_k[[1] \ \cdots \ [N-1]] \in \mathbf{R}^{n\times(N-1)} \tag{12}$$

Again, the matrix formed by the first N-1 rows of $\mathbf{X}_k$ be

$$\hat{\mathbf{X}}_{k+1} = X_k[[2] \ \cdots \ [N]] \in \mathbf{R}^{n\times(N-1)} \tag{13}$$

Data block from the measured data.

$$\mathbf{Y}_k = [\mathbf{y}(k) \ \cdots \ \cdots \ \mathbf{y}(k+N-1)] \in \mathbf{R}^{m\times N} \tag{14}$$

And matrix by deleting the last 1 rows

$$\hat{\mathbf{Y}}_k = [\mathbf{y}(k) \ \cdots \ \cdots \ \mathbf{y}(k+N-2)] \in \mathbf{R}^{m\times(N-1)} \tag{15}$$

Thus we find the state space equation combining the above matrixes

$$\begin{bmatrix} \hat{\mathbf{X}}_{k+1} \\ \hat{\mathbf{Y}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \hat{\mathbf{X}}_k + \begin{bmatrix} \boldsymbol{\rho}_w \\ \boldsymbol{\rho}_v \end{bmatrix} \tag{16}$$

$\boldsymbol{\rho}_w \in \mathbf{R}^{n\times(N-1)}$ and $\boldsymbol{\rho}_v \in \mathbf{R}^{m\times(N-1)}$ are error matrix

Right multiplying $\hat{\mathbf{X}}_k^T$ in the above equation and $\mathbf{A}$, $\mathbf{C}$ matrix is thus obtained as

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = \left(\begin{bmatrix} \hat{\mathbf{X}}_{k+1} \\ \hat{\mathbf{Y}}_{k|k} \end{bmatrix} \hat{\mathbf{X}}_k^T\right)(\hat{\mathbf{X}}_k \hat{\mathbf{X}}_k^T)^{-1} \tag{17}$$

Matrixes $\mathbf{A}$ and $\mathbf{C}$ thus obtained by the both methods. Dynamic characteristics of the system then found by eigenvalue analysis of the above two matrices.

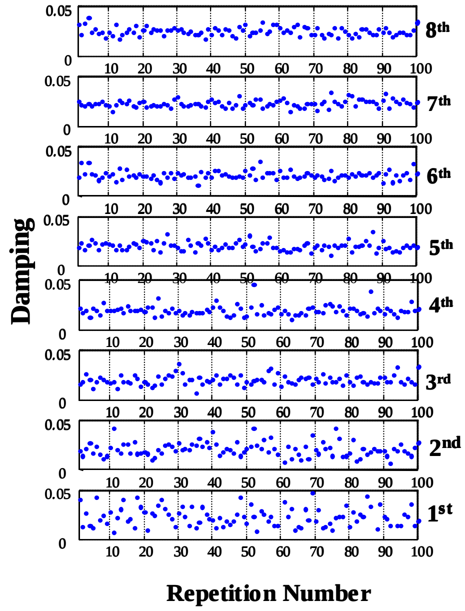

Fig. 5: Damping

Table-3: Estimation accuracy comparison

Mode order	Method	Frequency(Hz)					Damping				
		Analytical	mean	Est.err(%)	Std.	CV(%)	Assumed	mean	Est.err(%)	Std.	CV(%)
1st	method I	1.723	1.749	1.51	0.0178	1.016	0.02	0.0250	25.00	0.0121	48.509
	method II		1.788	3.77	0.0273	1.527	0.02	0.0465	132.50	0.0230	49.45
2nd	method I	2.864	2.894	1.05	0.0190	0.065	0.02	0.0214	7.00	0.0077	36.067
	method II		2.919	1.92	0.0274	0.938	0.02	0.0197	1.50	0.0081	40.84
3rd	method I	5.625	5.684	1.05	0.0329	0.578	0.02	0.0204	2.00	0.0050	24.416
	method II		5.698	1.29	0.0417	0.731	0.02	0.0173	13.50	0.0050	29.083
4th	method I	7.375	7.375	0.00	0.0346	0.469	0.02	0.0201	0.50	0.0052	25.821
	method II		7.376	0.01	0.0434	0.588	0.02	0.0155	22.50	0.0041	26.723
5th	method I	9.491	9.411	0.84	0.0400	0.425	0.02	0.0207	3.50	0.0044	21.046
	method II		9.380	1.17	0.0507	0.541	0.02	0.0188	6.00	0.0052	27.658
6th	method I	11.06	11.162	0.92	0.0423	0.379	0.02	0.0215	7.50	0.0041	19.227
	method II		11.099	0.35	0.0590	0.531	0.02	0.0215	7.50	0.0052	24.079
7th	method I	14.099	14.167	0.48	0.0521	0.367	0.02	0.0232	16.00	0.0037	15.927
	method II		14.048	0.36	0.0806	0.573	0.02	0.0257	28.50	0.0054	20.871
8th	method I	15.411	15.515	0.67	0.0597	0.385	0.02	0.0257	28.50	0.0043	16.709
	method II		15.400	0.07	0.0980	0.636	0.02	0.0260	30.00	0.0053	20.289

3. <u>Ambient vibration simulation</u>: The white noise is applied mutually on 8 node point of the model bridge (**Fig. 1**) other than the support point, and corresponding velocity response was recorded. The acceleration of white noise was shown in **Fig. 2**. **Table-1** and **Table-2** indicates the model bridge properties and the natural frequency respectively. For dynamic analysis modal damping was assumed as 0.02.

4. <u>Discussion on simulation result</u>: 3000 ambient vibration data were used to estimate dynamic characteristics for each time and the process was repeated for 100 times. **Fig. 4** is the graphical representation of the estimated frequency which shows the frequencies up to 8$^{\text{th}}$ mode order can be estimated automatically. Estimated damping by each method has shown in **Fig. 5**. The estimation accuracy of frequency and damping were evaluated in **Table-3**. The estimation error for the frequency as well as damping for lower order vibration is a little bit higher than upper mode order. The coefficient of variation for damping within 50%, which indicates also, a good accuracy as the value of damping is too small. Result of natural vibration mode and estimated vibration mode has been shown together in **Fig. 3** and the estimated vibration mode has excellent similarity to the natural vibration mode.

5. <u>Conclusion:</u> From this study, it has been understood that each method can effectively estimate frequency, damping, and vibration mode. Reasonable estimation error for both methods was observed. So the balanced probabilistic realization theory based on the ambient vibration measurement can be used structural identification.

Reference: (1) Akaike, H.**:** Markovian representation of stochastic processes by canonical variables, SIAM J. Control, Vol. 13, No. 1, 1975. **(2)** Desai, U. B., Pal, D., and Kirkpatrick, R.D.**:** A realization approach to stochastic model reduction, Int. J. Cont., Vol.42, No. 4, pp. 821-838, 1985.