A Fast Multipole Boundary Element Method for Multi-crack Problems in 2D Orthotropic Elastic Materials

○ DeFa Wang, Student Member, Graduate School of Fukui University
Takaaki Kobayashi, Student Member, Graduate School of Fukui University
Takuo Fukui, Member, Fukui University

1 Introduction

Recently, Fukui[1] has presented a fast multipole moment boundary element method for 2D orthotropic materials. Numerical examples show the efficiency of FM-BIEM in solving large-scale problem for orthotropic materials. On the other hand, it is well known that micro-cracks in brittle materials often lead to macrocrack initiation and induce progressive damage. So, the study on the crack problems is of significant importance. For 2D isotropic elastic materials, Fukui[2] has presented a fast multipole boundary element method for the multi-crack problems. The objective of this paper is to study the applications of FM-BIEM for large-scale crack problems in orthotropic materials. This paper could be regarded as an extension of Fukui's work[2].

2 Basic Equations and Crack Problem

In orthotropic elastic materials, the strain-displacement relation, the equlibrium equation and the constitutive equation are given as [3]:

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1}$$

$$\sigma_{ij,j} + X_i = 0 \tag{2}$$

$$\sigma_{ij} = c_{ij}^{kl}\epsilon_{kl} \quad \text{or} \quad \epsilon_{ij} = s_{ij}^{kl}\sigma_{kl} \tag{3}$$

where ϵ_{ij}, u_i, σ_{ij} and X_i are strain, displacement, stress and body force, respectively. c_{ij}^{kl} and s_{ij}^{kl} are the elastic and compliance tensors, respectively.

Based on the complex potential functions ϕ and χ [3], the displacement and the stress fields of orthotropic materials can be expressed as

$$\begin{aligned} D = u_1 + iu_2 &= \delta_1\phi'(z_1) + \rho_1\overline{\phi}'(\overline{z}_1) \\ &+\delta_2\chi'(z_2) + \rho_2\overline{\chi}'(\overline{z}_2) \end{aligned} \tag{4}$$

$$\begin{aligned} \Phi = \sigma_{11} - \sigma_{22} + 2i\sigma_{12} &= -4\gamma_1^2\phi''(z_1) \\ &-4\overline{\phi}''(\overline{z}_1) - 4\gamma_2^2\chi''(z_2) - 4\overline{\chi}''(\overline{z}_2) \end{aligned} \tag{5}$$

$$\begin{aligned} \Theta = \sigma_{11} + \sigma_{22} &= 4\gamma_1\phi''(z_1) + 4\overline{\gamma}_1\overline{\phi}''(\overline{z}_1) \\ &+4\gamma_2\chi''(z_2) + 4\overline{\gamma}_2\overline{\chi}''(\overline{z}_2) \end{aligned} \tag{6}$$

where $z_\alpha = z + \gamma_\alpha\overline{z}$ $(\alpha = 1, 2)$. γ_α is the characteristic root of the characteristic equation. δ_α、ρ_α are parameters associated with the material constants[1].

As an initial attempt, only the infinite, multi-crack problems are considered in this paper, which can be described as

$$\begin{aligned} &c_{ij}^{kl}u_{k,lj} + X_i = 0 \qquad &&\text{(in domain } B\text{)} \\ &s_i = \sigma_{ij}n_j = 0 \qquad &&\text{(on } S_1, S_2, ..., S_M\text{)} \end{aligned} \tag{7}$$

where s_i is the traction vector on the boundaries. $S_1, S_2, ..., S_M$ are the crack face boundaries.

3 Boundary Element Method

3.1 Boundary Integral Equations

Assume that the body force components are ignored and the boundaries under consideration are smooth. Based on Somigliana formula, the traction solutions to the crack problem (7) can be expressed as

$$0 = n_j\sigma_{ji}^0 - \sum_{K=1}^{M} \mathrm{Pf}\int_{S_K} U_{ij}(\boldsymbol{x}, \boldsymbol{y})[u_j](\boldsymbol{y})\, dS_y \tag{8}$$

where σ_{ji}^0 is the initial stress components. n_j is the unit normal outward vector. $[u_i] = u_i^+ - u_i^-$ is the crack opening displacement, and

$$U_{ij}(\boldsymbol{x}, \boldsymbol{y}) = T_{ik}^x S_{kj}(\boldsymbol{x}, \boldsymbol{y}) \tag{9}$$

where S_{ij} is the associated fundamental solution, which can be expressed in terms of complex functions [1].

3.2 Discretization of the BIEs

Based on the collocation method, Eq.(8) can be discretized into the following system:

$$0 = s_i^{0I} - \sum_{K=1}^{M}\sum_{I}^{N_K} D_{ij}^I(\boldsymbol{x})[u_j]^I \tag{10}$$

where $s_i^{0I} = n_j\sigma_{ji}^0(\boldsymbol{x})$, and

$$D_{ij}^I(\boldsymbol{x}) = \int_{E_I} U_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_I(\boldsymbol{y})\, ds_y \tag{11}$$

where $f_I(\boldsymbol{y})$ is the basic function. For the crcak face boundaries, the constant density element is adopted in which $f_I(\boldsymbol{y}) = 1$. The influence functions of the constant density element have been obtained by Fukui [1]. However, as shown in Fig.1, the displacement components near the crack tip are in proportion to $\sqrt{s}$ [4]where s is the distance from the crack tip, so we define the basic function as $f_I(\boldsymbol{x}) = \sqrt{2s/a}$. Thus, we have

$$D_{ij}^I(\boldsymbol{x}) = \int_{E_I} U_{ij}(\boldsymbol{x}, \boldsymbol{y})\sqrt{2s/a}\, ds_y \tag{12}$$

The hyper-singular kernel $U_{ij}(\boldsymbol{x}, \boldsymbol{y})$ is the traction field at $\boldsymbol{x}$ due to the double layer kernel, which can be defined by

$$\begin{aligned} T(\boldsymbol{x}, \boldsymbol{y}) = T_1 + iT_2 &= U_{1j}(\boldsymbol{x}, \boldsymbol{y})U_j + iU_{2j}(\boldsymbol{x}, \boldsymbol{y})U_j \\ &= \frac{1}{2}\left(\Theta^S\nu^x + \Phi^S\overline{\nu}^x\right) \end{aligned} \tag{13}$$

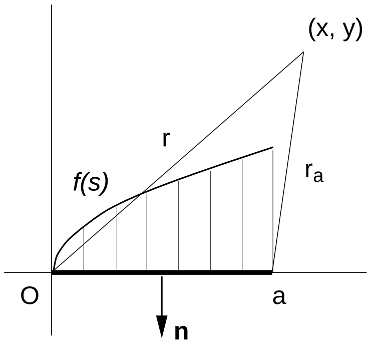

Fig.1 Crack-tip element

where ν^x is a direction vector. Then, without going into details, the influence function of $D_{ij}(\boldsymbol{x},\boldsymbol{y})$ due to a crack-tip element can be expressed as

$$\begin{aligned}&\int_E T(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y\\&=U_j\left[\int_E U_{1j}(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y+i\int_E U_{2j}(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y\right]\\&=\frac{1}{2}\left[\nu^x\int_E \Theta^S(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y+\overline{\nu}^x\int_E \Phi^S(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y\right]\\&=2[\nu_1^x\gamma_1 Vk_1(z_1)-\overline{\nu}_1^x\overline{Vk}_1(\overline{z}_1)+\nu_2^x\gamma_2 Wk_2(z_2)\\&\quad-\overline{\nu}_2^x\overline{Wk}_2(\overline{z}_2)]\end{aligned}\tag{14}$$

where $\nu_\alpha^x=\nu^x-\gamma_\alpha\overline{\nu}^x$, and $k_\alpha(z)$ is defined by

$$k_\alpha(z)=\frac{1}{2\pi}\sqrt{\frac{2}{a\tau_\alpha^3}}\left(\frac{1}{2\sqrt{z}}\log\frac{\sqrt{z}+\sqrt{a_\alpha}}{\sqrt{z}-\sqrt{a_\alpha}}-\frac{\sqrt{a_\alpha}}{z-a_\alpha}\right)\tag{15}$$

4 Fast Multipole Method

In this paper, Fast Multipole Method (FMM) is adopted to reduce the computational complexity for the multi-crack problems. To implement this fast method, the multipole expansion, the local expansion and the translation formulae are necessary, which were discussed in detail by Fukui[1] and omitted here. In this section only the multipole moment due to the crack-tip element is discussed.

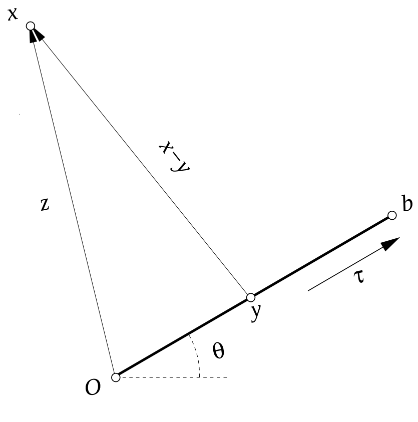

Fig.2 Crack tip element and multipole point

Consider a crack tip element $[0,b]$, as shown in Fig.2. Origin O is at the crack tip. Then we have $\sqrt{s/a}=\sqrt{\zeta_\alpha/b_\alpha}$. By integrating the multipole coefficients through the crack-tip element, i.e.,

$$\begin{aligned}\tilde{M}_n^S&=\int_E M_n^S(\zeta_1)\sqrt{2s/a}\,ds_y\\&=\frac{\sqrt{2}}{\tau_1\sqrt{b_1}}\int_0^{b_1}M_n^S(\zeta_1)\sqrt{\zeta_1}\,d\zeta_1\end{aligned}\tag{16}$$

we can get the multipole moments of crack tip element due to the double layer potential kernel as

$$\tilde{M}_{-1}^S=0,\qquad \tilde{M}_0^S=\frac{-2\sqrt{2}V}{3\tau_1}b_1$$

$$\tilde{M}_n^S=\frac{-\sqrt{2}V}{n(n+\frac{3}{2})\tau_1}b_1^{n+1}\tag{17}$$

$$\tilde{N}_{-1}^S=0,\qquad \tilde{N}_0^S=\frac{-2\sqrt{2}W}{3\tau_2}b_2$$

$$\tilde{N}_n^S=\frac{-\sqrt{2}W}{n(n+\frac{3}{2})\tau_2}b_2^{n+1}\tag{18}$$

where $\tau_\alpha=e^{i\theta}+\gamma_\alpha e^{-i\theta}$. The components V and W can be found in Fukui's work[1].

5 Conclusions

Based on the complex potential functions, a hyper-singular boundary integral equation for crack problems has been presented in which a crack tip element is specially introduced to improve the accuracy of the crack solution. FMM (fast multipole method) is adopted to reduce the computational complexity of the boundary integral equations, for which the multipole moment of the influence functions of the crack-tip element has been presented. Then, with the aid of the work by Fukui[1], the hyper-singular boundary integral equation can be solved numerically in connection with FMM, i.e., a fast multipole boundary element method for large-scale crack problems in 2D orthotropic elastostatic materials is presented.

References

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