An Approximate Representation for Statistics of Maximum Responses of One Degree-of-Freedom System

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1. Introduction

The objective of this study is to estimate the statistics of the maximum values of the structural response when the system is excited by a non-stationary process. To provide basic information for the design of structures, the response of a one degree-of-freedom (1DOF) system subjected to a non-stationary Gaussian white noise is considered. This type of problems has been discussed on the basis of the theory for stationary processes (for example, Vanmarcke[1)] and Kiureghian[2)]). However, we introduce the approximate representation for probabilistic properties of the maximum values of non-stationary and zero-mean white noise, whose standard deviation varies temporally under some limited conditions[3)]. Then, a method is proposed to estimate the stochastic properties of the maximum response of a given 1DOF system excited by a non-stationary white noise whose time-varying standard deviation are known. Furthermore, the response spectra will be estimated easily using the proposed method and the appropriateness of the obtained results is confirmed through Monte Carlo simulation (MCS).

2. Problem Setting and Outline of the Analysis

We will deal with the approximate distribution, $F_Y(y)$ for maximum value of i.n.n.i.d. Gaussian variables X_i ($i = 1, 2, \ldots$), as the simplest non-stationary process: that is,

$$X_i \equiv X(t_i) = \eta(t_i) \cdot W(t_i) \tag{1}$$

where, t_i stands for i-th discrete time, $W(t_i)$ for Gaussian white noise with zero mean and unit variance, and $\eta(t_i)$ for standard deviation which depends on time and varies smoothly with one extreme peak and $\eta(t; t \le 0) = \eta(\infty) = 0$. The variations of $\eta(t_i)$ will be set much smaller than the time increments Δt. It is noted that $\eta(t_i)$ will play the role of a kind of envelop function of $X(t_i)$.

We will discuss the stochastic characteristics of maximum response of a 1DOF system which are excited by Eq.(1). In a case where the damping factor, h is small, namely $h \ll 1$, the local maxima of the response are random variables following a Rayleigh distribution. Using this property and the results from the discussion about the statistics of extreme for i.n.n.i.d. (independent not necessarily identically distributed) variables[3)], we will derive an approximate representation for the probability density function (PDF) of maximum response of 1DOF system.

3. Stochastic Properties for Local Maxima of 1DOF-System Response

Firstly, we will derive analytically the statistics for the amplitude of the response. Let us introduce $X_R(t)$ which is the random response of 1DOF system excited by non-stationary Gaussian white noise:

$$X_R(t_i) = \zeta(t_i) * X(t_i) \qquad (i = 1, 2, \ldots, N), \tag{2}$$

where $*$ stands for the convolution operation, N for the number of discrete time, and $\zeta(t_i)$ for the impulse response function of 1DOF system whose natural period and damping factor are T_0 and h, respectively.

Cartwright and Longuet-Higgins[4)] have shown that the probability distribution of the local maxima can be described by Rayleigh distribution in a case where the process is narrow band. Although our problem satisfies the properties of the narrow band process because of the small damping factor of 1DOF system, $X_R(t_i)$ is not stationary. This means that we cannot apply directly the descriptions by Cartwright and Longuet-Higgins.

Thus, we will deal with the stochastic properties for the local maxima at each time t_i in the meaning of ensembles instead of the statistics over the time. A time, which $X_R(t_i)$ takes the local maximum, can vary for every sample, though the time intervals of the local maxima must be nearly equal to the natural period T_0 of the system. Therefore, we introduce the local maxima process defined as $\hat{X}_R(s_\ell) = X_R(\hat{t}_i) \quad (s_\ell < \hat{t}_i \le s_\ell + T_0;\ \ell = 1, 2, \ldots, N_{T_0})$, where $s_{\ell+1} - s_\ell = T_0$, $N_{T_0} = N\Delta t / T_0$, and

$$\dot{X}_R(t_i) = \left.\frac{dX_R(t)}{dt}\right|_{t=t_i} \begin{cases} > \ 0 & (s_\ell \le t_i < \hat{t}_i) \\ = \ 0 & (t_i = \hat{t}_i) \\ < \ 0 & (\hat{t}_i < t_i \le s_\ell). \end{cases} \tag{3}$$

In a case where $\eta(t_i)$ of Eq.(1) is constant over time, the probability distribution for $\hat{X}_R(s_\ell)$ coincides with Rayleigh distribution and its parameter is determined by

Key Words: *Extreme values, Gaussian white noise, non-stationary process, Gumbel's distribution, one degree-of-freedom system*
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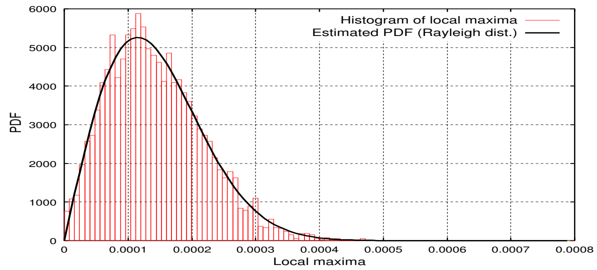

Fig. 1 Comparison of probability distributions for local maxima of response of 1DOF system at a specific time $s_\ell = 300$.

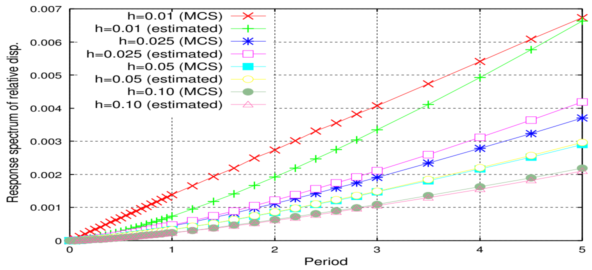

Fig. 2 Means of response spectra with various damping factor h. The estimated response spectra are compared with the results of MCS.

the root mean squares (rms), namely $\eta(t_i)$ [4]. From this, we can expect that $\hat{X}_R(s_\ell)$ follows the Rayleigh distribution at every time s_ℓ in the meaning of ensembles.

Actually, $\hat{X}_R(s_\ell)$ follows the Rayleigh distribution whose standard deviation $\sigma_{\hat{X}_R}(s_\ell)$ depends on time. Then, $\sigma_{\hat{X}_R}(s_\ell)$ can be written as

$$\sigma_{\hat{X}_R}(s_\ell) = \sqrt{\frac{G_0 T_0^3}{32\pi^2 h}} \cdot \eta(s_\ell), \tag{4}$$

where the parameter G_0 is a constant value and should be determined analytically. However, we determine empirically this value as $G_0 = \sqrt{2}\Delta t/10$ on the basis of many numerical calculations, because to derive analytically the value for G_0 is difficult by now. We confirmed the stability of this value within the scope of our calculations. This value for G_0 is used in the following numerical examples.

$\hat{X}_R(s_\ell)$ is simulated 5000 times by MCS and the histogram of $\hat{X}_R(s_\ell)$ at $s_\ell = 300$ is obtained as shown in Figure 1. The smooth line is the shape of Rayleigh distribution with the parameter determined by $\sigma_{\hat{X}_R}(300)$ of Eq.(4).

From this, we can describe analytically the stochastic properties for the local maxima of response. However, in a case where the band width of $X_R(t_i)$ is not enough narrow, namely h is not so small, there may be the negative local maxima.

4. Approximate Distribution for the Maximum Values of 1DOF-System Response

(1) Analytical derivation

This section is devoted to derive the approximate distribution for the maximum response of 1DOF system excited by non-stationary Gaussian white noise. For this, we will assume that the local maxima are mutually independent. Our problem can be rewritten as follows: we will approximate the PDF for maximum values of Rayleigh distributed variables whose standard deviation varies slowly over time.

This problem can be solved using the same way as the procedure to find the approximate distribution for the maximum value of i.n.n.i.d. Gaussian variable[3]. As a result, the cumulative distribution function of the extremes Y can be represented approximately by Gumbel's distribution[5] such as $F_Y(y) \approx \exp[-\exp[-\alpha(y-u)]]$, where $\alpha \approx \frac{\sqrt{(4-\pi)\ln n}}{\sigma}$, $u \approx \frac{2\sigma\sqrt{\ln n}}{4-\pi}$, in which n and σ can be obtained by the procedure in the reference by Morikawa[3].

(2) Estimation of response spectra

Using the above result, we can estimate the response spectra stochastically. The response spectrum is defined as a diagram of the maximum response of 1DOF system versus its natural period T_0. Therefore, after estimating the maximum response for some T_0 and h, we can obtain the response spectra easily.

Under the same conditions as the previous examples, we calculate the response spectra and the results are shown in Figure 2. In this figure, we compare the results from the proposed method with the results from 5000-time MCS. The estimation for the case $h = 0.01$ is not good, but other cases may be acceptable.

5. Conclusions

We have shown the probability distribution for the local maxima of response of 1DOF system excited by non-stationary Gaussian white noise can be described by Rayleigh distribution whose standard deviation varies with similar shape as the standard deviation of Gaussian white noise as the input motion. Using this result, furthermore, we have proposed a method to estimate the approximate distribution of maximum response and the response spectrum.

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