Approximate representation for extreme values of non-stationary Gaussian white noise

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1. Introduction

The statistics for extreme values of non-stationary processes are critical to designing structures in some engineering fields, such as earthquake engineering, coastal engineering, and so on. Thus, many researchers have proposed various methods for this purpose. For example, Vanmarcke[1)] has developed a method to estimate the extreme values of a given system's response to random excitation using the spectral moments in a frequency domain. Furthermore, this method was extended by Kiureghian[2)].

While we may deal with this problem in time domain, most research on this type of problem has been limited to stationary processes. Especially, in a case where a time series is stationary Gaussian white noise with zero mean, we can directly apply the asymptotic representation for the extreme values of i.i.d. (independent identically distributed) Gaussian variables. However, it is not easy to estimate the extreme values of non-stationary processes whose stochastic properties depend on time, because we have to deal with the i.n.n.i.d. (independent not necessarily identically distributed) random variables. Although general representations for this problem can be obtained[5),6)], it is difficult to derive the closed form or asymptotic solutions for any specific distributions such as Gaussian distribution, etc.

Therefore, we will discuss a probabilistic distribution for the extreme values of non-stationary Gaussian white noise as the simplest and most primary problem.

2. Problem setting

We will treat the probabilistic distribution, $F_Y(y)$ for maximum value of i.n.n.i.d. Gaussian variables $X_i = X(t_i)$ $(i = 1, 2, \ldots)$, as the simplest non-stationary process: that is,

$$X_i = X(t_i) = \eta(t_i)W(t_i), \tag{1}$$

where, t_i stands for i-th discrete time, $W(t_i)$ for Gaussian white noise with zero mean and unit variance, and $\eta(t_i)$ for standard deviation which depends on time and varies smoothly with one extreme peak. $\eta(t_i)$ will play the role of a kind of envelop function of $X(t_i)$.

Generally speaking, the order statistics of i.n.n.i.d. random variables can be represented by using that of i.i.d. random variables[5)]. However, it is difficult to obtain the closed form solutions for any specific distributions of X_i. Thus, firstly, the closed form solutions are derived for the extreme values of i.n.n.i.d. Gaussian variables with two distributions and the qualitative properties are determined. Next, we will propose an approximate representation of the distribution for the extreme values of $X(t_i)$ using these obtained properties, and confirm the appropriateness of the result through Monte Carlo simulations.

3. Distribution for extreme values of Gaussian variables with two different properties

Let us consider X_i $(i = 1, 2, \ldots, n_1 + n_2)$ which consists of n_j independent Gaussian variables, X_{jk} with zero mean and variance σ_j^2 $(j = 1, 2,\ k = 1, 2, \ldots, n_j)$. The distribution function for extreme values of X_{jk} for each $j = 1, 2$ can be written as following[6)]:

$$F_{Y_j}(y) = P(X_{jk} < y) = \exp\left[-\exp[-\alpha_j(y - u_j)]\right], \tag{2}$$

where $P(A)$ denotes the probability of A, $\alpha_j = \sqrt{2\ln n_j}/\sigma_j$, and $u_j = \left\{\sqrt{2\ln n_j} - \frac{\ln(\ln n_j)+\ln(4\pi)}{2\sqrt{2\ln n_j}}\right\}\sigma_j$. Thus, the distribution function for extreme values of X_i becomes

$$F_Y(y) = P(X_i < y) = \prod_{j=1}^{2} P(X_{jk} < y) = \prod_{j=1}^{2} F_{Y_j}(y) \tag{3}$$

Our objective is to derive an approximate representation for Eq.(3) as the form of $\exp[-\exp[-\alpha(y-u)]]$, where α and u should be determined from the parameters n_j and σ_j $(j = 1, 2)$.

As a result, some numerical calculations of Eq.(3) suggest that we can use $\alpha = \sqrt{2\ln n}/\sigma$ and $u = \left\{\sqrt{2\ln n} - \frac{\ln(\ln n)+\ln(4\pi)}{2\sqrt{2\ln n}}\right\}\sigma$, where $n = \sum_{j=1}^{2} n_j$, and $\sigma = \frac{\sum_{j=1}^{2} n_j\sigma_j}{n}$, if $\sigma_1 \approx \sigma_2$. In other cases such as $\sigma_i \ll \sigma_j$ $(i, j = 1, 2,\ i \neq j)$, $\alpha \approx \alpha_j$ and $u \approx u_j$ can be applied.

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This means that the distribution function, $F_Y(y)$ for the maximum values of i.n.n.i.d. Gaussian values, X_i is approximately rewritten by the distribution for the maximum values of i.i.d. Gaussian values with n and σ for $\sigma_1 \approx \sigma_2$. This properties can be supported by a theorem by Guilbaud in the general meaning[5)]. We should point out that the area of $\sum_{j=1}^{2}(n_j\sigma_j)$ can be replaced by a rectangular area of $\sigma(\sum_{j=1}^{2} n_j)$ using an equivalent standard deviation σ.

Figs. 1 and 2 show examples of the approximation of $F_Y(y)$ for $\sigma_1 \approx \sigma_2$ and $\sigma_1 \ll \sigma_2$, respectively. It is observed that the approximate distribution agrees with exact one in a case of $\sigma_1 \approx \sigma_2$, though the exact distribution coincides with one for σ_2 in a case of $\sigma_1 \ll \sigma_2$.

4. Approximate distribution for extreme values of non-stationary Gaussian white noise

We will apply the properties obtained in the previous section to determine the probabilistic properties for extreme values of Eq.(1). Since we assume the standard deviation, $\eta(t)$ varies smoothly, we can use the approximation for the case of $\sigma_i \approx \sigma_j$: that is, to obtain an appropriate i.i.d. random variables instead of i.n.n.i.d. random variable, X_i of Eq.(1), σ and n for the approximation should satisfy a relation:

$$\int_a^b \eta(t)dt = \sigma n, \tag{4}$$

where $n = (b - a)/\Delta t$, Δt stands for an equi-increment of discrete time t_i, the interval $[a, b]$ includes the peak value of $\eta(t)$, and $\eta(a) = \eta(b)$.

Although the approximate distribution for the extreme value is independent of the interval $[a, b]$, a too small value of r may give less approximation for $\eta(c) = r \cdot \eta(a)$ $(0 < r < 1)$, where $\eta(t)$ takes the peak value at c $(a < c < b)$.

Fig. 3 shows a comparison between the approximate distribution for extreme values of Eq.(1) and the histogram obtained by the Monte Carlo simulation. We can observe that the approximate distribution fairly good agreement with the numerical simulation.

5. Conclusions

We have derived the exact representation for extreme values of Gaussian variable with two different properties and found qualitative properties. Using these properties, an approximate representation were proposed for extreme values of no-stationary Gaussian white noise and the appropriateness was also confirmed through the numerical simulation.

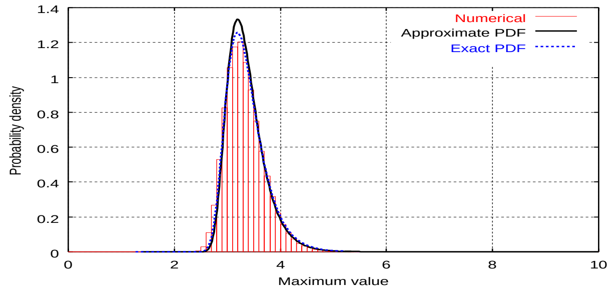

Fig. 1 An example of distribution for extreme values of Gaussian white noise ($n_1 = n_2 = 500$, $\sigma_1 = 1.0$, $\sigma_2 = 1.05$).

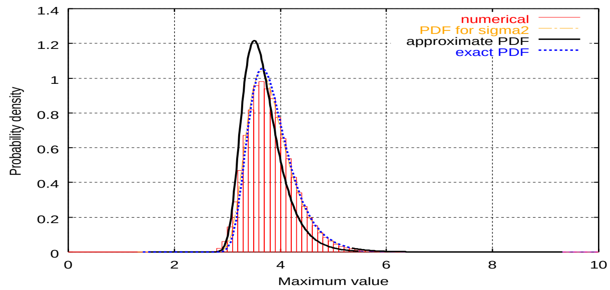

Fig. 2 An example of distribution for extreme values of Gaussian white noise ($n_1 = n_2 = 500$, $\sigma_1 = 1.0$, $\sigma_2 = 1.25$).

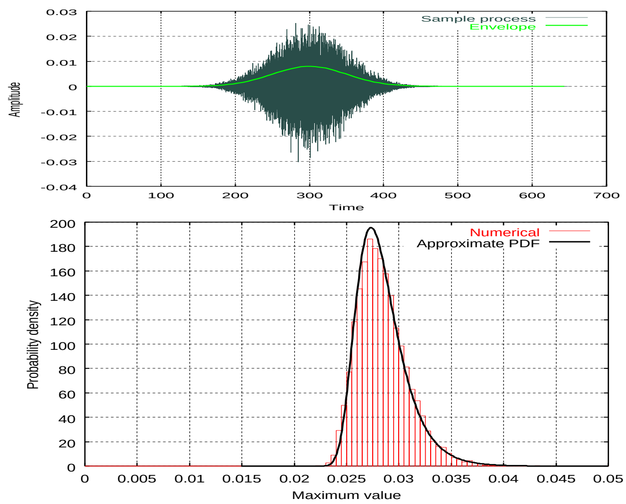

Fig. 3 An example of the approximate distribution for extreme values of non-stationary Gaussian white noise. The upper panel shows a sample process and $\eta(t)$, and the approximate distribution is compared with the result of numerical simulation in the lower panel.

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