Guided waves in a monoclinic layered half-space and a monoclinic layer between monoclinic solids

○Yosuke YOSHIZAWA　Student Member　Tokyo Institute of Technology

Anil C. Wijeyewickrema　Member　Tokyo Institute of Technology

1. Introduction

The propagation of elastic waves along the interfaces in a layered half-space and a layer between solids is considered. These media consists of monoclinic material with the symmetry plane at $x_3 = 0$ and $x_2 = 0$, all interfaces are rigidly bonded. The dispersion equation relating the wave speed to the wave number is obtained. Numerical examples are presented to illustrate the dispersion curves, and displacement and stress curves.

2. Waves in the monoclinic materials with the symmetry plane at $x_3 = 0$

When a monoclinic material with the symmetry plane at $x_3 = 0$ is subjected to a two-dimensional deformation, the in-plane displacements and the anti-plane displacement are uncoupled. So, for deformations where $u_i = u_i(x_1, x_2, t)$, $i = 1, 2, 3$, the equations of motion are

$$\begin{cases} \rho\ddot{u}_1 = C_{11}u_{1,11} + C_{66}u_{1,22} + 2C_{16}u_{1,12} + C_{16}u_{2,11} \\ \qquad + C_{26}u_{2,22} + (C_{12} + C_{66})u_{2,12}, & (2.1) \\ \rho\ddot{u}_2 = C_{16}u_{1,11} + C_{26}u_{1,22} + (C_{12} + C_{66})u_{1,12} \\ \qquad + C_{66}u_{2,11} + C_{22}u_{2,22} + 2C_{26}u_{2,12}, & (2.2) \\ \rho\ddot{u}_3 = C_{55}u_{3,11} + C_{44}u_{3,22} + 2C_{45}u_{3,12}. & (2.3) \end{cases}$$

A plane wave propagating in the x_1 direction is given by

$$\begin{aligned} u_1(x_1, x_1) &= A_1 e^{-bkx_2} e^{ik(x_1 - ct)}, \\ u_2(x_1, x_2) &= A_2 e^{-bkx_2} e^{ik(x_1 - ct)}, \\ u_3(x_1, x_2) &= 0. \end{aligned} \qquad (2.4)$$

for in-plane problem, and

$$\begin{aligned} u_1(x_1, x_1) &= 0, \\ u_2(x_1, x_2) &= 0, \\ u_3(x_1, x_2) &= A_3 e^{-bkx_2} e^{ik(x_1 - ct)}. \end{aligned} \qquad (2.5)$$

for anti-plane problem, where $i = \sqrt{-1}$, $c = \omega / k$ is the phase velocity, ω is the circular frequency, k is the wave number A_i, is the displacement amplitudes, and b is the decay factor that is obtained from the equations of motion.

3. Layered half-space problem

Consider a monoclinic layer of thickness h perfectly bonded to a different monoclinic half-space (Fig. 1).

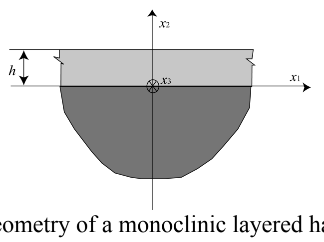

Fig. 1. Geometry of a monoclinic layered half-space.

Displacements and stresses in a layer are obtained from eqn (2.4) and Hooke's law, and are given as

$$\begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ V_1 & V_2 & V_3 & V_4 \\ D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \end{bmatrix} \begin{bmatrix} E_1(x_2)A_{11} \\ E_2(x_2)A_{12} \\ E_3(x_2)A_{13} \\ E_4(x_2)A_{14} \end{bmatrix}, \qquad (3.1)$$

for in-plane problem, and

$$\begin{bmatrix} u_3 \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ D_{31} & D_{32} \end{bmatrix} \begin{bmatrix} A_{31}E_1(x_2) \\ A_{32}E_2(x_2) \end{bmatrix}, \qquad (3.2)$$

for anti-plane problem, where

$$\begin{aligned} D_{1q} &= k\left\{(iC_{12} - b_q C_{26}) + (iC_{26} - b_q C_{22})V_q\right\} \\ D_{2q} &= k\left\{(iC_{16} - b_q C_{66}) + (iC_{66} - b_q C_{26})V_q\right\}, \\ D_{3q} &= k\left\{iC_{45} - b_q C_{44}\right\} \end{aligned} \qquad (3.3)$$

$$V_q = \frac{A_{2q}}{A_{1q}} = -\frac{C_{11} + 2b_q iC_{16} - b_q^2 C_{66} - \rho c^2}{C_{16} + b_q i(C_{12} + C_{66}) + b_q^2 C_{26}}. \qquad (3.4)$$

For decaying waves in the half-space Re(b)<0. Then displacements and stresses in half-space are

$$\begin{bmatrix} u_1^* \\ u_2^* \\ \sigma_{22}^* \\ \sigma_{21}^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ V_1^* & V_2^* & V_3^* & V_4^* \\ D_{11}^* & D_{12}^* & D_{13}^* & D_{14}^* \\ D_{21}^* & D_{22}^* & D_{23}^* & D_{24}^* \end{bmatrix} \begin{bmatrix} E_1^*(x_2)A_{11}^* \\ 0 \\ E_3^*(x_2)A_{13}^* \\ 0 \end{bmatrix}, \qquad (3.5)$$

for in-plane problem, and

$$\begin{bmatrix} u_3^* \\ \sigma_{23}^* \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ D_{31}^* & D_{32}^* \end{bmatrix} \begin{bmatrix} A_{31}^* E_1^*(x_2) \\ 0 \end{bmatrix}, \qquad (3.6)$$

for anti-plane problem.

At the interface, each material is rigidly bonded, and the displacements and stresses are continuous. While the surface $x_2 = h$ is traction-free. From these boundary conditions, the dispersion relations are given as

$$\begin{vmatrix} D_{11}E_1(h) & D_{12}E_2(h) & D_{13}E_3(h) & D_{14}E_4(h) & 0 & 0 \\ D_{21}E_1(h) & D_{22}E_2(h) & D_{23}E_3(h) & D_{24}E_4(h) & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ V_1 & V_2 & V_3 & V_4 & -V_1^* & -V_3^* \\ D_{11} & D_{12} & D_{13} & D_{14} & -D_{11}^* & -D_{13}^* \\ D_{21} & D_{22} & D_{23} & D_{24} & -D_{21}^* & -D_{23}^* \end{vmatrix} = 0, \qquad (3.7)$$

for in-plane problem, and

Keywords: wave propagation, monoclinic materials, layered half-space, layer between solids, dispersion relation, dispersion curve
2-12-1, O-okayama, Meguro, Tokyo 152-8552. Tel.: 03-5734-2595, Fax: 03-5734-3578, e-mail y-yosuke@cv.titech.ac.jp

$$\left(-1+e^{2kh\sqrt{-\left(\frac{C_{45}}{C_{44}}\right)^2+\frac{C_{55}}{C_{44}}-\xi}}\right)C_{44}^2\left(-\left(\frac{C_{45}}{C_{44}}\right)^2+\frac{C_{55}}{C_{44}}-\xi\right)$$
$$+\left(1+e^{2kh\sqrt{-\left(\frac{C_{45}}{C_{44}}\right)^2+\frac{C_{55}}{C_{44}}-\xi}}\right)C_{44}\sqrt{-\left(\frac{C_{45}}{C_{44}}\right)^2+\frac{C_{55}}{C_{44}}-\xi}\sqrt{-\left(\frac{C_{45}^*}{C_{44}}\right)^2+\frac{C_{44}^*C_{55}}{C_{44}^2}-\frac{\xi C_{44}^*\rho^*}{C_{44}\rho}}. \tag{3.8}$$

for anti-plane problem, where $\xi=\rho c^2/C_{44}$ is the non-dimensional phase speed

4. Layer between solids problem

Consider a monoclinic layer of thickness $2h$ sandwiched between two different types of monoclinic solids. Both interfaces are rigidly bonded (Fig.2).

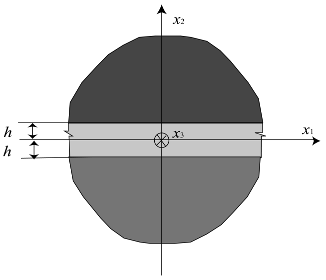

Fig. 2. Geometry of a monoclinic layer between monoclinic solids.

For the in-plane problem, displacements and stresses in the layer are same as for the layered half-space problem. Now, for convenience rewrite the formal solution (3.1) in the compact form as

$$\boldsymbol{Y}_{(2)}(x_2)=\boldsymbol{X}_{(2)}\boldsymbol{E}_{(2)}(x_2)\boldsymbol{A}_{(2)}\ . \tag{4.1}$$

In this problem, the propagator matrix approach is used. This matrix relates the upper face of layer to the lower face of layer. Specializing eqn (4.1) to the upper and lower face of layer, leads respectively to

$$\boldsymbol{Y}_{(2)}(h)=\boldsymbol{X}_{(2)}\boldsymbol{E}_{(2)}(h)\boldsymbol{A}_{(2)},$$
$$\boldsymbol{Y}_{(2)}(-h)=\boldsymbol{X}_{(2)}\boldsymbol{E}_{(2)}(-h)\boldsymbol{A}_{(2)}. \tag{4.2}$$

From eqn (4.2), the relationship

$$\boldsymbol{Y}_{(2)}(-h)=\boldsymbol{P}_{(2)}\boldsymbol{Y}_{(2)}(-h), \tag{4.3}$$

is obtained where

$$\boldsymbol{P}_{(2)}=\boldsymbol{X}_{(2)}\boldsymbol{E}_{(2)}(2h)\boldsymbol{X}_{(2)}^{-1}. \tag{4.4}$$

From the boundary conditions at the interface

$$\boldsymbol{Y}_{(1)}(h)=\boldsymbol{Y}_{(2)}(h),$$
$$\boldsymbol{Y}_{(2)}(-h)=\boldsymbol{Y}_{(3)}(-h). \tag{4.5}$$

From eqns (4.3) and (4.5) the dispersion relation for in-plane problem is given as

$$\begin{vmatrix} Z_{11} & Z_{13} \\ Z_{31} & Z_{33} \end{vmatrix}=0, \tag{4.6}$$

where

$$\boldsymbol{Z}=\boldsymbol{X}_{(1)}^{-1}\boldsymbol{P}_{(2)}\boldsymbol{X}_{(3)}\ . \tag{4.7}$$

The anti-plane problem is solved using the same method used for the layered half-space. The dispersion relation is obtained as

$$\begin{vmatrix} E_1^{(2)}(h) & E_2^{(2)}(h) & -E_1^{(1)}(h) & 0 \\ D_{31}^{(2)}E_1^{(2)}(h) & D_{32}^{(2)}E_2^{(2)}(h) & -D_{31}^{(1)}E_1^{(1)}(h) & 0 \\ E_1^{(2)}(-h) & E_2^{(2)}(-h) & 0 & -E_2^{(3)}(-h) \\ D_{31}^{(2)}E_1^{(2)}(-h) & D_{32}^{(2)}E_2^{(2)}(-h) & 0 & -D_{32}^{(3)}E_2^{(3)}(-h) \end{vmatrix}=0. \tag{4.8}$$

5. Numerical results

Material parameter given in Table 1 are used to illustrate the dispersion curves, and displacement and stress curves.

Table. 1. Values of the elastic constants and density.

Material	C_{11}	C_{22}	C_{12}	C_{16}	C_{26}	C_{66}	C_{44}	C_{55}	C_{45}	ρ
aegirite-augite	216	156	66	19	25	46.5	49.2	40.0	4	3420
augite	218	182	72	25	20	51.1	55.8	69.7	4	3320
diallage	211	154	37	12	15	62.2	52.3	63.9	-9	3300

C_{ij} (10^3Mpa), ρ (kg m^{-3})

Dispersion curves are illustrated from the eqns (3.8) and (4.8) as

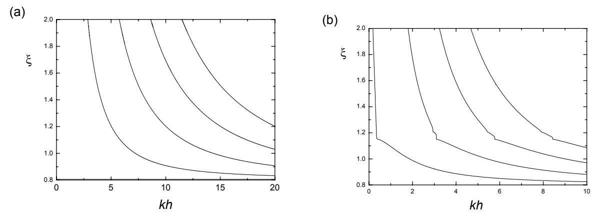

Fig. 3. Dispersion curves for (a) layered-half space and (b) layer between solids.

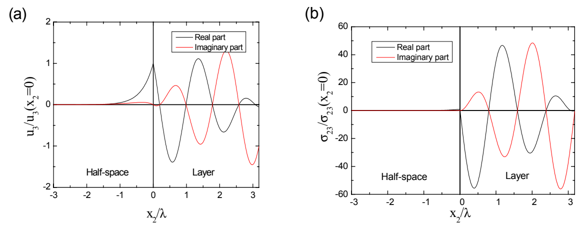

Fig. 4. (a) Displacement and (b) stress amplitudes for layered half-space. (kh=20)

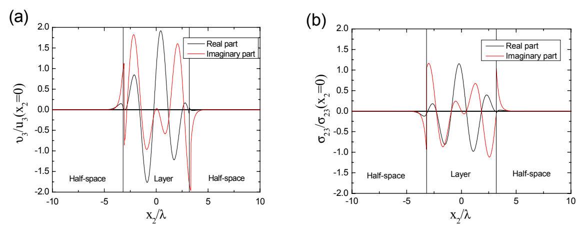

Fig. 5. (a) Displacement and (b) stress amplitudes for layer between solids. (kh=20)

6. References

Qu, J., 1994, "Guided waves in an anisotropic layer sandwiched between anisotropic media", *Review of Progress in Quantitative Nondestructive Evaluation,* Vol. 13, pp. 1391-1398.

Sotiropoulos, D. A., 1999, "The effect of anisotropy on guided elastic waves in a layered half-space", *Mechanics of Materials,* Vol. 31, pp. 215-223.

Yoshizawa Yosuke, 2003, "Guided waves in a monoclinic layered half-space and a monoclinic layer between monoclinic solids", *Bachelor Thesis, Department of Civil Engineering, Tokyo Institute of Technology, Japan.*