WAVE PROPAGATION IN AN ANISOTROPIC ELASTIC LAYERED HALF-SPACE

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1. Introduction

The geometric model dealt with in this research is a transversely isotropic layer bonded to an orthotropic half-space. Elastic wave propagation in anisotropic layered media is useful for earthquake engineering and non-destructive evaluation studies. Recently, Sotiropoulos (1999) obtained the dispersion equation for an orthotropic layer on an orthotropic half-space.

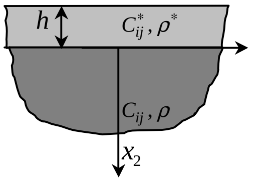

Fig. 1. The anisotropic elastic layered half-space.

2. Basic equations

The equations of motion in an orthotropic half-space $(x_2 \geq 0)$ and a transversely isotropic layer $(-h \leq x_2 \leq 0)$ that has x_2-axis as an axis of symmetry are written as

$$x_2 \geq 0: \quad \begin{aligned} &C_{11}u_{1,11} + C_{12}u_{2,21} + C_{66}\left(u_{1,22} + u_{2,12}\right) = \rho \ddot{u}_1, \\ &C_{66}\left(u_{1,21} + u_{2,11}\right) + C_{12}u_{1,12} + C_{22}u_{2,22} = \rho \ddot{u}_2, \\ &C_{55}u_{3,11} + C_{44}u_{3,22} = \rho \ddot{u}_3, \end{aligned} \qquad (2.1)$$

$$-h \leq x_2 \leq 0: \quad \begin{aligned} &C^*_{11}u^*_{1,11} + C^*_{12}u^*_{2,21} + C^*_{44}\left(u^*_{1,22} + u^*_{2,12}\right) = \rho^* \ddot{u}^*_1, \\ &C^*_{44}\left(u^*_{1,21} + u^*_{2,11}\right) + C^*_{12}u^*_{1,12} + C^*_{22}u^*_{2,22} = \rho^* \ddot{u}^*_2, \\ &C^*_{55}u^*_{3,11} + C^*_{44}u^*_{3,22} = \rho^* \ddot{u}^*_3, \end{aligned} \qquad (2.2)$$

where C_{ij} and C^*_{ij} are the elastic constants of orthotropic media and transversely isotropic media respectively.

When $u_i = u_i\left(x_1, x_2, t\right), (i = 1, 2, 3)$, only six and five independent elastic constants are required for orthotropic and transversely isotropic media respectively.

From eqns (2.1) and (2.2), it can be seen that the layered half-space problem can be decoupled into the in-plane problem ($u_\alpha \neq 0, \alpha = 1, 2, u_3 = 0$), and the anti-plane problem ($u_3 \neq 0, u_\alpha = 0, \alpha = 1, 2$).

3. In-plane problem of a transversely isotropic layer on an orthotropic half-space

3.1. Displacements

The displacements in the orthotropic half-space and in the transversely isotropic layer are respectively,

$$\begin{aligned} x_2 \geq 0: &\quad u_\alpha = U_\alpha \exp\left[-qkx_2 + ik\left(x_1 - ct\right)\right], \\ -h \leq x_2 \leq 0: &\quad u^*_\alpha = U^*_\alpha \exp\left[-q^*kx_2 + ik\left(x_1 - ct\right)\right], \end{aligned} \qquad (3.1.1)$$

where U_α, U^*_α are arbitrary constants and q, q^* are the decay factors and k, c are the wave number and the phase velocity. Substituting eqn (3.1.1) into the equations of motion, the equation of q in each material is obtained. The equation for the orthotropic half-space is

$$F_1 q^4 + F_2 q^2 + F_3 = 0, \qquad (3.1.2)$$

where

$$\begin{aligned} &F_1 = C_{22}C_{66}, \\ &F_2 = -C_{11}C_{22} - {C_{66}}^2 + \rho c^2\left(C_{22} + C_{66}\right) + \left(C_{12} + C_{66}\right)^2, \\ &F_3 = \left(\rho c^2 - C_{11}\right)\left(\rho c^2 - C_{66}\right), \end{aligned}$$

and for the transversely isotropic layer is

$$F^*_1 {q^*}^4 + F^*_2 {q^*}^2 + F^*_3 = 0, \qquad (3.1.3)$$

where

$$\begin{aligned} &F^*_1 = C^*_{22}C^*_{44}, \\ &F^*_2 = -C^*_{11}C^*_{22} - {C^*_{44}}^2 + \rho^* c^2\left(C^*_{22} + C^*_{44}\right) + \left(C^*_{12} + C^*_{44}\right)^2, \\ &F^*_3 = \left(\rho^* c^2 - C^*_{11}\right)\left(\rho^* c^2 - C^*_{44}\right). \end{aligned}$$

From eqns (3.1.2) and (3.1.3), each equation has four solutions, however, it is necessary that the real part of the decay factor of the half-space is positive. Therefore, assuming that

$$\begin{aligned} &q_1 = -q_3,\ q_2 = -q_4,\ \mathrm{Re}(q_1),\ \mathrm{Re}(q_2) > 0, \\ &q^*_1 = -q^*_3,\ q^*_2 = -q^*_4,\ \mathrm{Re}(q^*_1),\ \mathrm{Re}(q^*_2) > 0, \end{aligned}$$

the displacements in each material are written as

$$\begin{aligned} x_2 \geq 0: &\quad u_\alpha = \left\{\sum_{\beta=1}^{2} U^{(\beta)}_\alpha e^{-q_\beta k x_2}\right\} \exp\left[ik\left(x_1 - ct\right)\right], \\ -h \leq x_2 \leq 0: &\quad u^*_\alpha = \left\{\sum_{\gamma=1}^{4} U^{(\gamma)*}_\alpha e^{-q^*_\gamma k x_2}\right\} \exp\left[ik\left(x_1 - ct\right)\right], \end{aligned} \qquad (3.1.4)$$

where

$$\begin{aligned} U^{(\beta)}_2 / U^{(\beta)}_1 &= \frac{-C_{11} + {q_\beta}^2 C_{66} + \rho c^2}{iq_\beta\left(C_{12} + C_{66}\right)}, \\ U^{(\gamma)*}_2 / U^{(\gamma)*}_1 &= \frac{-C^*_{11} + {q^*_\gamma}^2 C^*_{44} + \rho^* c^2}{iq^*_\gamma\left(C^*_{12} + C^*_{44}\right)}. \end{aligned} \qquad (3.1.5)$$

3.2. Dispersion equation

Since the transversely isotropic layer and the orthotropic half-space are rigidly bonded, the displacements and the stresses of both materials at the interface $(x_2 = 0)$ should be continuous, therefore, the boundary conditions at the interface are

$$u_1 = u^*_1,\ u_2 = u^*_2,\ \sigma_{22} = \sigma^*_{22},\ \sigma_{12} = \sigma^*_{12}, \quad (x_2 = 0). \qquad (3.2.1)$$

The boundary conditions at the free surface are

$$\sigma^*_{22} = \sigma^*_{12} = 0, \qquad (x_2 = -h). \qquad (3.2.2)$$

The dispersion equation is obtained from the determinant of the 6×6 matrix representing the six boundary conditions. Using Laplace expansion, the dispersion equation is simplified as

$$\begin{aligned} &A\frac{\sinh\left[kh\left(q^*_1 + q^*_2\right)\right]}{\left(q^*_1 + q^*_2\right)} - B\frac{\sinh\left[kh\left(q^*_1 - q^*_2\right)\right]}{\left(q^*_1 - q^*_2\right)} \\ &+ C\frac{\sinh^2\left[\frac{1}{2}kh\left(q^*_1 + q^*_2\right)\right]}{\left(q^*_1 + q^*_2\right)^2} - D\frac{\sinh^2\left[\frac{1}{2}kh\left(q^*_1 - q^*_2\right)\right]}{\left(q^*_1 - q^*_2\right)^2} + E = 0. \end{aligned} \qquad (3.2.3)$$

4. Anti-plane problem of a transversely isotropic layer on an orthotropic half-space

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4.1. Displacements

Adopting the same procedure used for the in-plane problem, from eqns (2.1) and (2.2), the displacements are

$$x_2 \ge 0: \quad u_3 = U_3 e^{-qkx_2} \exp\left[ik\left(x_1 - ct\right)\right], \qquad (a)$$

$$-h \le x_2 \le 0: \quad u_3^* = \left\{\sum_{j=1}^{2} U_3^{(j)*} e^{-q_j^* k x_2}\right\} \exp\left[ik\left(x_1 - ct\right)\right], \qquad (b) \qquad (4.1.1)$$

where

$$q = \left(\frac{C_{55} - \rho c^2}{C_{44}}\right)^{1/2}, \quad q_1^* = -q_2^* = \left(\frac{C_{55}^* - \rho^* c^2}{C_{44}^*}\right)^{1/2}.$$

Equation (4.1.1a) is also written as

$$u_3 = U_3 e^{-\mathrm{Re}(q)kx_2} e^{ik\{-\mathrm{Im}(q)x_2 + x_1 - ct\}}.$$

$\mathrm{Re}(q)$ and $\mathrm{Im}(q)$ are the real part and the imaginary part of q respectively. The displacement decays with the distance from the interface x_2, therefore, q must have the real part, i.e.,

$$C_{55} > \rho c^2. \qquad (4.1.2)$$

4.2. Dispersion equation

The boundary conditions are

$$u_3 = u_3^*, \quad \sigma_{23} = \sigma_{23}^*, \qquad (x_2 = 0),$$
$$\sigma_{23}^* = 0, \qquad (x_2 = -h). \qquad (4.2.1)$$

For plotting the dispersion curves, it is necessary to normalize the phase velocity. The dispersion equation is written as

$$\tan\left[\left\{\left(\frac{\rho^* C_{55}}{\rho C_{55}^*}\zeta - 1\right)\frac{C_{55}^*}{C_{44}^*}\right\}^{1/2} kh\right] - \frac{C_{44}\left\{(1-\zeta)\frac{C_{55}}{C_{44}}\right\}^{1/2}}{C_{44}^*\left\{\left(\frac{\rho^* C_{55}}{\rho C_{55}^*}\zeta - 1\right)\frac{C_{55}^*}{C_{44}^*}\right\}^{1/2}} = 0, \qquad (4.2.2)$$

where $\zeta = \rho c^2 / C_{55}$, $(\zeta < 1)$.

5. Numerical results

Table 1. Material properties.

	Material	$\rho\left(g/cm^3\right)$	C_{44}	C_{55}
Layer	Graphite-epoxy	1.7	7.07	3.5
	Beta-quartz	2.65	36.1	49.95
	Carbon-epoxy	1.58	6.2	3.6
	Austenite	8.1	128.4	82.4
Half-space	Graphite-epoxy	1.6	3.52	12.08
	Composition $(Mg_{91.7}Fe_{8.3})O.SiO_2$	3.324	667	810

C_{ij}:elastic constants $\left(\times 10^3 MPa\right)$

Table 2. Combinations of materials.

	Layer	Half-space
Case 1	Graphite-epoxy	Graphite-epoxy
Case 2	Carbon-epoxy	Graphite-epoxy
Case 3	Beta-quartz	Composition
Case 4	Austenite	Composition

5.1. Dispersion curves

For the anti-plane problem, the dispersion curves of the first 5 modes are shown in Figs. 2-3.

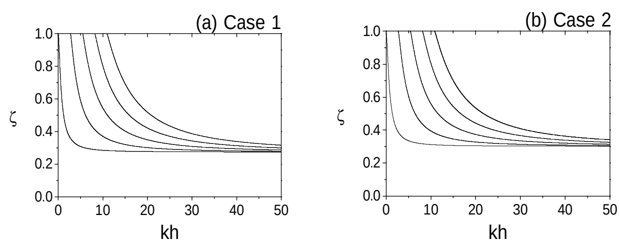

Fig. 2. Dispersion curves for (a) Case 1 and (b) Case 2.

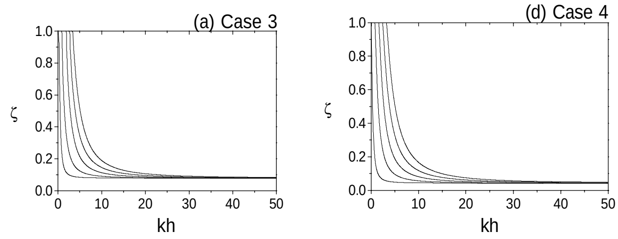

Fig. 3. Dispersion curves for (a) Case 3 and (b) Case 4.

5.2. Displacement and stress distribution

For the anti-plane problem, the figures of the displacements and the stresses in terms of the distance from the interface x_2 are shown in Fig. 4. The following figures are for three modes of Case 1 (Table 1) at $kh = 20$.

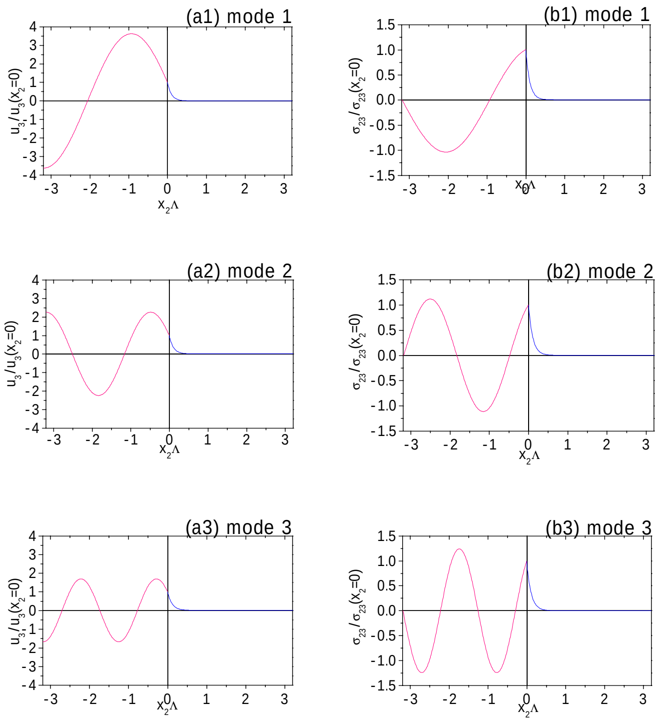

Fig. 4. (a) Displacements and (b) stresses for mode 1 to 3.

6. Summary and Conclusions

For the in-plane problem, the simplification of the dispersion equation is not enough because each term of the explicit equation is still very large. For the anti-plane problem, the dispersion equation is obtained explicitly for drawing the dispersion curve. In addition, the figures of the displacement and the stress are shown.

7. References

Sotiropoulos, D. A., **1999**, "The effect of anisotropy on guided elastic waves in a layered half-space", *Mechanics of Materials*, Vol. 31, 215-223.

Usui Kanako, **2003**, "Wave propagation in an anisotropic elastic layered half-space", *Bachelor Thesis, Department of Civil Engineering, Tokyo Institute of Technology, Japan.*