# Large deformation theory by the logarithmic strain tensor and the subloading surface model with tangential plasticity

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**Keywords:** Corotational rate, Elastoplasticity, Logarithmic strain, Large deformation, Subloading surface model

## INTRODUCTION

Constitutive equation for the large elastoplastic deformation is formulated in this article by refining the large deformation theory of Naghdabadi and Saidi (2002) adopting the *corotational logarithmic (Hencky) strain rate tensor* and incorporating it into the *subloading surface model* of Hashiguchi (1980) falling within the framework of the unconventional plasticity.

## CONSTITUTIVE EQUATIONS

The deformation gradient $\mathbf{F}$ can be led to the polar decomposition:

$$\mathbf{F} = \mathbf{V}\mathbf{R}\,, \tag{1}$$

where

$$\mathbf{V} = (\mathbf{F}\mathbf{F}^{\mathrm{T}})^{1/2},\quad \mathbf{R} = \{(\mathbf{F}\mathbf{F}^{\mathrm{T}})^{1/2}\}^{-1}\mathbf{F}\,, \tag{2}$$

whilst $\mathbf{V}$ can be written in the principal directions as

$$\mathbf{V} = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha, \tag{3}$$

denoting the principal values and directions as $\lambda_\alpha$ and $\mathbf{n}_\alpha$, respectively.

Throughout this paper the *corotational rate* $\overset{\circ}{\mathbf{T}}$ with objectivity for an arbitrary second-order tensor $\mathbf{T}$ is given as

$$\overset{\circ}{\mathbf{T}} \equiv \overset{\bullet}{\mathbf{T}} - \mathbf{\Lambda}\mathbf{T} + \mathbf{T}\mathbf{\Lambda}\,, \tag{4}$$

where $(^\bullet)$ stands for the material-time derivative and $\mathbf{\Lambda}$ is the proper *spin tensor of material-substructure*.

Let the logarithmic (Hencky) strain rate $(\ln\mathbf{V})^\circ$ be additively decomposed into the elastic strain rate $((\ln\mathbf{V})^\circ)^e$ and the inelastic strain rate $((\ln\mathbf{V})^\circ)^i$, i.e.

$$(\ln\mathbf{V})^\circ = ((\ln\mathbf{V})^\circ)^e + ((\ln\mathbf{V})^\circ)^i\,. \tag{5}$$

Further, let the inelastic strain rate $((\ln\mathbf{V})^\circ)^i$ be additively decomposed into the plastic strain rate $((\ln\mathbf{V})^\circ)^p$ and the tangential strain rate $((\ln\mathbf{V})^\circ)^t$, i.e.

$$((\ln\mathbf{V})^\circ)^i = ((\ln\mathbf{V})^\circ)^p + ((\ln\mathbf{V})^\circ)^t\,, \tag{6}$$

provided that $((\ln\mathbf{V})^\circ)^p$ and $((\ln\mathbf{V})^\circ)^t$ are induced by the stress rate component normal and tangential, respectively, to the yield and/or loading surface.

Assume that the elastic logarithmic (Hencky) strain tensor $(\ln\mathbf{V})^e$ is derived from the following *complementary energy* (or *Gibbs function*) $w^e$, i.e.

$$(\ln\mathbf{V})^e = \frac{\partial W^e(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}, \tag{7}$$

where $\boldsymbol{\sigma}$ is Cauchy stress. It results from Eq. (7) that

$$((\ln\mathbf{V})^e)^\circ = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}, \tag{8}$$

putting the elastic modulus $\mathbf{E}$ as

$$\mathbf{E} \equiv \left(\frac{\partial^2 W^e}{\partial\boldsymbol{\sigma}^2}\right)^{-1}, \tag{9}$$

where $(\ )^{-1}$ stands for the inverse tensor.

The complementary energy function $W^e$ may be given by

$$W^e = \frac{1+\nu}{2E}\mathrm{tr}\boldsymbol{\sigma}^2 - \frac{\nu}{2E}\mathrm{tr}^2\boldsymbol{\sigma} \tag{10}$$

for metals and

$$W^e = \gamma p\left\{\ln\frac{p}{p_0} - 1\right\} + \frac{1}{4G(p)}\mathrm{tr}\boldsymbol{\sigma}^{*2},\quad G(p) = c\left(\frac{p}{p_0}\right)^n \tag{11}$$

for soils, where $E$, $\nu$, $\gamma$, $c$ and $n$ are material constants, and $p$ and $p_0$ are the pressure and its initial value, respectively.

The following tensors may be substituted for the spin tensor $\mathbf{\Lambda}$ of material-substructure.

$$\mathbf{\Lambda} = \begin{cases} \mathbf{W} = (\mathbf{L} - \mathbf{L}^{\mathrm{T}})/2: \text{ continuum sipn for Jaumann rate} \\ \mathbf{\Omega} = \overset{\bullet}{\mathbf{R}}\mathbf{R}^{\mathrm{T}}: \text{ polar spin for Green - Naghdi rate} \\ \mathbf{\Omega}^E = \overset{\bullet}{\mathbf{n}}_\alpha \otimes \mathbf{n}_\alpha: \text{ Eulerian spin for Eulerian rate} \\ \mathbf{\Omega}^p = \mathbf{W} - \mathbf{W}^p: \text{ continuum-plastic sipn for Dafalias rate} \end{cases} \tag{12}$$

$$\mathbf{W}^p \equiv \xi\{\boldsymbol{\sigma}((\ln\mathbf{V})^\circ)^p - ((\ln\mathbf{V})^\circ)^p\boldsymbol{\sigma}\}, \tag{13}$$

where $\xi$ is the material parameter.

Assume that the elastic rotation is far smaller than the plastic one and thus the rotation is induced plastically, i.e.

$$\left.\begin{aligned} (\ln\mathbf{V})^\circ &= ((\ln\mathbf{V})^\circ)^p \\ &= ((\ln\mathbf{V})^\bullet)^p - \mathbf{\Lambda}(\ln\mathbf{V})^p + (\ln\mathbf{V})^p\mathbf{\Lambda}, \\ ((\ln\mathbf{V})^\circ)^e &= ((\ln\mathbf{V})^\bullet)^e \end{aligned}\right\} \tag{14}$$

Let the following yield condition be assumed.

$$f(\boldsymbol{\sigma}, \mathbf{H}) = F(H), \tag{15}$$

where the scalar $H$ and the second-order tensor $\mathbf{H}$ are the isotropic and the anisotropic hardening variables, respectively. The function $f$ is assumed to be homogeneous of degree one in the stress $\boldsymbol{\sigma}$.

In the subloading surface model the conventional yield surface is renamed as the *normal-yield surface*. Then, the following *subloading surface* is introduced, which always passes through the current stress point and also keeps a shape similar to the normal-yield surface and the orientation of similarity to the normal-yield surface with respect to the origin of stress space, i.e. $\boldsymbol{\sigma} = \mathbf{0}$.

$$f(\boldsymbol{\sigma}, \mathbf{H}) = RF(H), \tag{16}$$

where $R$ is the *normal-yield ratio* denoting the ratio of the size of subloading surface to that of normal-yield surface. Eq. (16) leads to

$$\mathrm{tr}\left\{\frac{\partial f(\boldsymbol{\sigma}, \mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\right\} + \mathrm{tr}\left\{\frac{\partial f(\boldsymbol{\sigma}, \mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\right\} = \overset{\bullet}{R}F + R\overset{\bullet}{F}. \tag{17}$$

Let the following evolution equation of the normal-yield ratio $R$ be assumed.

$$\dot{R} = U(R)\left\|((\ln \mathbf{V})^{\circ})^{p}\right\| \quad \text{for } ((\ln \mathbf{V})^{\circ})^{p} \neq \mathbf{0}\,, \tag{18}$$

where $U$ is a monotonically decreasing function of the normal-yield ratio $R$, fulfilling

$$U = \begin{cases} \infty & \text{for } R = 0, \\ 0 & \text{for } R = 1, \end{cases} \tag{19}$$
$$(U < 0 \quad \text{for } R > 1).$$

Let the function $U$ satisfying Eq. (19) be simply given by

$$U(R) = -u \ln R\,, \tag{20}$$

where $u$ is the material constant.

The substitution of Eq. (18) into Eq. (17) leads to the *consistency condition* extended to the subloading surface:

$$\mathrm{tr}\Big\{\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\Big\} + \mathrm{tr}\Big\{\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \mathbf{H}}\overset{\circ}{\mathbf{H}}\Big\} = U\left\|((\ln \mathbf{V})^{\circ})^{p}\right\|F + R\dot{F}\,. \tag{21}$$

Assume the plastic flow rule

$$((\ln\mathbf{V})^{\circ})^{p} = \lambda \mathbf{N}\,, \tag{22}$$

where $\lambda$ is the positive proportionality factor and

$$\mathbf{N} \equiv \frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \boldsymbol{\sigma}} \bigg/ \left\|\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \boldsymbol{\sigma}}\right\| \quad (\|\mathbf{N}\| = 1)\,. \tag{23}$$

The substitution of Eq. (22) into Eq. (21) leads to

$$\mathrm{tr}\Big\{\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\Big\} + \mathrm{tr}\Big\{\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \mathbf{H}}\overset{\circ}{\mathbf{H}}\Big\} = U\lambda F + RF'\lambda h\,, \tag{24}$$

where

$$F' \equiv dF/dH\,. \tag{25}$$
$$h \equiv \dot{H}/\lambda, \quad \mathbf{h} \equiv \overset{\circ}{\mathbf{H}}/\lambda\,, \tag{26}$$

from which one has

$$\lambda \equiv \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^{p}}\,, \tag{27}$$

where

$$M^{p} \equiv \Big\{\frac{F'}{F}h - \frac{1}{RF}\mathrm{tr}\Big(\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \mathbf{H}}\mathbf{h}\Big) + \frac{U}{R}\Big\}\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma}) \tag{28}$$

by use of the following relation based on Euler's theorem for a homogeneous function.

$$\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \boldsymbol{\sigma}} = \frac{\mathrm{tr}\Big(\frac{\partial f(\boldsymbol{\sigma},\mathbf{H})}{\partial \boldsymbol{\sigma}}\boldsymbol{\sigma}\Big)}{\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma})}\mathbf{N} = \frac{f(\boldsymbol{\sigma},\mathbf{H})}{\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma})}\mathbf{N} = \frac{RF}{\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma})}\mathbf{N}\,. \tag{29}$$

Adopt the plastic flow rule

$$((\ln \mathbf{V})^{\circ})^{p} = \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^{p}}\mathbf{N}\,. \tag{30}$$

Further, modifying the tangential strain rate of Hashiguchi (1998) or Hashiguchi and Tsutsumi (2001), the tangential plastic strain rate was given by Hashiguchi (2003) as

$$((\ln\mathbf{V})^{\circ})^{t} = \frac{1}{T}\mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}\,, \tag{31}$$

where

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t} = \overset{\circ}{\boldsymbol{\sigma}}{}^{*} - \overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{n}\,, \quad \overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{n} \equiv \mathrm{tr}(\mathbf{n}^{*}\overset{\circ}{\boldsymbol{\sigma}})\mathbf{n}^{*}\,, \tag{32}$$

$$\mathbf{n}^{*} \equiv \Big(\frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\Big)^{*} \bigg/ \left\|\Big(\frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\Big)^{*}\right\| = \frac{\mathbf{N}^{*}}{\|\mathbf{N}^{*}\|} \quad (\|\mathbf{n}^{*}\| = 1)\,, \tag{33}$$

$$T = \xi / R^{b}\,. \tag{34}$$

$b$ is a material constant and $\xi$ is a material function of the stress and the plastic internal variables in general. $(\ )^{*}$ designates a deviatoric component.

The logarithmic strain rate is given from Eqs. (5), (6), (8), (30) and (31) as follows:

$$(\ln \mathbf{V})^{\circ} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^{p}}\mathbf{N} + \frac{1}{T}\mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}\,. \tag{35}$$

From Eq. (35) one has

$$\mathrm{tr}\{\mathbf{NE}\ln(\mathbf{V})^{\circ}\} = \mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}}) + \mathrm{tr}(\mathbf{NEN})\frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^{p}} = \{M^{p} + \mathrm{tr}(\mathbf{NEN})\}\frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^{p}}\,, \tag{36}$$

The positive proportionality factor $\lambda$ in the flow rule (22) is expressed in terms of strain rate, rewriting $\lambda$ by $\Lambda$, from Eq. (36) as follows:

$$\Lambda = \frac{\mathrm{tr}\{\mathbf{NE}(\ln \mathbf{V})^{\circ}\}}{M^{p} + \mathrm{tr}(\mathbf{NEN})}\,. \tag{37}$$

The loading criterion is given as follows (Hashiguchi, 2000):

$$\left.\begin{aligned} ((\ln \mathbf{V})^{\circ})^{p} \neq \mathbf{0}: \ \Lambda > 0, \\ ((\ln \mathbf{V})^{\circ})^{p} = \mathbf{0}: \ \Lambda \leq 0 \end{aligned}\right\} \tag{38}$$

It holds from Eqs. (35) and (37) that

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}(\ln \mathbf{V})^{\circ} - \mathbf{E}\frac{\mathrm{tr}\{\mathbf{NE}(\ln \mathbf{V})^{\circ}\}}{M^{p} + \mathrm{tr}(\mathbf{NEN})}\mathbf{N} - \frac{1}{T}\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}\,. \tag{39}$$

Eqs. (32) and (39) leads to

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t} = \overset{\circ}{\boldsymbol{\sigma}} - \mathrm{tr}\Big[\mathbf{n}^{*}\Big\{\mathbf{E}(\ln \mathbf{V})^{\circ} - \mathbf{E}\Big(\frac{\mathrm{tr}\{\mathbf{NE}(\ln \mathbf{V})^{\circ}\}}{M^{p} + \mathrm{tr}(\mathbf{NEN})}\Big)\mathbf{N}\Big\}\Big]\mathbf{n}^{*}. \tag{40}$$

Substituting Eq. (40) into Eq. (39), one has

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}(\ln \mathbf{V})^{\circ} - \mathbf{E}\Big[\frac{\mathrm{tr}\{\mathbf{NE}(\ln \mathbf{V})^{\circ}\}}{M^{p} + \mathrm{tr}(\mathbf{NEN})}\Big]\mathbf{N} - \frac{1}{T}\overset{\circ}{\boldsymbol{\sigma}} + \frac{1}{T}\mathrm{tr}\Big[\mathbf{n}^{*}\Big\{\mathbf{E}(\ln \mathbf{V})^{\circ} - \mathbf{E}\Big(\frac{\mathrm{tr}\{\mathbf{NE}(\ln \mathbf{V})^{\circ}\}}{M^{p} + \mathrm{tr}(\mathbf{NEN})}\Big)\mathbf{N}\Big\}\Big]\mathbf{n}^{*} \tag{41}$$

which results in the expression of stress rate in terms of strain rate.

$$\overset{\circ}{\boldsymbol{\sigma}} = \frac{1}{1+T}\Big[T\mathbf{E}(\ln \mathbf{V})^{\circ} + \mathrm{tr}\{\mathbf{n}^{*}\mathbf{E}(\ln \mathbf{V})^{\circ}\}\mathbf{n}^{*} - \frac{\mathrm{tr}\{\mathbf{NE}(\ln \mathbf{V})^{\circ}\}}{M^{p} + \mathrm{tr}(\mathbf{NEN})}\{T\mathbf{EN} + \mathrm{tr}(\mathbf{n}^{*}\mathbf{EN})\mathbf{n}^{*}\}\Big]. \tag{42}$$

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