INTERFACIAL WAVES IN IMPERFECTLY BONDED INCOMPRESSIBLE ELASTIC HALF-PLANES WITH FINITE PRIMARY DEFORMATIONS

○Yamaguchi Keiichi Student Member Tokyo Institute of Technology

Anil C. Wijeyewickrema Member Tokyo Institute of Technology

1. Introduction

In the present analysis, interfacial waves along an imperfectly bonded interface between two pre-stressed incompressible elastic half-planes are examined (Fig.1). The pre-strained material parameters of the half-planes are in general different from each other. The underlying finite strain in the half-planes is homogeneous with common principal axes, one axis being normal to the planar interface and another along the direction of propagation. A shear spring type resistance model that can accommodate the extreme cases of perfect bonding or a sliding interface simulates the imperfectly bonded interface.

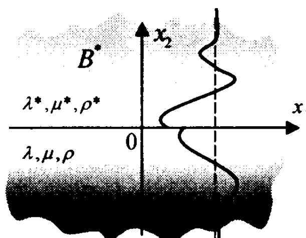

Fig. 1. Interfacial waves.

2. The governing equations

The equation of motion is expressed as

$$\alpha \psi_{,1111} + 2\beta \psi_{,1122} + \gamma \psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}), \quad (2.1)$$

where ρ is the mass density of the material in the region B and a dot denotes differentiation with respect to t. The coefficients α, β and γ on the left-hand side of eqn (2.1) are material parameters defined in terms of the strain-energy function W and the principal stretches λ_1, λ_2 by

$$\alpha \lambda_2^2 = \gamma \lambda_1^2 = (\lambda_1 W_1 - \lambda_2 W_2)\lambda_1^2 \lambda_2^2 / (\lambda_1^2 - \lambda_2^2), \quad (2.2)$$

$$2\beta + 2\gamma = \lambda_1^2 W_{11} + \lambda_2^2 W_{22} - 2\lambda_1 \lambda_2 W_{12} + 2\lambda_2 W_2, \quad (2.3)$$

where $W_i = \partial W / \partial \lambda_i, W_{ij} = \partial W / \partial \lambda_i \partial \lambda_j$.

The scalar function ψ is the velocity potential such that

$$v_1 = \psi_{,2}, \quad v_2 = -\psi_{,1}. \quad (2.4)$$

For the region B^*, material parameters α^*, β^* and γ^* are defined analogous to eqns (2.2) and (2.3), and the velocity components v_1^* and v_2^* are given by

$$v_1^* = \psi^*_{,2}, \quad v_2^* = -\psi^*_{,1}, \quad (2.5)$$

and ψ^* satisfies the equation

$$\alpha^* \psi^*_{,1111} + 2\beta^* \psi^*_{,1122} + \gamma^* \psi^*_{,2222} = \rho^*(\ddot{\psi}^*_{,11} + \ddot{\psi}^*_{,22}), \quad (2.6)$$

where ρ^* is the mass density in B^*.

3. Boundary conditions

The shear spring model is used for the imperfectly bonded interface. In this model, the velocities of upper and lower half-planes in the x_1-direction are assumed to be discontinuous but the velocities in the x_2-direction are assumed to be continuous at $x_2 = 0$. The nominal stress rate $\dot{S}_{21}$, which is assumed to be proportional to the velocity jump in the x_1-direction, is given by

$$\dot{S}_{021} = \mu k_x (v_1 - v_1^*), \quad (3.1)$$

where the spring constant k_x has the dimension $[1/L]$, and μ is the Lamé parameter.

Other boundary conditions assumed at $x_2 = 0$ are that the velocity in x_2-direction and traction rates are continuous, i.e.,

$$v_2 = v_2^*, \quad \dot{S}_{021} = \dot{S}_{021}^*, \quad \dot{S}_{022} = \dot{S}_{022}^*. \quad (3.2)$$

4. Derivation of the dispersion equation

An interfacial wave is characterized by the fact that its amplitude decays rapidly away from the interface. Therefore solutions are sought for which $\psi \to 0$ as $x_2 \to -\infty$ and $\psi^* \to 0$ as $x_2 \to \infty$. For simplicity, waves are considered time-harmonic propagating in the x_1-direction and in B the velocity potential ψ is taken as

$$\psi = e^{skx_2} e^{-ik(x_1 - ct)}. \quad (4.1)$$

Substitution of eqn (4.1) into eqn (2.1) yields

$$\gamma s^4 - (2\beta - \rho c^2) s^2 + (\alpha - \rho c^2) = 0, \quad (4.2)$$

and therefore the solution for ψ in B may be written as

$$\psi = \left(A_1 e^{s_1 k x_2} + A_2 e^{s_2 k x_2}\right) e^{-ik(x_1 - ct)}, \quad (4.3)$$

where A_1 and A_2 are constants, and s_1 and s_2 are the solutions of eqn (4.2) with positive real parts.

Similarly in B^*

$$\psi^* = \left(A_1^* e^{-s_1^* k x_2} + A_2^* e^{-s_2^* k x_2}\right) e^{-ik(x_1 - ct)}, \quad (4.4)$$

where A_1^* and A_2^* are constants and s_1^* and s_2^* are the solutions with positive real part of analogue of eqn (4.2) for B^*.

In terms of boundary conditions eqns (3.1) and (3.2) and general solutions eqns (4.3) and (4.4), the dispersion equation can be obtained as

$$F(c,k,k_x) \equiv \mu k_x \left\{ \begin{array}{l} \gamma^2 f(\eta) + \gamma^{*2} f^*(\eta^*) + 2\gamma\gamma^*(1-\eta)(1-\eta^*) \\ + \gamma\gamma^*(\eta + \eta^*)(s_1 + s_1)(s_1^* + s_2^*) \end{array} \right\} - k\left\{ \gamma^2 \gamma^* f(\eta)(s_1^* + s_2^*) + \gamma\gamma^{*2} f^*(\eta^*)(s_1 + s_1) \right\} = 0, \quad (4.5)$$

where η and η^* are non-dimensional parameters, which give the phase speed,

$$\eta = \left(\frac{\alpha - \rho c^2}{\gamma}\right)^{1/2}, \quad \eta^* = \left(\frac{\alpha^* - \rho^* c^2}{\gamma^*}\right)^{1/2}. \quad (4.6)$$

When $k_x \to \infty$, the solution for the perfectly bonded interface is obtained and agrees with Dowaikh and Ogden (1990). When $k_x = 0$, the bonding disappears and eqn (4.5) corresponds to the slipping interface.

5. Numerical results

The existence of interfacial waves is analyzed for neo-Hookean materials. The strain-energy function and relationships between material parameters and pre-stressed condition for neo-Hookean material are given by these equations,

Keywords: interfacial wave propagation, two-half-planes, imperfectly bonded interface, dispersion

2-12-1, O-okayama, Meguro, Tokyo 152-8552. Tel: 03-5734-2595. Fax: 03-5734-3578

$$W=\frac{1}{2}\mu\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right), \tag{5.1}$$

$$\alpha=\mu\lambda_1^2,\quad \gamma=\mu\lambda_2^2,\quad 2\beta=\alpha+\gamma, \tag{5.2}$$

and similarly for $\alpha^*,\beta^*,\gamma^*$.

For a fixed value of the shear modulus ratio μ/μ^*, the neutral surface N, given by eqn (4.5) when the interfacial wave speed $c=0$, is a surface in $(\lambda_1,\lambda_2,\lambda_1^*,\lambda_2^*)$ -space.

For fixed values of the shear modulus ratio μ/μ^* and the density ratio ρ/ρ^* the limiting surface, corresponding to $c_L \le c_L^*$, can also regarded as a surface in $(\lambda_1,\lambda_2,\lambda_1^*,\lambda_2^*)$ -space, and the surface L is obtained by setting $\eta=0$ and $\eta^*=\eta_L^*$; and if $c_L^* \le c_L$, the surface L^* is obtained when $\eta^*=0$ and $\eta=\eta_L$, where c_L and c_L^* are the maximum phase speeds defined by eqn (4.2).

The cases where both half-planes are under plane strain $\lambda_3=\lambda_3^*=1$ or equibiaxial strain $\lambda_1=\lambda_3,\lambda_1^*=\lambda_3^*$ are considered. Therefore in order to plot the neutral and limiting curves in (λ,λ^*) -plane, λ_1,λ_2 and λ_1^*,λ_2^* are given by the incompressibility condition that

$$\lambda_1=\lambda,\ \lambda_2=\lambda^{-1},\ \lambda_1^*=\lambda^*,\ \lambda_2^*=\lambda^{*-1}\quad \text{(plane strain)}, \tag{5.3}$$

$$\lambda_1=\lambda,\ \lambda_2=\lambda^{-2},\ \lambda_1^*=\lambda^*,\ \lambda_2^*=\lambda^{*-2}\ \text{(equibiaxial strain)}. \tag{5.4}$$

The neutral and limiting curves are illustrated as the dependence on the interface and the wave number conditions represented by the non-dimensional parameter k/k_x, where $k/k_x=0$ corresponds the perfect bond interface case and $k/k_x\to\infty$ corresponds the high frequency limit case or the sliding interface case. In Fig. 2-5 Interfacial wave motions exist in shaded regions.

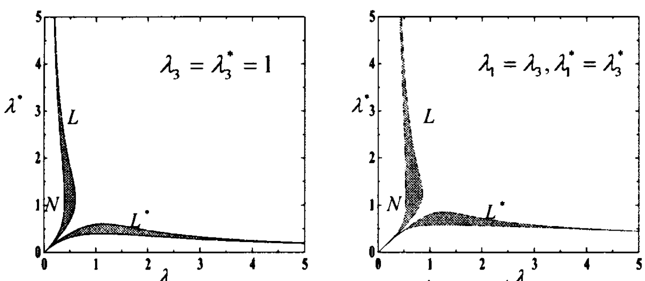

Fig. 2. Perfect bond $k_x\to\infty$, $\mu/\mu^*=\rho/\rho^*=1$.

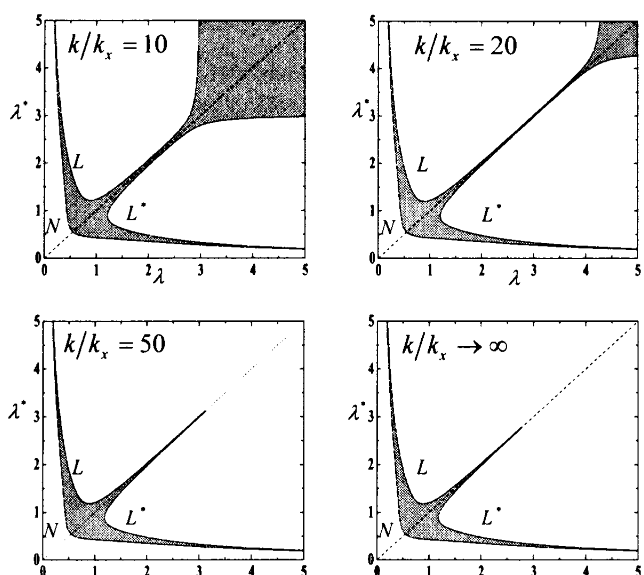

Fig. 3. Imperfect bond $\mu/\mu^*=\rho/\rho^*=1$, $\lambda_3=\lambda_3^*=1$.

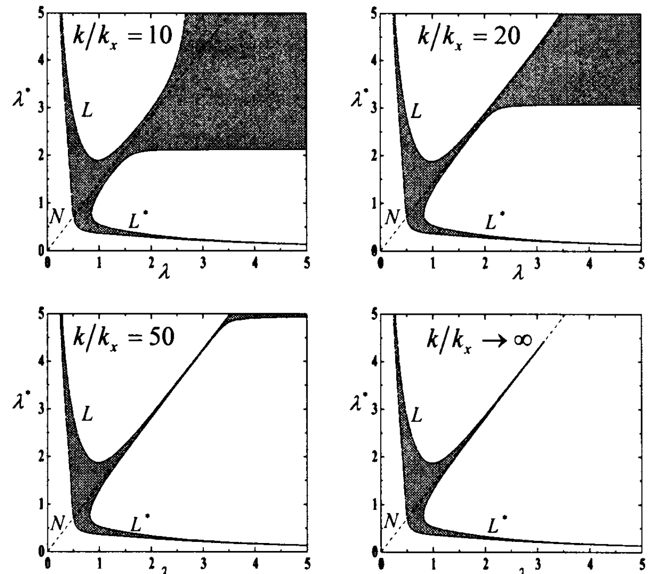

Fig. 4. Imperfect bond $\mu/\mu^*=2$, $\rho/\rho^*=1$, $\lambda_3=\lambda_3^*=1$.

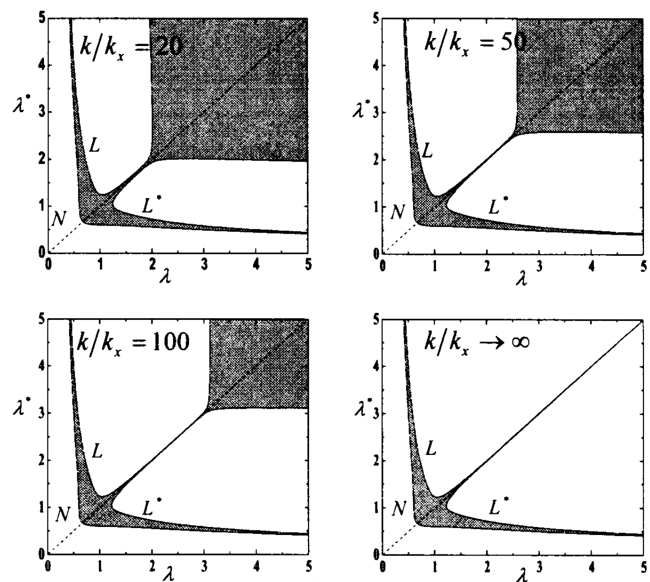

Fig. 5. Imperfect bond $\mu/\mu^*=\rho/\rho^*=1$, $\lambda_1=\lambda_3,\lambda_1^*=\lambda_3^*$.

6. Conclusion

The dispersion equation governing the phase speed of interfacial waves is obtained in explicit form. Effects of the imperfectly bonded interface can be shown using the neutral and limiting curves. In the imperfectly bonded interface case, the existence of interfacial wave motions has a large dependence on the interface and the wave number conditions and interfacial wave motions exist even for the sliding interface case.

7. Reference

Dowaikh, M. A. and Ogden, R. W., 1991, "Interfacial waves and deformations in pre-stressed elastic media", *Proceedings of the Royal Society*, Series A, Vol. 433, 313-328.