Analysis of compression of a finite composite elastic cylinder

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1. Introduction

The problem of a linear elastic finite composite cylinder with prescribed axisymmetric displacements at the plane ends are considered. Since the composite cylinder can be considered as the combination of a solid cylinder and a hollow cylinder, problems of solid and hollow cylinders are also considered. The inner material is perfectly bonded with the outer material. The cylinder considered has stress-free curved surfaces and the applied stress is axisymmetrical (Fig. 1).

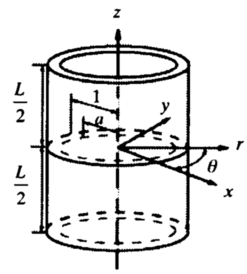

Fig. 1 Geometry of a finite composite circular cylinder.

2. The governing equations

The equilibrium equations expressed in terms of radial and axial displacements for axisymmetric deformation are

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{1-2\nu}{2(1-\nu)}\frac{\partial^2}{\partial z^2}\right]u+\left[\frac{1}{2(1-\nu)}\frac{\partial^2}{\partial r\partial z}\right]w=0,$$
$$\left[\frac{1}{1-2\nu}\left(\frac{1}{r}\frac{\partial}{\partial z}+\frac{\partial^2}{\partial r\partial z}\right)\right]u+\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2}{\partial z^2}\right]w=0, \tag{2.1}$$

where $u=u(r,z)$ and $w=w(r,z)$ are radial and axial displacements and ν is Poisson's ratio.

3. Displacements and stresses

The Poisson's ratios of the solid cylinder and hollow cylinder are ν_1 and ν_2, respectively.

3.1 The displacements and stresses of solid cylinder

The displacements and stresses are

$$\begin{bmatrix} u^s(r,z) \\ w^s(r,z) \end{bmatrix}=\left[A_{ij}^1\right]\left[C_j^1\right]e^{-\lambda z}+\left[A_{ij}^2\right]\left[C_j^2\right]e^{\lambda z},\rho$$
$$\begin{bmatrix} \sigma_{rr}^s(r,z) \\ \sigma_{zz}^s(r,z) \\ \sigma_{rz}^s(r,z) \end{bmatrix}=\left[B_{ij}^1\right]\left[C_j^1\right]e^{-\lambda z}+\left[B_{ij}^2\right]\left[C_j^2\right]e^{\lambda z}, \tag{3.1}$$

where

$$\left[C_j^1\right]=\left[C_1\ C_3\right]^T,\left[C_j^2\right]=\left[C_5\ C_7\right]^T,$$

$$\left[A_{ij}^1\right]=\begin{bmatrix} J_1(\lambda r) & \lambda r J_0(\lambda r) \\ J_0(\lambda r) & 2n_1J_0(\lambda r)-\lambda rJ_1(\lambda r) \end{bmatrix},\left[A_{ij}^2\right]=\begin{bmatrix} J_1(\lambda r) & \lambda r J_0(\lambda r) \\ -J_0(\lambda r) & -2n_1J_0(\lambda r)+\lambda rJ_1(\lambda r) \end{bmatrix},$$

$$\left[B_{ij}^1\right]=\begin{bmatrix} \lambda J_0(\lambda r)-J_1(\lambda r)/r & \lambda\{l_1J_0(\lambda r)-\lambda rJ_1(\lambda r)\} \\ -\lambda J_0(\lambda r) & \lambda\{m_1J_0(\lambda r)+\lambda rJ_1(\lambda r)\} \\ -\lambda J_1(\lambda r) & -\lambda\{n_1J_1(\lambda r)+\lambda rJ_0(\lambda r)\} \end{bmatrix},$$

$$\left[B_{ij}^2\right]=\begin{bmatrix} \lambda J_0(\lambda r)-J_1(\lambda r)/r & \lambda\{l_1J_0(\lambda r)-\lambda rJ_1(\lambda r)\} \\ -\lambda J_0(\lambda r) & \lambda\{m_1J_0(\lambda r)+\lambda rJ_1(\lambda r)\} \\ \lambda J_1(\lambda r) & \lambda\{n_1J_1(\lambda r)+\lambda rJ_0(\lambda r)\} \end{bmatrix},$$

$l_1=1-2\nu_1,\quad m_1=2(\nu_1-2),\quad n_1=2(1-\nu_1).$

The superscript "s" on the left hand side indicates the solution is for a solid circular cylinder.

3.2 The displacements and stresses of hollow cylinder

The displacements and stresses are

$$\begin{bmatrix} u^h(r,z) \\ w^h(r,z) \end{bmatrix}=\left[D_{ij}^1\right]\left[C_j^1\right]e^{-\lambda z}+\left[D_{ij}^2\right]\left[C_j^2\right]e^{\lambda z},$$
$$\begin{bmatrix} \sigma_{rr}^h(r,z) \\ \sigma_{zz}^h(r,z) \\ \sigma_{rz}^h(r,z) \end{bmatrix}=\left[E_{ij}^1\right]\left[C_j^1\right]e^{-\lambda z}+\left[E_{ij}^2\right]\left[C_j^2\right]e^{\lambda z}, \tag{3.2}$$

where

$$\left[C_j^1\right]=\left[C_1\ C_2\ C_3\ C_4\right]^T,\left[C_j^2\right]=\left[C_5\ C_6\ C_7\ C_8\right]^T,$$

$$\left[D_{ij}^1\right]=\begin{bmatrix} J_1(\lambda r) & J_0(\lambda r) \\ Y_1(\lambda r) & Y_0(\lambda r) \\ \lambda rJ_0(\lambda r) & 2n_2J_0(\lambda r)-\lambda rJ_1(\lambda r) \\ \lambda rY_0(\lambda r) & 2n_2Y_0(\lambda r)-\lambda rY_1(\lambda r) \end{bmatrix}^T,\left[D_{ij}^2\right]=\begin{bmatrix} J_1(\lambda r) & -J_0(\lambda r) \\ Y_1(\lambda r) & -Y_0(\lambda r) \\ \lambda rJ_0(\lambda r) & -2n_2J_0(\lambda r)+\lambda rJ_1(\lambda r) \\ \lambda rY_0(\lambda r) & -2n_2Y_0(\lambda r)+\lambda rY_1(\lambda r) \end{bmatrix}^T,$$

$$\left[E_{ij}^1\right]=\begin{bmatrix} \lambda J_0(\lambda r)-J_1(\lambda r)/r & -\lambda J_0(\lambda r) & -\lambda J_1(\lambda r) \\ \lambda Y_0(\lambda r)-Y_1(\lambda r)/r & -\lambda Y_0(\lambda r) & -\lambda Y_1(\lambda r) \\ \lambda\{l_2J_0(\lambda r)-\lambda rJ_1(\lambda r)\} & \lambda\{m_2J_0(\lambda r)+\lambda rJ_1(\lambda r)\} & -\lambda\{n_2J_1(\lambda r)+\lambda rJ_0(\lambda r)\} \\ \lambda\{l_2Y_0(\lambda r)-\lambda rY_1(\lambda r)\} & \lambda\{m_2Y_0(\lambda r)+\lambda rY_1(\lambda r)\} & -\lambda\{n_2Y_1(\lambda r)+\lambda rY_0(\lambda r)\} \end{bmatrix}^T,$$

$$\left[E_{ij}^2\right]=\begin{bmatrix} \lambda J_0(\lambda r)-J_1(\lambda r)/r & -\lambda J_0(\lambda r) & \lambda J_1(\lambda r) \\ \lambda Y_0(\lambda r)-Y_1(\lambda r)/r & -\lambda Y_0(\lambda r) & \lambda Y_1(\lambda r) \\ \lambda\{l_2J_0(\lambda r)-\lambda rJ_1(\lambda r)\} & \lambda\{m_2J_0(\lambda r)+\lambda rJ_1(\lambda r)\} & \lambda\{n_2J_1(\lambda r)+\lambda rJ_0(\lambda r)\} \\ \lambda\{l_2Y_0(\lambda r)-\lambda rY_1(\lambda r)\} & \lambda\{m_2Y_0(\lambda r)+\lambda rY_1(\lambda r)\} & \lambda\{n_2Y_1(\lambda r)+\lambda rY_0(\lambda r)\} \end{bmatrix}^T,$$

$l_2=1-2\nu_2,\quad m_2=2(\nu_2-2),\quad n_2=2(1-\nu_2).$

The superscript "h" on the left hand side indicates the solution is for a hollow circular cylinder.

4. Eigenvalues and eigenfunctions of the composite cylinder

Considering the interface at $r=a$ is perfectly bonded and at $r=1$, the stresses $\sigma_{rr}^h=\sigma_{rz}^h=0$, the equation for the eigenvalues of composite cylinder is given by

$$\frac{16\lambda(1-\nu_2)}{a^2\pi^2}\left\{\begin{matrix}(\nu_1-1)\left[\lambda^2J_0^2(\lambda)+(\lambda^2+2\nu_2-2)J_1^2(\lambda)\right] \\ +a^2\lambda^2(\nu_2-\nu_1)\left[J_0^2(\lambda)+J_1^2(\lambda)\right]\end{matrix}\right\}=0. \tag{4.1}$$

The composite cylinder with $\nu_1=\nu_2=1/3$ results in the solid cylinder. Figure 2 shows the first twenty eigenvalues of the solid cylinder in the first quadrant of the complex plane which are used in Section 6.

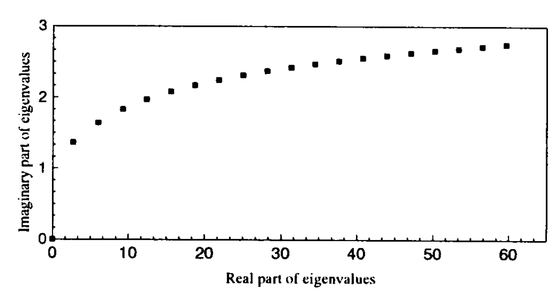

Fig. 2 Eigenvalues of solid cylinder, $\nu_1=\nu_2=1/3$.

Keywords: composite cylinder, eigenvalues, biorthogonal solution

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The relevant solutions can be expressed as

$$u^s(r,z)=u^*+\sum_k^\infty a_{1k}u_{1k}^s e^{-\lambda_k\left(z+\frac{L}{2}\right)}+\sum_k^\infty a_{2k}u_{1k}^s e^{-\lambda_k\left(\frac{L}{2}-z\right)},$$
$$w^s(r,z)=w^*+\sum_k^\infty a_{1k}w_{1k}^s e^{-\lambda_k\left(z+\frac{L}{2}\right)}+\sum_k^\infty a_{2k}w_{1k}^s e^{-\lambda_k\left(\frac{L}{2}-z\right)},\qquad (4.2)$$
$$\sigma_{zz}^s(r,z)=\sigma^*+\sum_k^\infty a_{1k}\sigma_{1k}^s e^{-\lambda_k\left(z+\frac{L}{2}\right)}+\sum_k^\infty a_{2k}\sigma_{1k}^s e^{-\lambda_k\left(\frac{L}{2}-z\right)},$$
$$\sigma_{rz}^s(r,z)=\tau^*+\sum_k^\infty a_{1k}\tau_{1k}^s e^{-\lambda_k\left(z+\frac{L}{2}\right)}+\sum_k^\infty a_{2k}\tau_{1k}^s e^{-\lambda_k z\left(\frac{L}{2}-z\right)},$$

where a_{1k}, a_{2k} are unknown coefficients and eigenfunctions are given by

$$u_{1k}^s=\left[\lambda_k^2J_0(\lambda_k)+2(1-\nu)\lambda_kJ_1(\lambda_k)\right]J_1(\lambda_kr)-\lambda_k^2rJ_1(\lambda_k)J_0(\lambda_kr),$$
$$w_{1k}^s=\left[\lambda_k^2J_0(\lambda_k)-2(1-\nu)\lambda_kJ_1(\lambda_k)\right]J_0(\lambda_kr)+\lambda_k^2rJ_1(\lambda_k)J_1(\lambda_kr),\qquad (4.3)$$
$$\sigma_{1k}^s=\left[-\lambda_k^3J_0(\lambda_k)+2\lambda_k^2J_1(\lambda_k)\right]J_0(\lambda_kr)-\lambda_k^3rJ_1(\lambda_k)J_1(\lambda_kr),$$
$$\tau_{1k}^s=-\lambda_k^3J_0(\lambda_k)J_1(\lambda_kr)+\lambda_k^3rJ_1(\lambda_k)J_0(\lambda_kr).$$

The superscript "*" on the right hand side indicates the eigenfunction for zero eigenvalue given by

$$u^*=-T\nu r,\quad w^*=Tz+K,\quad \sigma^*=T(1+\nu),\qquad (4.4)$$

where T, K are arbitrary constants that represent uniform tension in the z-direction and a rigid body translation parallel to the z-axis.

5. Biorthogonal relation of eigenfunctions

The biorthogonal relation obtained from the stress-free conditions, $\sigma_{rr}=\sigma_{rz}=0$, at $r=1$ is given by,

$$\int_0^1\left(w_{1j}^s\sigma_{1k}^s-\tau_{1j}^su_{1k}^s\right)rdr=0,\qquad (j\neq k).\qquad (5.1)$$

6. Determination of the coefficients in the eigen expansions

In the case of the finite solid cylinder with prescribed displacements at the plane ends, the coefficients are given by

$$a_{1k}+\sum_{j(j\neq k)}^\infty A_{jk}a_{1j}+B_ke^{-\lambda_kL}a_{2k}+\sum_{j(j\neq k)}^\infty C_{jk}e^{-\lambda_jL}a_{2j}=D_{1k},$$
$$a_{2k}+\sum_{j(j\neq k)}^\infty A_{jk}a_{2j}+B_ke^{-\lambda_kL}a_{1k}+\sum_{j(j\neq k)}^\infty C_{jk}e^{-\lambda_jL}a_{1j}=D_{2k},\qquad (6.1)$$
$$\int_0^1 w_{01}(r)rdr=\sum_{j(j\neq k)}^\infty\left(a_{1k}+a_{2k}e^{-\lambda_kL}\right)\int_0^1 w_{1k}^s(r)rdr+\frac{1}{2}\left(-T\frac{L}{2}+K\right),$$
$$\int_0^1 w_{02}(r)rdr=\sum_{j(j\neq k)}^\infty\left(a_{1k}e^{-\lambda_kL}+a_{2k}\right)\int_0^1 w_{1k}^s(r)rdr+\frac{1}{2}\left(T\frac{L}{2}+K\right),$$

where A_{jk} and C_{jk} are known matrix coefficients, B_k are known vector coefficients, D_{1k} and D_{2k} are known functions, $u_{01}(r), u_{02}(r), w_{01}(r)$ and $w_{02}(r)$ are prescribed end radial displacements at $z=-L/2,\ L/2$ respectively, and prescribed end axial displacements at $z=-L/2$, $L/2$, respectively.

7. Numerical results

For the finite solid cylinder the prescribed displacements at the plane ends are

$$u_{01}(r)=-0.008r,\quad w_{01}(r)=0.02\quad \text{at } z=-L/2,$$
$$u_{02}(r)=0.008r,\quad w_{02}(r)=-0.02\quad \text{at } z=L/2,$$

where now $L=1.2$.
Radial and axial displacements are shown in Figs. 3-6.

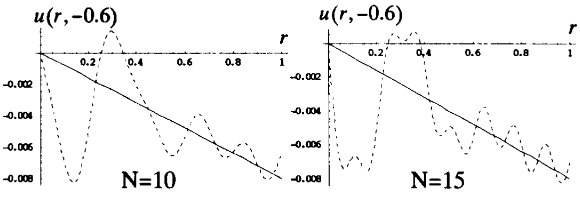

Fig. 3 Radial displacement u at $z=-0.6$; —— prescribed displacement; ----- biorthogonal solution.

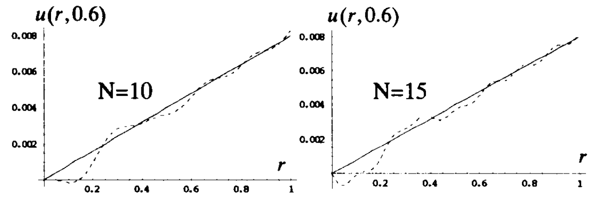

Fig. 4 Radial displacement u at $z=+0.6$; —— prescribed displacement; ----- biorthogonal solution.

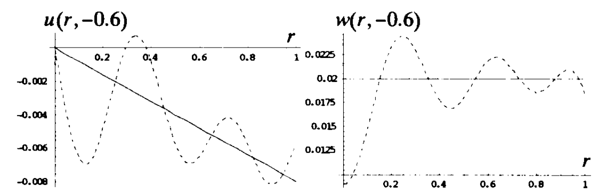

Fig. 5 Radial and axial displacements u, w at $z=-0.6$, N=5, —— prescribed displacement; ---- biorthogonal solution.

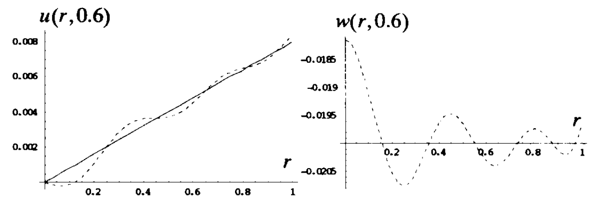

Fig. 6 Radial and axial displacements u, w at $z=+0.6$, N=5; —— prescribed displacement, ---- biorthogonal solution.

8. Conclusion and discussion

The main reason for ill convergent problems with biorthogonal solution are the factors $(\lambda_j-\lambda_k)^n$, $(n=1,2,\dots)$ in the denominators of all the terms in the matrix coefficients given in eqn (6.1). The effect of the factors $(\lambda_j-\lambda_k)^n$ becomes large as the number of pairs of eigenvalues become large.

References

Fama, M. E. D. (1972), "Radial eigenfunctions for the elastic circular cylinder," *Quarterly Journal of Mechanics*, Vol. XXV, pp. 479-495.

Robert, M. and Keer, L. M. (1987), "An elastic circular cylinder with displacement prescribed at the ends-axially symmetric case," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 40, pp. 339-363.