

INTRODUCTION OF REGULARIZATION TECHNIQUE IN MINIMIZATION METHODS TO IMPROVE ACCURACY OF BACKCALCULATION RESULTS

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INTRODUCTION

In most backcalculation analyses, pavement layer moduli, E_j , which will give computed deflection basin that is very close to the measured basin, are obtained by a minimization technique. In this technique, the sum of squares of the differences between measured and computed pavement surface deflections are minimized. Backcalculation analysis involves an iteration process where computations are repeated while modifying layer moduli until the rate of change of layer moduli is equal to or less than a tolerance value. There is, however, a problem on non-uniqueness of the solution. In this research, a regularization technique is introduced to improve accuracy of minimization method in order to obtain a stabilized solution.

GENERAL FORMULATION

The objective function, J , in the minimization could be written as:

$$J = \sum_{i=1}^n (u_i - z_i(E_j))^T (u_i - z_i(E_j)) \quad (1)$$

On employing Taylor series expansion:

$$z(E + dE) = z(E) + AdE \quad (2)$$

where: $A = \frac{\partial z}{\partial E}$ is $m \times n$ matrix with $m \geq n$

Substituting equation (2) into equation (1) and minimize by partially differentiating J with respect to dE yields:

$$AdE = u - z(E) \quad (3)$$

Substituting x for dE and b for $u - z(E)$ and simplify yields a general expression for linear least-squares problems:

$$\min_x \|Ax - b\|_2 \quad (4)$$

The singular value decomposition (SVD) of A is given as:

$$A = U \Sigma V^T = \sum_{i=1}^n u_i \sigma_i v_i^T \quad (5)$$

where $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_n)$ are matrices with orthonormal columns, $U^T U = V^T V = V V^T = I$, and where $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$ has non-negative diagonal elements appearing in non-increasing order such that $\sigma_1 \geq \dots \geq \sigma_n \geq 0$. σ_i are singular values of A , while u_i and v_i are left and right singular vectors of A , respectively. The condition number of A is equal to the ratio σ_1 / σ_n .

The solution to the least-squares problem, x_{LSQ} , is given by:

$$x_{LSQ} = \sum_{i=1}^n \frac{u_i^T b}{\sigma_i} v_i \quad (6)$$

It is worth noting that, if the singular value of A decay gradually to zero and the condition number is large, then this problem is called a discrete ill-posed problem. The solution, x_{LSQ} , is dominated by the terms in the sum corresponding to the smallest σ_i . As a consequence, the solution, x_{LSQ} , has many sign changes and appears completely random. It is difficult to obtain a meaningful solution to this problem in a straightforward manner using standard methods in numerical linear algebra such as LU, Cholesky or QR factorization. In case of small perturbation of the data used in an ill-posed problem, there will be arbitrarily large perturbation of the solution.

Key Words: Backcalculation, Minimization, Ill-posed Problem, Regularization.

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REGULARIZATION METHODS

These are practical methods for computing stabilized solutions to the discrete ill-posed problems by diminishing the influence of noise. The underlying argument in regularization is the idea that a regularized solution with small norm and suitably small residual norm is not too far from the desired, unknown solution to the unperturbed problem underlying the given problem. The most common and popular form of regularization is the one known as Tikhonov regularization. The regularized solution \mathbf{x} is defined as the minimizer of the following weighted combination of the residual norm and the side constraint

$$\mathbf{x} = \operatorname{argmin} \left\{ \|\mathbf{Ax} - \mathbf{b}\|_2^2 + \lambda^2 \|\mathbf{x}\|_2^2 \right\} \quad (7)$$

where the regularization parameter, λ , controls how much weight is given to minimization of $\|\mathbf{x}\|_2$ relative to minimization of residual norm.

Since the purpose of regularization is to dampen or filter out the contributions to the solution corresponding to the small singular values, a regularized solution, \mathbf{x}_{reg} , can be written as:

$$\mathbf{x}_{\text{reg}} = \sum_{i=1}^n f_i \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i \quad \text{and} \quad 0 \leq f_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \leq 1 \quad (8)$$

A good regularization parameter should strike a fair balance between the perturbation error and the regularization error in the regularized solution. A plot in log-log scale, for all valid regularization parameters, of the norm $\|\mathbf{x}_{\text{reg}}\|_2$ of the regularized solution versus the corresponding residual norm $\|\mathbf{Ax}_{\text{reg}} - \mathbf{b}\|_2$ displays the compromise between minimization of these two quantities. A distinct corner separating the vertical and the horizontal parts of the curve will give the optimum regularization parameter. This corner is a point of maximum slope in the plot. By using optimum regularization parameter, unique solution of the problem will be obtained.

EXAMPLE

A famous ill-posed matrix is used as an example to illustrate the extent of the problem discussed above.

$\mathbf{A} = (a_{11}, \dots, a_{ij}, \dots, a_{nn})$, where: $a_{ij} = 1/(i+j-1)$, where $i = 1, \dots, n$ while $j = 1, \dots, n$ and $n = 10$

$\mathbf{b} = (b_1, \dots, b_n)$, where $b_i = \sum_{j=1}^n a_{ij}$

The true solution, \mathbf{x} , to this problem is given as: $\mathbf{x} = (1, 1, \dots, 1)$

The condition number of \mathbf{A} is $1.6\text{E}+13$, which means smaller perturbation in \mathbf{b} will have a bigger effect in the minimization solution. In order to clarify this point small errors, $\Delta\mathbf{b} = 1\text{E}-06$ & $1\text{E}-05$ are introduced in \mathbf{b} . Straightforward use of standard numerical method and Tikhonov regularization technique were performed and results are shown in Table 1.

Table 1 Least-Square Solutions for Straightforward (S/F) Method and Regularization (R) Technique

Error $\Delta\mathbf{b}$	Method	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.E+00	S/F	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00
	R	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00
1.E-06	S/F	1.00E+00	1.00E+00	9.76E-01	1.24E+00	-2.61E-01	4.78E+00	-5.73E+00	8.00E+00	-2.94E+00	1.92E+00
	R	1.00E+00	1.00E+00	1.00E+00	9.97E-01	1.00E+00	1.00E+00	1.00E+00	9.96E-01	9.94E-01	1.01E+00
1.E-05	S/F	1.00E+00	1.01E+00	7.62E-01	3.40E+00	-1.16E+01	3.88E+01	-6.63E+01	7.10E+01	-3.84E+01	1.02E+01
	R	1.00E+00	9.98E-01	1.01E+00	9.72E-01	9.95E-01	1.05E+00	1.02E+00	9.55E-01	9.41E-01	1.06E+00

CONCLUSION

It has been shown that the use of Tikhonov regularization can improve accuracy of minimization method and a stabilized solution to the least-squares problem can be obtained.

REFERENCE

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