The vertex singularity in the Sekiguchi-Ohta model

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1. Introduction

The Critical state theory incorporated with normality criterion has released many numerical implementations in soil mechanics. Sekiguchi and Ohta (1977) proposed the constitutive model to address the stress anisotropy induced during the natural clay deposition in addition to those of Cambridge models by introducing the non-negative normalized shear stress η* taking principal stresses reorientation into account. The expression causes the inevitable discontinuity by accommodating the singular vertex in stress space. In recent days, Pipatpongsa et al. (2001a,b) developed the mathematical treatment for the intersecting corner of two continuously differentiable convex yield loci; namely, upper and lower yield loci, and evaluated theoretical K_o-value and Poisson's ratio in corresponding to the Sekiguchi-Ohta model. However, it is not clear the implementation, which is based on the triaxial condition, is valid for general conditions. The paper discusses the scope of method by considering the existence of the vertex in principal stress space and plane strain condition. This study may lead to a better understanding of the vertex singularity in the model and its implementation.

2. Deviatoric view of yield surface

In addition to three stress invariants, the stress-induced anisotropic yield function for an inherent isotropic media must depend on the state of stress at the completion of consolidation. Herein, the invicid form of yield function proposed by Sekiguchi and Ohta (1977) is shown by Eq.(1).

$$f(\boldsymbol{\sigma},\boldsymbol{\sigma}_o,\alpha)=f(p',\eta^*,\alpha)\equiv MD\ln\left(\frac{p'}{p'_o}\right)+D\eta^*-\alpha=0 \quad (1)$$

where $\alpha\equiv\varepsilon_v^p=\int\dot{\varepsilon}_v^p dt$; $\boldsymbol{\eta}\equiv\frac{\mathbf{s}}{p'}$; $\boldsymbol{\eta}_o\equiv\frac{\mathbf{s}_o}{p'_o}$; $\eta^*\equiv\sqrt{\frac{3}{2}}\|\boldsymbol{\eta}-\boldsymbol{\eta}_o\|$

The set of intersection of the yield surface with π-plane is yield curve, which is conveniently given by the expression transformed to polar coordinates where θ is angle measured anti-clockwise on π-plane. The substitution of θ=const. gives the meridional section relating $\|\mathbf{s}\|$ and p'.

$$\bar{s}(p',\theta)=\sqrt{\frac{2}{3}}p'\left[\cos(\theta)\eta_o+\sqrt{M^2\ln\left(\frac{p'}{p'_o}\right)^2-\sin(\theta)^2\eta_o^{\,2}}\right] \quad (2)$$

The major principal stress axis at θ=0 locally coincides with the major principal direction of stress-induced initial anisotropy, in general, the vertical stress direction. Fig. 1 shows the plot of Eq.(2). The physical meanings of the angle θ are given as following.

$\pi/3\geq\theta\geq0$ for $\sigma'_z\geq\sigma'_y\geq\sigma'_x$, compression test: θ=0
$2\pi/3\geq\theta\geq\pi/3$ for $\sigma'_y\geq\sigma'_z\geq\sigma'_x$
$\pi\geq\theta\geq2\pi/3$ for $\sigma'_y\geq\sigma'_x\geq\sigma'_z$, extension test: θ=π

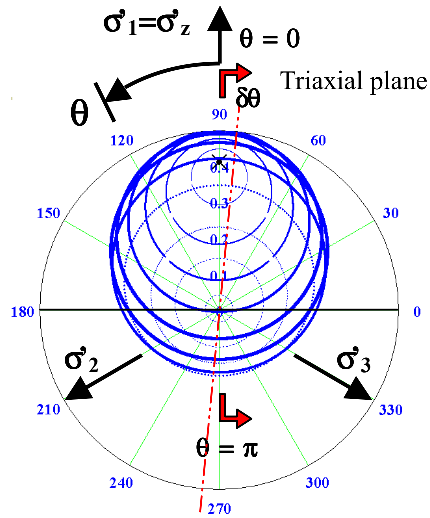

Figure 1: Circular yield curves formed by intersection of yield surface with planes of constant mean stress

Taking θ=0 and π will cut the Sekiguchi-Ohta yield surface by a triaxial plane relating to customary p'-q plane where upper and lower yield loci with intersecting corner can be observed in Fig. 2. It is clearly seen this particular state of stress totally passes the singular vertex (η*=0) where the serious numerical convergence occurs. The corner is rounded off for a small rotation δθ, indicating the special treatment is only required for state of stress under axis-symmetry in which the Sekiguchi-Ohta model is reduced to the Ohta-Hata model (1971). In the case of

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plane-strain, the intermediate effective stress is determined by Eq.(3), thus diverting the stress condition from the vertex. However, K_o-condition can be deduced from plane-strain when $\tau_{zx}=0$, $\sigma'_x = K_o\sigma'_z$.

$$\sigma'_2 = \sigma'_y = \nu'(\sigma'_x + \sigma'_z) = \frac{K_o}{1+K_o}(\sigma'_x + \sigma'_z) \qquad (3)$$

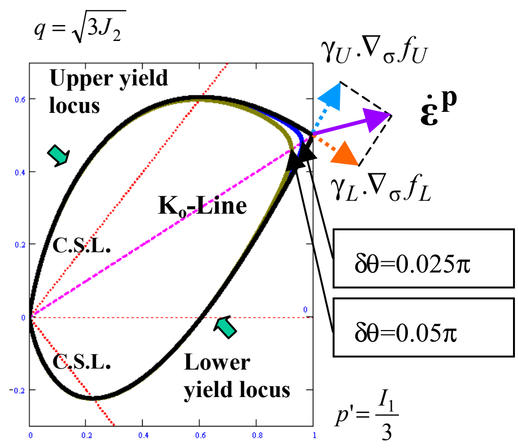

Figure 2: p'-q plane relating to meridional section at θ=0 and π

3. Implementation at the vertex singularity

Singular yield surfaces with edges or corners may be described by a finite number of yield functions based on Koiter's suggestion. Concerning with the Sekiguchi-Ohta model, the discontinuity is observed on triaxial plane where upper and lower yield locus expressed by Eq.(4),(5) intersect each other to form the corner. At the singular stress $\boldsymbol{\sigma}$ in which $f_U(\boldsymbol{\sigma},\alpha) = f_L(\boldsymbol{\sigma},\alpha) = 0$, a consistency requirement guarantees the actual values of $\gamma_U \geq 0$ and $\gamma_L \geq 0$ can be determined, then $\boldsymbol{\sigma}$ must keep on the hardening vertex so that $\dot{f}_U(\boldsymbol{\sigma},\alpha) = \dot{f}_L(\boldsymbol{\sigma},\alpha) = 0$. For a certain imposed strain rate in which either $\gamma_U = 0$ or $\gamma_L = 0$ is evaluated, this particular method will reduce to the ordinary method applicable to the Sekiguchi-Ohta model and the stress point $\boldsymbol{\sigma}$ will move out of the singularity. The basic equations in tensor notation are available below.

$$f_U(p',q,\alpha) \equiv MD\ln\left(\frac{p'}{p'_o}\right) + D\left(\frac{q}{p'} - \eta_o\right) - \alpha = 0 \qquad (4)$$

$$f_L(p',q,\alpha) \equiv MD\ln\left(\frac{p'}{p'_o}\right) - D\left(\frac{q}{p'} - \eta_o\right) - \alpha = 0 \qquad (5)$$

Incremental elastic stress-strain relations:

$$\dot{\boldsymbol{\sigma}} \equiv \mathbf{C} \bullet \left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{\mathbf{p}}\right) \qquad (6)$$

Evolution of associated flow rule: Koiter (1953)

$$\dot{\boldsymbol{\varepsilon}}^{\mathbf{p}} \equiv \gamma_U \nabla_{\boldsymbol{\sigma}} f_U(\boldsymbol{\sigma},\alpha) + \gamma_L \nabla_{\boldsymbol{\sigma}} f_L(\boldsymbol{\sigma},\alpha) \qquad (7)$$

Consistency requirement at the corner: (8),(9)

$$\dot{f}_U = \nabla_{\boldsymbol{\sigma}} f_U \bullet \dot{\boldsymbol{\sigma}} + \frac{\partial f_U}{\partial \alpha}\dot{\alpha} \equiv 0 \quad \text{and} \quad \dot{f}_L = \nabla_{\boldsymbol{\sigma}} f_L \bullet \dot{\boldsymbol{\sigma}} + \frac{\partial f_L}{\partial \alpha}\dot{\alpha} \equiv 0$$

From (6)-(9), the manipulation for unknowns is shown by

$$\begin{bmatrix} \mathbf{C}^{-1} & \{\nabla_{\boldsymbol{\sigma}} f_U\} & \{\nabla_{\boldsymbol{\sigma}} f_L\} \\ \{\nabla_{\boldsymbol{\sigma}} f_U\}^T & \frac{\partial f_U}{\partial \alpha}\frac{\partial f_U}{\partial p'} & \frac{\partial f_U}{\partial \alpha}\frac{\partial f_L}{\partial p'} \\ \{\nabla_{\boldsymbol{\sigma}} f_L\}^T & \frac{\partial f_L}{\partial \alpha}\frac{\partial f_U}{\partial p'} & \frac{\partial f_L}{\partial \alpha}\frac{\partial f_L}{\partial p'} \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\sigma}} \\ \gamma_U \\ \gamma_L \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{\varepsilon}} \\ 0 \\ 0 \end{pmatrix} \qquad (10)$$

To solve Eq.(10), $\mathbf{X}_{(2x2)}$, $\mathbf{L}_{(2x1)}$ and consistency parameters must be primarily obtained by calculating Eq.(11-13)

Coupled hardening matrix: (11)

$$\mathbf{X} = \begin{bmatrix} \nabla_{\boldsymbol{\sigma}} f_U \bullet \mathbf{C} \bullet \nabla_{\boldsymbol{\sigma}} f_U - \frac{\partial f_U}{\partial \alpha}\frac{\partial f_U}{\partial p'} & \nabla_{\boldsymbol{\sigma}} f_U \bullet \mathbf{C} \bullet \nabla_{\boldsymbol{\sigma}} f_L - \frac{\partial f_U}{\partial \alpha}\frac{\partial f_L}{\partial p'} \\ \nabla_{\boldsymbol{\sigma}} f_L \bullet \mathbf{C} \bullet \nabla_{\boldsymbol{\sigma}} f_U - \frac{\partial f_L}{\partial \alpha}\frac{\partial f_U}{\partial p'} & \nabla_{\boldsymbol{\sigma}} f_L \bullet \mathbf{C} \bullet \nabla_{\boldsymbol{\sigma}} f_L - \frac{\partial f_L}{\partial \alpha}\frac{\partial f_L}{\partial p'} \end{bmatrix}$$

Loading parameters: Consistency parameters:

$$\mathbf{L} = \begin{pmatrix} \nabla_{\boldsymbol{\sigma}} f_U \bullet \mathbf{C} \bullet \dot{\boldsymbol{\varepsilon}} \\ \nabla_{\boldsymbol{\sigma}} f_L \bullet \mathbf{C} \bullet \dot{\boldsymbol{\varepsilon}} \end{pmatrix} \quad (12) \qquad \begin{pmatrix} \gamma_U \\ \gamma_L \end{pmatrix} = \mathbf{X}^{-1} \cdot \mathbf{L} \quad (13)$$

Tangent elastoplastic moduli: $\dot{\boldsymbol{\sigma}} \equiv \mathbf{C}^{ep} \bullet \dot{\boldsymbol{\varepsilon}}$

$$\mathbf{C}^{ep} = \mathbf{C} - \sum_{\alpha,\beta \in \{1,2\}} \mathbf{X}^{-1}{}_{\alpha,\beta}\left(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right) \quad \text{where } \mathbf{g} = \begin{bmatrix} \mathbf{C} \bullet \nabla_{\boldsymbol{\sigma}} f_U \\ \mathbf{C} \bullet \nabla_{\boldsymbol{\sigma}} f_L \end{bmatrix} \qquad (14)$$

4. Conclusion

A generalized concept to the Sekiguchi-Ohta yield surface possessing the singular point where the gradients of yield surface (or potential) to stress space are indeterminate is implemented. Though Koiter's method does not apply to the Sekiguchi-Ohta model in stress space, it is particularly applied to the intersecting corner of two yield loci characterized by the Sekiguchi-Ohta model on Rendulic's stress plane or triaxial plane, where the plane of induced anisotropy is coincided, resulting in simple formulation.

5. References

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