Flexural Wave Propagation in Pre-stressed Imperfectly Bonded Incompressible Layered Elastic Composites

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1. Introduction

The problem of flexural wave propagation in 4-ply laminated composites is studied (Fig. 1). The wave is traveling along x_1-direction and the pre-stress is in x_2-direction. The imperfect interface is simulated by a spring type resistance model, and the effect of the interface on the dispersion relation is investigated.

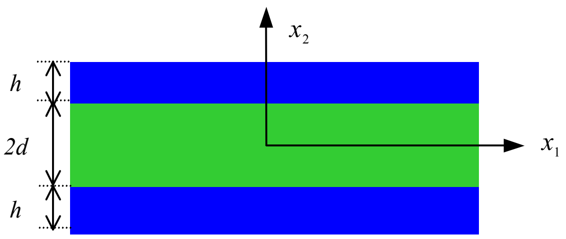

Figure 1. The 4-ply composite.

2. The governing equations

The equations of motion are expressed as

$$B_{1111}u_{1,11}+(B_{1122}+B_{2112})u_{2,21}+B_{2121}u_{1,22}-p^*_{,1}=\rho\ddot{u}_1\ , \qquad (2.1)$$

$$(B_{1221}+B_{2211})u_{1,12}+B_{1212}u_{2,11}+B_{2222}u_{2,22}-p^*_{,2}=\rho\ddot{u}_2\ , \qquad (2.2)$$

where B_{ijkl} are components of the appropriate fourth order elasticity tensor, u_i are the displacements superimposed upon a pre-stressed equilibrium state, ρ is the density of the material, p^* is a time-dependent pressure increment and comma indicates differentiation with respect to the implied spatial coordinate component. In addition, the non-zero linearized traction increments are

$$\tau_1=B_{2121}u_{1,2}+(B_{2112}+\bar{p})u_{2,1}\,, \qquad (2.3)$$

$$\tau_2=B_{2211}u_{1,1}+(B_{2222}+\bar{p})u_{2,2}-p^*. \qquad (2.4)$$

The flexural wave is expressed in the form

$$(u_1,u_2,p^*)=(U_1,\ U_2\quad kP)\quad e^{kq_2h}e^{ik(x_1-vt)}\,, \qquad (2.5)$$

where q is determined by satisfying eqns (2.1)-(2.4).

3. Propagator Matrix

The parameter q in the eqn (2.5) has four roots, so the displacements are expressed in the form

$$u_l(x_1,x_2,t)=\sum_j^4\{U_l^{(j)}e^{kq'_jx_2}\}e^{ik(x_1-vt)},\quad l=1,\ 2\,. \qquad (3.1)$$

At arbitrary x_1 and t, let

$$U_l(x_2)=\sum_j^4\{U_l^{(j)}e^{kq'_jx_2}\},\quad l=1,\quad 2\,, \qquad (3.2)$$

within which

$$q'_1=q_1,\qquad q'_2=q_2,\qquad q'_3=-q_1,\qquad q'_4=-q_2\,. \qquad (3.3)$$

Then the vector of the displacements and tractions at that point is expressed as functions of x_2

$$\mathbf{Y}(x_2)=(-iU_1(x_2),\ U_2(x_2),\ \frac{\tau_1(x_2)}{ik},\ \frac{\tau_2(x_2)}{k})^{\mathrm{T}}. \qquad (3.4)$$

This vector is related to the arbitrary constants $\mathbf{A}$,

$$\mathbf{Y}(x_2)=\mathbf{HE}(x_2)\mathbf{A}\,, \qquad (3.5)$$

$$\mathbf{A}=(U_2^1,U_2^2,U_2^3,U_2^4)^{\mathrm{T}}\,. \qquad (3.6)$$

Therefore the appropriate propagator matrix is derived to get the vector $\mathbf{Y}(x_2)$ at an arbitrary point of x_2-axis from a known vector.

$$\mathbf{Y}(x_2)=\mathbf{P}(x_2-\bar{x}_2)\mathbf{Y}(\bar{x}_2)\,, \qquad (3.7)$$

$$\mathbf{P}(x_2-\bar{x}_2)=\mathbf{H}(x_2-\bar{x}_2)\mathbf{H}^{-1}\,, \qquad (3.8)$$

$$\mathbf{E}(x_2)=diag(E_1^+,E_1^-,E_2^+,E_2^-),\qquad E_m^{\pm}=\exp(\pm kq_mh)\,. \qquad (3.9)$$

Equations (2.1)-(3.9) are equations for the outer layer. Equations for the inner core are obtained from the previous equations for the outer layer by interchanging $q\to p$ and denoting quantities associated with the inner core by a tilde, for example $B_{ijkl}\to\tilde{B}_{ijkl}$, $\mathbf{Y}(x_2)\to\tilde{\mathbf{Y}}(\tilde{x}_2)$.

4. Boundary Conditions

For the imperfect interface the interfacial shear stress is given by,

$$\sigma_{21}(d)=\frac{k_x\mu}{d}=\{U_1(d)-\tilde{U}_1(d)\}\,, \qquad (4.1)$$

where k_x is the spring constant, μ is the Lamé parameter and d is the thickness of the inner core. Then,

$$\tau_1(d)=\tilde{\tau}_1(d)=\sigma_{21}\,. \qquad (4.2)$$

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At the upper surface of the material,

$$Y(d+h) = (-iU_1(d+h),\ U_2(d+h), 0, 0)^{\mathrm{T}} . \qquad (4.3)$$

For the flexural wave at the mid plane

$$\tilde{Y}(0) = (0,\ \tilde{U}_2(0), \frac{\tilde{\tau}_1(0)}{ik}, 0)^{\mathrm{T}} . \qquad (4.4)$$

5. Dispersion equation

With the use of the propagator matrix, the boundary conditions are introduced and the dispersion equation for the imperfectly bonded composite is obtained in the form

$$\begin{aligned} &P_{3i}\tilde{P}_{i2}P_{4j}\tilde{P}_{i3} - P_{3i}\tilde{P}_{i3}P_{4j}\tilde{P}_{j2} + \\ &\frac{d\ \ k}{k_x\mu}(P_{41}\tilde{P}_{33}P_{3i}\tilde{P}_{i2} + P_{31}\tilde{P}_{32}P_{4i}\tilde{P}_{i3} - P_{41}\tilde{P}_{32}P_{3i}\tilde{P}_{i3} \\ &\quad -P_{31}\tilde{P}_{33}P_{4i}\tilde{P}_{i2}) = 0. \end{aligned} \qquad (5.1)$$

For a perfectly bonded composite $k_x \to \infty$, and epn (5.1) agrees with Rogerson and Sandiford (1997). In the case of $k_x = 0$ the bonding disappears, and eqn (5.1) becomes that for the slipping case.

6. Numerical Results

The material parameters are defined as

$$\alpha = B_{1212},\ 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221},\ \gamma = B_{2121}, \qquad (6.1)$$

and given in Table 1.

Table 1. The material parameters

	α	2β	γ	σ_2	ρ
Material 1	3.0	4.5	0.5	1.2	1
Material 2	1.0	2.5	1.2	1.2	1
Material 3	4.0	5.0	1.0	1.2	1
Material 4	3.0	1.5	0.5	1.2	1

There are four cases of limiting wave speeds, given by

$$\rho v_L^2 = \begin{cases} \alpha & \alpha \le 2\beta \\ 2\beta - 2\gamma + 2\sqrt{\gamma}\sqrt{\alpha+\gamma-2\beta} & \alpha > 2\beta \end{cases} . \qquad (6.2)$$

The results are shown for all four cases in Yamamoto (2001). In Case 1, Material 1 and 2 are used for outer layer and inner core respectively. Similarly in Case 3, Material 3 and Material 4 are used. This abstract contains Case 1 and Case 3。

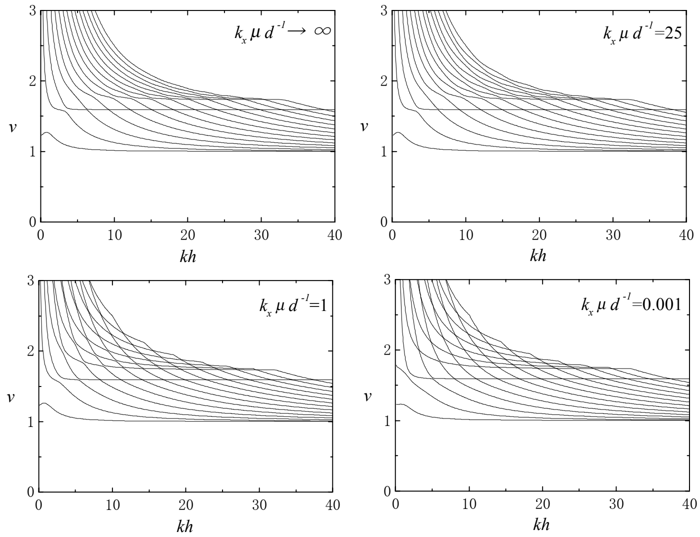

Figure 2 The dispersion curves for Case 1.

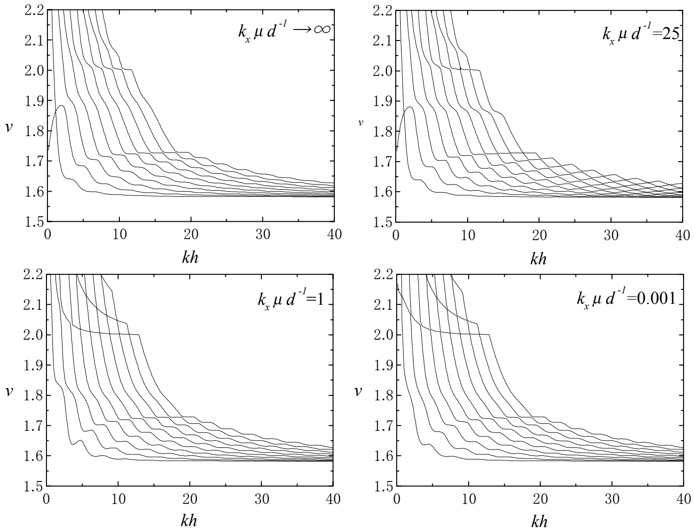

Figure 3 The dispersion curves for Case3

7. Conclusion

The main aspect observed in the graphs is the difference between Case 1 and Case 3. In addition, the graphs for Case 2 are similar to those of Case 1, and Case 3 and Case 4 have the same tendency. The effect of the changing value of spring constant is clearly seen. The difference in the dispersion curves are small when $\frac{k_x\mu}{d} = 25$ and $\frac{k_x\mu}{d} \to \infty$.

8. Reference

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